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Araştırma Makalesi / Research Article

Family of Surfaces with A Common Mannheim D- Isoasymptotic Curve

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Abstract

In this study, we have given the necessary and sufficient conditions for a Mannheim D pair curve to be both isoparametric and asymptotic on a surface. By using this, we have formed the parametrization for the family of surfaces accepting the basis curve as a Mannheim D-pair as an asymptotic curve. Moreover, we have extended the idea to the ruled surfaces and examined the conditions for those to be developable. Finally, we have presented some examples to illustrate the surfaces with a common Mannheim D- asymptotic curve.

AMS Subject Classification: 53A04, 53A05, 53C22

Keywords: Mannheim D-curves; Darboux Frame; Asymptotic curve; Parametric Surface; Ruled Surface

Ortak Mannheim D- İsoasimptotik Eğrili Yüzeyler Ailesi

Öz

Bu çalışmada, herhangi bir Mannheim D- eğri çiftinin bir yüzey üzerinde hem izoparametrik hem de asimtotik olması için gerekli ve yeterli şartlar verildi. Bundan yola çıkarak dayanak eğrisini asimtotik Mannheim D- eğrisi kabul eden yüzeyler ailesinin parametrik formu oluşturuldu. İlgili fikir aynı zamanda regle yüzeylere de genişletilerek, bu regle yüzeyler için açılabilir olma şartları incelendi. Son olarak, ortak Mannheim D- isoasimtotik eğrili yüzey aileleri için bazı örnekler sunuldu.

Anahtar Kelimeler: Mannheim D- eğrileri; Darboux çatısı, Asimtotik eğri; Parametrik Yüzey, Regle Yüzey

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1. Introduction

In differential geometry, the theory of curves is an important subject. It is well known that the curves are named as associated if there exist any relations to be established among them. The Mannheim pair is the one kind of associated curves having the property of principal normal and binormal parallelization. It was first introduced by Mannheim (1878). The characterizations of the curve were studied in the Euclidean and Minkowski space by Liu & Wang (2008) and Lee (2011) respectively. The Mannheim D- partner curves were first introduced by Kazaz et al (2015) in Euclidean 3-space. They defined the Mannheim D- curve first and investigated the properties of these curves to reach the certain characterizations. On the other hand, the asymptotic curve is coined as one of the significant curves on a surface. Bayram et al (2012) introduced the parametric representations for the family of surfaces with a common asymptotic curve. In addition to these, the parametric representation of the surfaces possessing a Mannheim pair curve as an asymptotic was studied by Atalay (2018). The main aim of this study is to form the parameterization of surfaces having the basis Mannheim D pair curve as asymptotic.

2. Preliminaries

Let E^3 denote 3-dimensional Euclidean space defined with the usual inner product as $\langle u,v\rangle = u_1v_1 + u_2v_2 + u_3v_3$ and vector product as $u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ where e_i is the standard basis, u_i and v_i are two vector elements of the rectangular coordinate system of E^3 $(i = 1,2,3)$. Define the norm of any given vector, $V \in E^3$ as $\|V\| = \sqrt{\langle V,V\rangle}$. Now, take $\alpha = \alpha(s) : I \subset IR \to E^3$ any curve and denote $\{\mathrm{T(s), N(s), B(s)}\}$ as the Frenet frame moving along α. The well-known Frenet formulae is given by

$$\begin{aligned} T'(s) &= \kappa(s)N(s) \\ N'(s) &= -\kappa(s)T(s) + \tau(s)B(s) \\ B'(s) &= -\tau(s)N(s) \end{aligned} \qquad \textbf{(2.1)}$$

where $\kappa(s) = \dfrac{\|\alpha'(s) \times \alpha''(s)\|}{\|\alpha'(s)\|^3}$ and $\tau(s) = \dfrac{\langle \alpha'(s) \times \alpha''(s), \alpha'''(s)\rangle}{\|\alpha'(s) \times \alpha''(s)\|^2}$ are called the curvature and torsion of the curve α, respectively. Now let T be the tangent vector of the curve α on a surface φ, n be the principal normal at point P of surface and g be defined as $g = n \times T$. Such frame denoted by $\{T, g, n\}$ is called as Darboux frame (O'Neill 1966). Relations between the Frenet and Darboux frame are given as

$$\begin{aligned} T &= T \\ N &= \cos\theta g - \sin\theta n, \qquad \sphericalangle(g,N) = \theta. \\ B &= \sin\theta g + \cos\theta n \end{aligned} \qquad \textbf{(2.2)}$$

Derivative changes of the Darboux frame are shown as below:

$$
\begin{aligned}
T' &= \kappa_g g + \kappa_n n \\
g' &= -\kappa_g T + \tau_g n \\
n' &= -\kappa_n T - \tau_g g
\end{aligned}
\tag{2.3}
$$

where the geodesic curvature κ_g, the normal curvature κ_n, and the geodesic torsion τ_g are defined to be as

$$
\kappa_g = \kappa\cos\theta, \quad \kappa_n = \kappa\sin\theta, \quad \tau_g = \tau + \frac{d\theta}{ds}. \tag{2.4}
$$

Let α and α^* be taken on two directed surfaces φ and φ^*, respectively and $\{T, g, n\}$ and $\{T^*, g^*, n^*\}$ denote the Darboux frame of each. If g and n^* are linearly dependent then (α, α^*) pair is called to be Mannheim D- pair curves, (Kazaz et al 2015). According to this definition we write

$$
\alpha^*(s) = \alpha(s) + \lambda(s)g(s). \tag{2.5}
$$

Relationships between the Darboux frame of the Mannheim D- partner curves (α, α^*) are given as:

$$
\begin{aligned}
T^* &= \cos\psi T - \sin\psi\sin\theta g - \sin\psi\cos\theta n, \\
g^* &= \sin\theta^*\sin\psi T + (\cos\theta^*\cos\theta + \sin\theta^*\sin\theta\cos\psi)g \\
&\quad + (-\sin\theta\cos\theta^* + \sin\theta^*\cos\theta\cos\psi)n, \\
n^* &= \cos\theta^*\sin\psi T + (-\sin\theta^*\cos\theta + \cos\theta^*\sin\theta\cos\psi)g \\
&\quad + (\sin\theta\sin\theta^* + \cos\theta^*\cos\theta\cos\psi)n,
\end{aligned}
\tag{2.6}
$$

where $\sphericalangle(T, T^*) = \psi, \quad \sphericalangle(g, N) = \theta, \quad \sphericalangle(g^*, N^*) = \theta^*$, (Kazaz et al 2015).

A curve on a surface is called as asymptotic provided its velocity always points in an asymptotic direction in which the normal curvature is zero (O'Neill 1966). On the other hand, the curve $\alpha(s)$ lying on the surface, $\varphi = \varphi(s, v)$ is called as isoparametric if any one of the parameters, $(s \text{ or } v)$ is constant. In other words, there exist a constant parameter, s_0 (or v_0) such that $\alpha(s) = \varphi(s, v_0)$ (or $\alpha(v) = \varphi(s_0, v)$). Now, if the curve, $\alpha(s)$ is both isoparametric and asymptotic on the surface, φ then it is called as an isoasymptotic curve (Bayram et al 2012).

Moreover, given a surface $\varphi_r(s, v)$ is called as ruled if it is formed by a straight line that moves along the curve. The parametric form of such is given as:

$$
\varphi_r(s, v) = \alpha(s) + ve(s) \tag{2.7}
$$

where $\alpha(s)$ is the base curve and $e(s)$ is the director. A sufficient condition for a ruled surface to be a developable one is that $\det(\alpha', e, e') = 0$, (O'Neill 1966).

3. Family of Surfaces with a Common Mannheim D- Isoasymptotic Curve

Let $\alpha^*(s)$ be a Mannheim D- partner curve of $\alpha(s)$. Surface family that interpolates $\alpha^*(s)$ as a common curve is given in general with the parametric form as

$$\varphi^*(s,v) = \alpha^*(s) + \left(x(s,v)T^*(s) + y(s,v)g^*(s) + z(s,v)n^*(s)\right) \quad \textbf{(3.1)}$$

where $L_1 \le s \le L_2,\ K_1 \le v \le K_2$ and $x(s,v),\ y(s,v),\ z(s,v)$ are named to be differentiable marching scale functions. Since $\alpha^*(s)$ is a Mannheim D- partner of $\alpha(s)$ we know that $\lambda(s) = \lambda$ is a constant in the equation (2.5), so we rewrite it as following

$$\alpha^*(s) = \alpha(s) + \lambda g(s). \quad \textbf{(3.2)}$$

Now by recalling the equation (2.2), we reparametrize the surface family with a common Mannheim D- asymptotic curve as

$$\begin{aligned} \varphi^*(s,v) &= \alpha^*(\mathrm{s}) + x(s,v)\mathrm{T}^*(\mathrm{s}) + (y(s,v)\cos\theta^* - z(s,v)\sin\theta^*)N^*(\mathrm{s}) \\ &\quad + (y(s,v)\sin\theta^* + z(s,v)\cos\theta^*)\,\mathrm{B}^*(\mathrm{s}). \end{aligned} \quad \textbf{(3.3)}$$

Theorem 3.1. Let $\alpha(s)$, be a unit speed curve with nonvanishing curvature and $\alpha^*(s)$ be its Mannheim D- partner. $\alpha^*(s)$ is an isoasymptotic curve on the surface if and only if

$$\begin{cases} x(s,v_0) = y(s,v_0) = z(s,v_0) = 0 \\ y(s,v) = (v-v_0)\beta(s)\cos\theta^*(s)\ , \qquad \beta(s) \ne 0. \\ z(s,v) = -(v-v_0)\beta(s)\sin\theta^*(s) \end{cases} \quad \textbf{(3.4)}$$

Proof: Since $\alpha^*(s)$ is defined to be as isoparametric on the surface $\varphi^*(s,v)$, there exists a parameter v_0 such that

$$x(s,v_0) = y(s,v_0) = z(s,v_0) = 0. \quad \textbf{(3.5)}$$

On the other hand the condition for $\alpha^*(s)$ to be asymptotic on the surface $\varphi^*(s,v)$ is $\left\langle \frac{\partial \mathrm{n}^*}{\partial \mathrm{s}}(s,v_0), \mathrm{T}^*(\mathrm{s}) \right\rangle = 0$, where n is the normal of the surface, φ^* which is calculated as

$$\begin{aligned} n^*(s,v) &= \frac{\partial \varphi^*(s,v)}{\partial s} \times \frac{\partial \varphi^*(s,v)}{\partial v} \\ &= \left(-\frac{\partial y(s,v)}{\partial v}\sin\theta^* - \frac{\partial z(s,v)}{\partial v}\cos\theta^*\right)N^*(s). \\ &\quad + \left(\frac{\partial y(s,v)}{\partial v}\cos\theta^* - \frac{\partial z(s,v)}{\partial v}\sin\theta^*\right)B^*(s) \end{aligned} \quad \textbf{(3.6)}$$

For the sake of simplicity, we can rewrite the latter equation as

$$n^*(s,v_0) = \phi_1(s,v_0)T^*(s) + \phi_2(s,v_0)N^*(s) + \phi_3(s,v_0)B^*(s) \quad \textbf{(3.7)}$$

where

$$\begin{cases} \phi_1(s,v_0)=0 \\ \phi_2(s,v_0)=-\left.\dfrac{\partial y(s,v)}{\partial v}\right|_{v=v_0}\sin\theta^*-\left.\dfrac{\partial z(s,v)}{\partial v}\right|_{v=v_0}\cos\theta^* \\ \phi_3(s,v_0)=\left.\dfrac{\partial y(s,v)}{\partial v}\right|_{v=v_0}\cos\theta^*-\left.\dfrac{\partial z(s,v)}{\partial v}\right|_{v=v_0}\sin\theta^* \end{cases}. \qquad \textbf{(3.8)}$$

Now recalling the asymptoticity condition, we get

$$\left\langle \frac{\partial n^*}{\partial s}(s,v_0),T^*(s)\right\rangle=0 \Leftrightarrow \frac{\partial(\phi_1(s,v_0))}{\partial s}-\kappa^*\phi_2(s,v_0)=0$$
$$\Leftrightarrow \phi_2(s,v_0)=0 \qquad \textbf{(3.9)}$$

where κ^* is the curvature of the curve α^*. Since $\kappa^*(s)=\|\alpha''(s)\|\neq 0$, by taking into account the latter equation, we can simplify the asymptoticity condition for α^* as

$$\phi_2(s,v_0)=-\left.\frac{\partial y(s,v)}{\partial v}\right|_{v=v_0}\sin\theta^*-\left.\frac{\partial z(s,v)}{\partial v}\right|_{v=v_0}\cos\theta^*=0$$

From here we simply get the following equations meeting those conditions with an arbitrary function ($\beta(s)\neq 0$) as

$$\begin{cases} y(s,v)=(v-v_0)\beta(s)\cos\theta^* \\ z(s,v)=-(v-v_0)\beta(s)\sin\theta^* \end{cases}.$$

4. Ruled Surfaces with A Common Mannheim D- Isoasymptotic Curve

In this section, the parametrization of the family of ruled surfaces are defined such that it has the base as a common Mannheim D- isoasymptotic curve.

Theorem 4.1: Given an arc-length Mannheim D- curve, $\alpha^*(s)$, there exists a ruled surface family possessing this curve as isoasymptotic.

Proof: By choosing marching-scale functions as

$$\begin{aligned} x(s,v)&=(v-v_0)f(s), \\ y(s,v)&=(v-v_0)\beta(s)\cos\theta^*, \qquad \beta(s)\neq 0 \\ z(s,v)&=-(v-v_0)\beta(s)\sin\theta^*, \end{aligned} \qquad \textbf{(4.1)}$$

the equation (3.1) takes the following form of a ruled surface

$$\varphi^*(s,v)=\alpha^*(s)+(v-v_0)\left(f(s)T^*(s)+\beta(s)N^*(s)\right), \qquad \beta(s)\neq 0 \qquad \textbf{(4.2)}$$

which satisfies equation (3.4) interpolating $\alpha^*(s)$ as a common Mannheim D- asymptotic.

Corollary 4.2. The ruled surface given in (4.2) is developable if and only if

$$\tau^*(s)=0\,,\ \beta(s)\neq 0. \qquad \textbf{(4.3)}$$

Proof: $\varphi^*(s,v)=\alpha^*(s)+(v-v_0)\left(f(s)T^*(s)+\beta(s)N^*(s)\right)$ is developable if and only if $\det(\alpha',R,R')=0$, where $R(s)=f(s)T^*(s)+\beta(s)N^*(s)$. If necessary, calculations are made and determinants are used we get

$$\beta^2(s)\tau^*(s)=0 \qquad \textbf{(4.4)}$$

Since $\beta(s)\neq 0$, we have $\tau^*(s)=0$, which completes the proof.

Example 1. Let $\alpha(s)=\frac{1}{\sqrt{2}}(-\cos s,-\sin s,s)$ be a unit speed curve and take the surface as $\varphi(s,v)=\frac{1}{\sqrt{2}}(-\cos s,-\sin s,s+v)$. Then it is easy to show that

$$\begin{cases} T(s)=\frac{1}{\sqrt{2}}(\sin s,-\cos s,1) \\ g(s)=\frac{1}{\sqrt{2}}(-\sin s,\cos s,1) \\ n(s)=(-\cos s,-\sin s,0) \\ \kappa=\frac{1}{\sqrt{2}},\quad \tau=\frac{1}{\sqrt{2}} \end{cases}$$

By taking $\lambda=\sqrt{2}$ and recalling the equation (3.2) we write the parametric representation of the Mannheim D- partner curve $\alpha^*(s)$ as

$$\alpha^*(s)=(-\frac{\sqrt{2}}{2}\cos(s)-\sin(s),-\frac{\sqrt{2}}{2}\sin(s)+\cos(s),\frac{\sqrt{2}}{2}s+1)$$

The Frenet apparatus of $\alpha^*(s)$ is

$$\begin{cases} T^*(s)=\left(-\frac{\sqrt{2}}{4}\left(-\sqrt{2}\sin(s)+2\cos(s)\right),-\frac{\sqrt{2}}{4}\left(\sqrt{2}\cos(s)+2\sin(s)\right),\frac{1}{2}\right) \\ N^*(s)=\left(\frac{\sqrt{3}}{3}\left(\sqrt{2}\sin(s)+\cos(s)\right),-\frac{\sqrt{3}}{3}\left(\sqrt{2}\cos(s)-\sin(s)\right),0\right) \\ B^*(s)=\left(\frac{\sqrt{3}}{6}\left(\sqrt{2}\cos(s)-\sin(s)\right),\frac{\sqrt{3}}{6}\left(\sqrt{2}\sin(s)+\cos(s)\right),\frac{\sqrt{3}}{2}\right) \\ \kappa^*(s)=\frac{\sqrt{6}}{4},\quad \tau^*(s)=\frac{\sqrt{2}}{4} \end{cases}$$

i) If we choose the marching scale functions satisfying the condition given in equation (3.4) as $x(s,v)=0,\ y(s,v)=vs\cos(s),\ z(s,v)=-vs\sin(s)\ \textit{and}\ \beta(s)=s,\ v_0=0$ then we get one member of the family of surfaces with a common Mannheim D- iso asymptotic curve $\alpha^*(s)$ as

$$\varphi_i(s,v)=\Big(-\frac{\sqrt{2}}{2}\cos(s)-\sin(s)+\frac{\sqrt{3}}{3}vs\left(\sqrt{2}\sin(s)+\cos(s)\right),$$
$$-\frac{\sqrt{2}}{2}\sin(s)+\cos(s)-\frac{\sqrt{3}}{3}vs\left(\sqrt{2}\cos(s)-\sin(s)\right),$$
$$\frac{\sqrt{2}}{2}s+1\Big).$$

The picture of the surface $\varphi_i(s,v)$ is given in Figure 1 where $-\pi\le s\le\pi,\ -0.75\le v\le 0.75.$

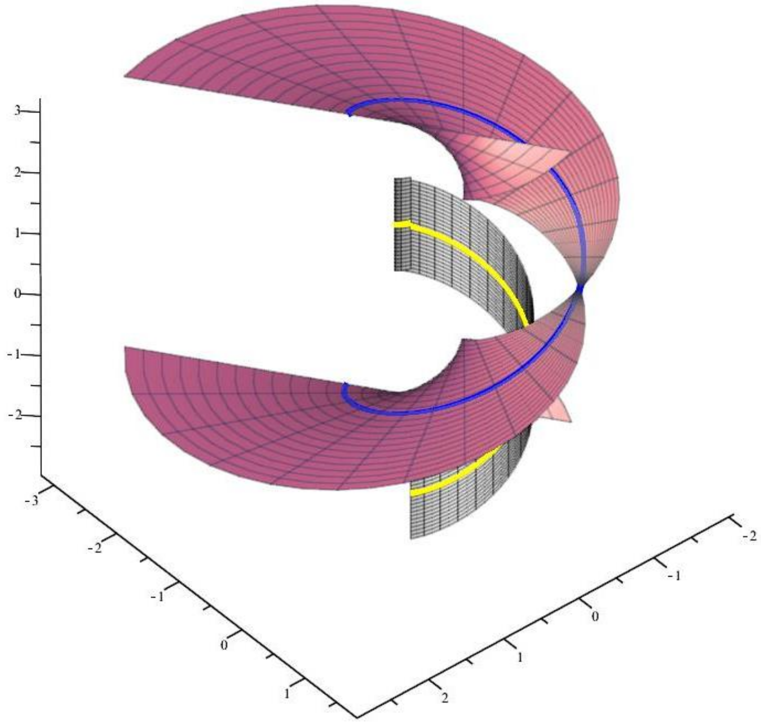

Figure.1: The surface, $\varphi(s,v)$ (gray) and the curve, $\alpha(s)$ (yellow) together with the surface, $\varphi_i(s,v)$ (purple) and isoasymptotic Mannheim D- partner curve, $\alpha^*(s)$ (blue).

ii) When chosen another set of scale functions that meet the condition in (3.4) as $x(s,v)=v,\ y(s,v)=vs^2\cos(s),\ z(s,v)=-vs^2\sin(s)$ *and* $\beta(s)=s^2,\ v_0=0$ results another member as

$$\varphi_{ii}(s,v)=(-\frac{\sqrt{2}\cos(s)}{2}-\sin(s)-\frac{\sqrt{2}\left(-\sqrt{2}\sin(s)+2\cos(s)\right)v}{4}+\frac{\sqrt{3}\left(\sqrt{2}\sin(s)+\cos(s)\right)vs^2}{3},$$
$$-\frac{\sqrt{2}\sin(s)}{2}+\cos(s)-\frac{\sqrt{2}\left(\sqrt{2}\cos(s)+2\sin(s)\right)v}{4}-\frac{\sqrt{3}\left(\sqrt{2}\cos(s)-\sin(s)\right)vs^2}{3},$$
$$\frac{\sqrt{2}s}{2}+\frac{v}{2}+1).$$

The layout of this surface, $\varphi_{ii}(s,v)$ can be seen in Figure 2 where $-\pi \le s \le \pi$ and $-0.75 \le v \le 0.75$.

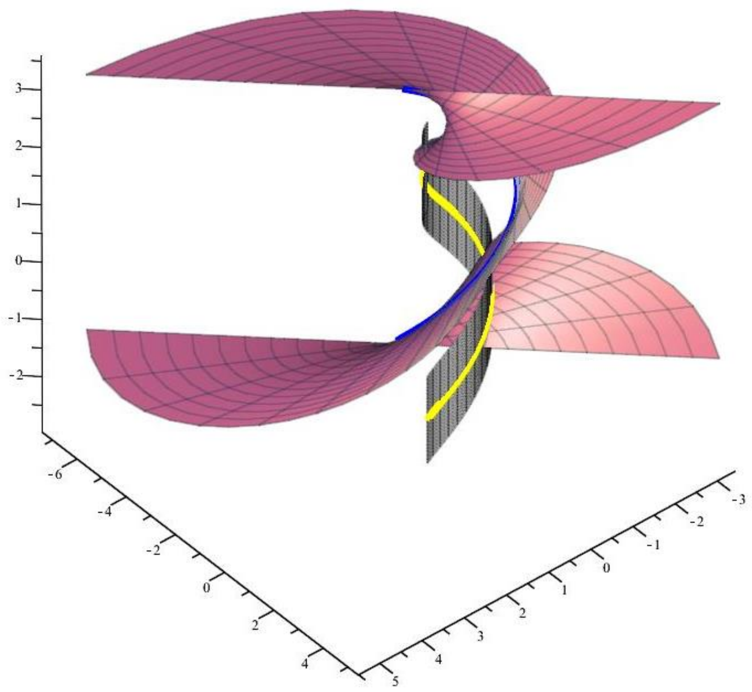

Figure 2: The surface, $\varphi(s,v)$ (gray) and the curve, $\alpha(s)$ (yellow) together with the surface, $\varphi_{ii}(s,v)$ (purple) and isoasymptotic Mannheim D- partner curve, $\alpha^{*}(s)$ (blue).

iii) To illustrate more, if we take $x(s,v)=\sin v,\ y(s,v)=vs^{3}\cos(s),\ z(s,v)=-vs^{3}\sin(s)$ and $\beta(s)=s^{3},\ v_{0}=0$ again satisfying the asymptoticity condition we have the following parametric form of yet another member of the family (Figure 3).

$$\begin{aligned}\varphi_{iii}(s,v)=(&-\frac{\sqrt{2}\cos(s)}{2}-\sin(s)+\frac{\sqrt{2}\left(\sqrt{2}\sin(s)-2\cos(s)\right)\sin(v)}{4}+\frac{\sqrt{3}\left(\sqrt{2}\sin(s)+\cos(s)\right)vs^{3}}{3},\\ &-\frac{\sqrt{2}\sin(s)}{2}+\cos(s)-\frac{\sqrt{2}\left(\sqrt{2}\cos(s)+2\sin(s)\right)\sin(v)}{4}-\frac{\sqrt{3}\left(\sqrt{2}\cos(s)-\sin(s)\right)vs^{3}}{3},\\ &\frac{\sqrt{2}s}{2}+\frac{\sin(v)}{2}+1).\end{aligned}$$

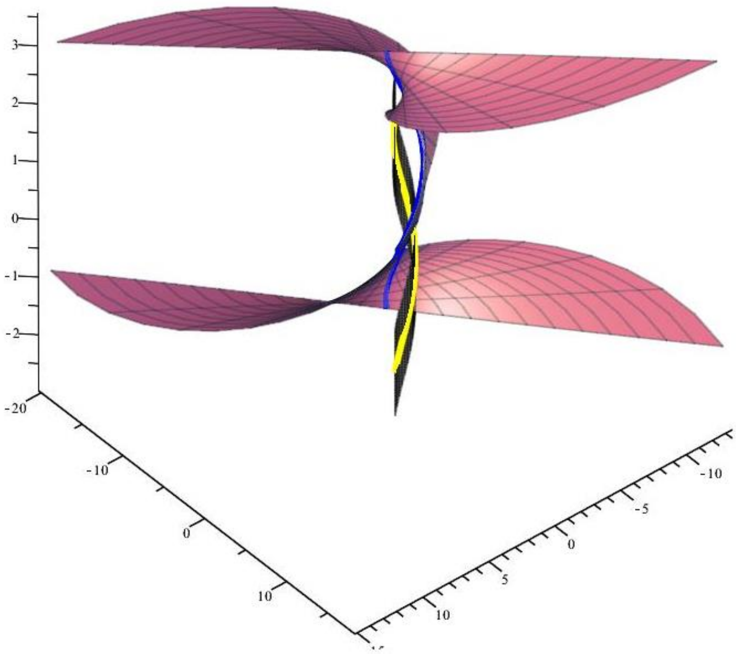

Figure 3: The surface, $\varphi(s,v)$ (gray) and the curve, $\alpha(s)$ (yellow) together with the surface, $\varphi_{iii}(s,v)$ (purple) and isoasymptotic Mannheim D- partner curve, $\alpha^*(s)$ (blue).

iv) As an example of a ruled surface if we choose $f(s)=0,\ \beta(s)=1$ and $v_0=0$ and substitute those in equation (4.2), we get the following nondevelopable ruled surface with a common Mannheim D- asymptotic curve, α^* as

$$\begin{aligned}\varphi_{iv}(s,v)=(&\frac{\sqrt{6}}{3}\sin(s)v+\frac{\sqrt{3}}{3}\cos(s)v-\frac{\sqrt{2}}{2}\cos(s)-\sin(s),\\ &-\frac{\sqrt{6}}{3}\cos(s)v+\frac{\sqrt{3}}{3}\sin(s)v-\frac{\sqrt{2}}{2}\sin(s)+\cos(s),\\ &\frac{\sqrt{2}}{2}s+1).\end{aligned}$$

By choosing the same range as before $-\pi\le s\le\pi,\ -0.75\le v\le 0.75$ the illustration of this surface is given in Figure 4.

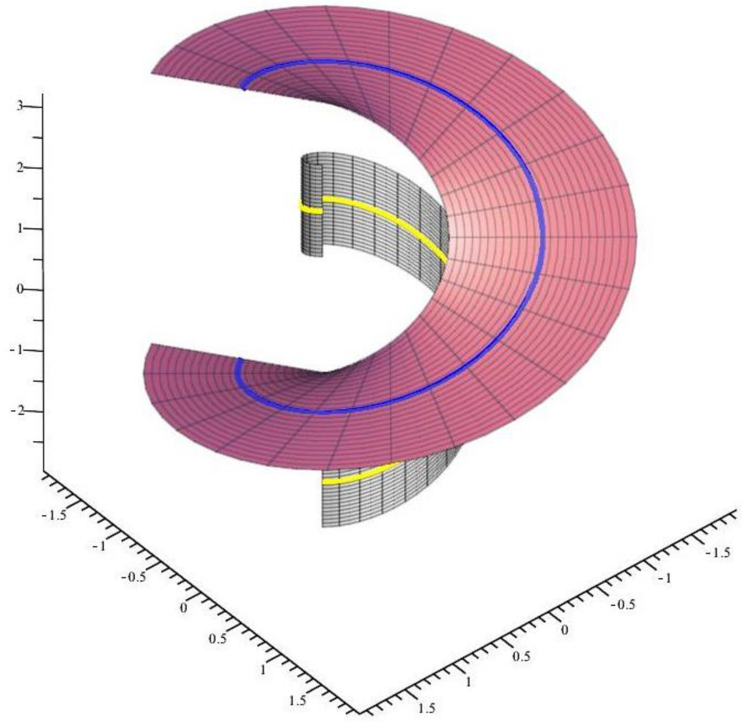

Figure 4: The surface, $\varphi(s,v)$ (gray) and the curve, $\alpha(s)$ (yellow) together with the nondevolapable ruled surface, $\varphi_{iv}(s,v)$ (purple) and isoasymptotic Mannheim D- partner curve, $\alpha^{*}(s)$ (blue).

v) In equation (4.2), if we take $f(s)=s$, $\beta(s)=1$ and $v_0=0$ then we obtain another nondevelopable ruled surface as

$$\varphi_v(s,v)=\Big(\frac{\sin(s)sv}{2}+\frac{\sqrt{6}\sin(s)v}{3}-\frac{\sqrt{2}\cos(s)sv}{2}+\frac{\sqrt{3}\cos(s)v}{3}-\frac{\sqrt{2}\cos(s)}{2}-\sin(s),$$
$$-\frac{\cos(s)sv}{2}-\frac{\sqrt{6}\cos(s)v}{3}-\frac{\sqrt{2}\sin(s)sv}{2}+\frac{\sqrt{3}\sin(s)v}{3}-\frac{\sqrt{2}\sin(s)}{2}+\cos(s),$$
$$\frac{\sqrt{2}s}{2}+\frac{sv}{2}+1\Big).$$

The picture of this surface is given in Figure 5 with the same range for s and v as before.

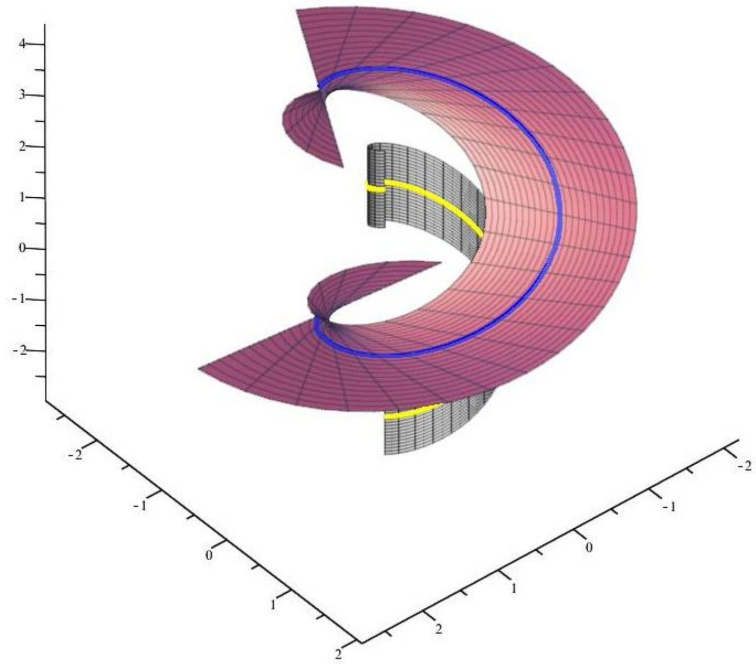

Figure 5: The surface, $\varphi(s,v)$ (gray) and the curve, $\alpha(s)$ (yellow) together with the non-devolapable ruled surface, $\varphi_v(s,v)$ (purple) and isoasymptotic Mannheim D- partner curve, $\alpha^*(s)$ (blue).

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