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Research Article / Araştırma Makalesi

On the Surfaces with Common Mannheim D- Isogeodesic Curve

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Abstract

In this paper, we construct the parameterization of surface family possessing a Mannheim D pair of a given curve as a geodesic. By using the Darboux frame, we present the surface as a linear combination of this frame and analyze the necessary and sufficient condition for a given curve such that its Mannheim D pair is both isoparametric and geodesic on a parametric surface. The extension to ruled surfaces is also outlined. Finally, examples are given to show the family of surfaces with common Mannheim D isogeodesic curve.

Keywords: Mannheim D-curves; Darboux frame; Geodesic curve; Parametric surface; Ruled surface

Ortak Mannheim D- İzojeodezik Egrili Yüzey Ailesi

Öz

Bu çalışmada, verilen bir eğrinin Mannheim D- çiftini üzerinde jeodezik olarak kabul eden yüzeylerin parametrik formu inşa edildi. Yüzey, Darbaux çatısının lineer bir bileşimi olarak ifade edilerek, üzerinde bulundurduğu Mannheim D- eğri çiftinin izoparametrik ve jeodezik olması için gerekli ve yeterli şartlar tanımlandı. Mevcut tanımlamalar regle yüzeyler için ayrıca ele alındı. Son olaraksa ortak Mannheim D- izogeodezik eğrili yüzeylere bazı örnekler verildi.

Anahtar Kelimeler: Mannheim D- eğrileri; Darboux çatısı; Jeodezik eğri; Parametrik yüzey; Regle Yüzey

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1. Introduction

In the literature of differential geometry, the subject of curves plays an important role. As a result, there exist many studies done by researchers to find some new special curves, to characterize those and to associate one with another (Lockwood 1967; Guan et al 1997; Petrovic et al 2000; Izumiya & Takeuchi 2003). One of the special curves is known to be a Mannheim curve. The special characteristic of this curve is that it satisfies the equation $\kappa = \lambda / (\kappa^2 + \tau^2)$ with a nonzero constant, λ, where κ is the curvature and τ is the torsion of the curve (Mannheim 1878; Blum 1966). There are several studies covering Mannheim curves in different spaces, as well such as the Euclidean and Minkowski space (Liu & Wang 2008; Lee 2011). From a different point of view, Kazaz et al (2015) introduced the Mannheim D-pair curves in three-dimensional Euclidean space by taking into account the surface theory. They investigated the properties of these curves to give its characteristics. On the other hand, the concept of family of surfaces with a characteristic curve lying on it was first introduced by Wang et al (2004). They constructed the parametric form of surfaces possessing a given curve as a common geodesic. Then, Atalay (2018) parametrized the surfaces with a common Mannheim geodesic curve. Moreover, Ayvacı and Atalay (2020) studied the surfaces with a common Bertrand B isogeodesic curve. By this study, we introduce a parametric representation of the family of surfaces having the basis Mannheim D- pair curve as a geodesic.

2. Preliminaries

Let E^3 be a 3-dimensional Euclidean space provided with the metric given by

$$\langle\,,\,\rangle = dx_1^{\,2} + dx_2^{\,2} + dx_3^{\,2}$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Recall that, the norm of an arbitrary vector $X \in E^3$ is given by $\|X\| = \sqrt{\langle X, X\rangle}$. Let $\alpha = \alpha(s): I \subset IR \to E^3$ is an arbitrary curve of arc-length parameter s. The curve α is called a unit speed curve if velocity vector, α' of α satisfies $\|\alpha'\| = 1$. Let $\{\mathrm{T(s)}, \mathrm{N(s)}, \mathrm{B(s)}\}$ be the moving Frenet frame along α, then the Frenet formulae is given by

$$\frac{d}{ds}\begin{vmatrix} T(s) \\ N(s) \\ B(s) \end{vmatrix} = \begin{vmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{vmatrix}\begin{vmatrix} T(s) \\ N(s) \\ B(s) \end{vmatrix} \qquad \textbf{(2.1)}$$

where the function $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = -\langle B'(s), N(s)\rangle$ are called the curvature and torsion of the curve α, respectively. Now let T be the tangent vector of the curve α on a surface φ, n be the principal normal at the point P of φ and g be defined as $g = n \times T$. Such frame denoted by $\{T, g, n\}$ is called as Darboux frame (O'Neill 2014). Relations between the Frenet frame and the Darboux frame could be given in the figure below:

$$\begin{vmatrix} T \\ N \\ B \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{vmatrix}\begin{vmatrix} T \\ g \\ n \end{vmatrix}, \qquad \measuredangle(g, N) = \theta. \qquad \textbf{(2.2)}$$

Derivative change of the Darboux frames are as shown below:

$$\frac{d}{ds}\begin{vmatrix} T \\ g \\ n \end{vmatrix} = \begin{vmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{vmatrix}\begin{vmatrix} T \\ g \\ n \end{vmatrix} \tag{2.3}$$

where $\kappa_g = \kappa\cos\theta$ is the geodesic curve, $\kappa_n = \kappa\sin\theta$ is the normal curve and $\tau_g = \tau + \dfrac{d\theta}{ds}$ is known as geodesic torsion.

Let α and α^* be given on two directed surfaces φ and φ^*, respectively and let $\{T, g, n\}$ and $\{T^*, g^*, n^*\}$ denote the Darboux frame of each. If g and n^* are linearly dependent then the pair, (α, α^*) is called to be Mannheim-D pair curves. Relationship between the Darboux frames of the Mannheim D- mate curves are given in the following:

$$\begin{aligned} T^* &= \cos\psi T - \sin\psi\sin\theta g - \sin\psi\cos\theta n, \\ g^* &= \sin\theta^*\sin\psi T + (\cos\theta^*\cos\theta + \sin\theta^*\sin\theta\cos\psi)g \\ &\quad + (-\sin\theta\cos\theta^* + \sin\theta^*\cos\theta\cos\psi)n, \\ n^* &= \cos\theta^*\sin\psi T + (-\sin\theta^*\cos\theta + \cos\theta^*\sin\theta\cos\psi)g \\ &\quad + (\sin\theta\sin\theta^* + \cos\theta^*\cos\theta\cos\psi)n, \end{aligned} \tag{2.4}$$

where $\sphericalangle(T, T^*) = \psi, \quad \sphericalangle(g, N) = \theta, \quad \sphericalangle(g^*, N^*) = \theta^*$ (Kazaz et al 2015).

The curve, α is called as s- parametric (isoparametric), if there exist a constant v_0 such that $\alpha(s) = \varphi(s, v_0)$ (Wang et al 2004). If the curve is both geodesic and parametric then we name it as isogeodesic curve. On the other hand, the surface $\varphi(s, v)$ is ruled if it is formed by a straight line that moves along the curve, α. The parametric equation of a ruled surface is given as:

$$\varphi(s, v) = \alpha(s) + ve(s) \tag{2.5}$$

where α is the base curve and e is the director. A sufficient condition for a ruled surface to be a developable one is that $\det(\alpha', e, e') = 0$ (Izumiya & Takeuchi 2003).

3. Surfaces Family with a Common Mannheim D-Geodesic Curve

Let α^*, be Mannheim D- partner curve of the α. A general parametric form of the surface interpolating α^* as a common curve is given as

$$\varphi^*(s, v) = \alpha^*(s) + \left(x(s, v)T^*(s) + y(s, v)g^*(s) + z(s, v)n^*(s)\right) \tag{3.1}$$

where $L_1 \le s \le L_2,\ K_1 \le v \le K_2$ and $x(s, v),\ y(s, v),\ z(s, v)$ are called as marching-scale functions Since α^* is a Mannheim D- partner curve of α, then by definition we can write

$$\alpha^*(s) = \alpha(s) + \lambda(s)g(s) \tag{3.2}$$

where λ is a non-zero constant. Now by using equation (2.2) we derive the parametric form of surface family with a common Mannheim D-geodesic curve as

$$\begin{aligned}\varphi(s,v) &= \alpha^*(s)+x(s,v)\mathrm{T}^*(s)+(y(s,v)\cos\theta^*-z(s,v)\sin\theta^*)N^*(s) \\ &\quad +(y(s,v)\sin\theta^*+z(s,v)\cos\theta^*)\,\mathrm{B}^*(s)\end{aligned} \qquad \textbf{(3.3)}$$

Theorem 3.1. Let $\alpha(s)$ be a unit speed curve with nonvanishing curvature and $\alpha^*(s)$, be its Mannheim D- partner. $\alpha^*(s)$ is isogeodesic on the surface if and only if

$$\begin{cases} x(s,v_0)=y(s,v_0)=z(s,v_0)=0 \\ y(s,v_0)=(v-v_0)\beta(s)\sin\theta^*(s)\ , \qquad \beta(s)\neq 0. \\ z(s,v_0)=(v-v_0)\beta(s)\cos\theta^*(s) \end{cases} \qquad \textbf{(3.4)}$$

Proof: Since $\alpha^*(s)$ is isoparametric on the surface $\varphi^*(s,v)$, then it is clear that there exists a parameter v_0 such that

$$x(s,v_0)=y(s,v_0)=z(s,v_0)=0. \qquad \textbf{(3.5)}$$

On the other hand if Mannheim D- partner is geodesic on the surface, then $n^*(s,v_0)\,||\,N^*(s)$ where n^* is the normal vector of the surface, $\varphi^*(s,v)$ and $N^*(s)$ is the principal normal vector of $\alpha^*(s)$.

By recalling the definition of normal vector of a surface, we write

$$\begin{aligned} n^*(s,v) &= \frac{\partial\varphi^*(s,v)}{\partial s}\times\frac{\partial\varphi^*(s,v)}{\partial v} \\ &= \left(-\frac{\partial x(s,v)}{\partial v}\sin\theta^* - \frac{\partial z(s,v)}{\partial v}\cos\theta^*\right)N^*(s) \\ &\quad +\left(\frac{\partial y(s,v)}{\partial v}\cos\theta^* - \frac{\partial z(s,v)}{\partial v}\sin\theta^*\right)B^*(s) \end{aligned} \qquad \textbf{(3.6)}$$

Thus we may write

$$n(s,v_0)=\phi_1(s,v_0)T^*(s)+\phi_2(s,v_0)N^*(s)+\phi_3(s,v_0)B^*(s) \qquad \textbf{(3.7)}$$

where

$$\begin{cases} \phi_1(s,v_0)=0 \\ \phi_2(s,v_0)=-\dfrac{\partial y(s,v_0)}{\partial v}\sin\theta^* - \dfrac{\partial z(s,v_0)}{\partial v}\cos\theta^* \\ \phi_3(s,v_0)=\dfrac{\partial y(s,v_0)}{\partial v}\cos\theta^* - \dfrac{\partial z(s,v_0)}{\partial v}\sin\theta^* \end{cases} \qquad \textbf{(3.8)}$$

Referring now the geodesicity condition results following relations

$$\begin{aligned} -\left.\frac{\partial y(s,v)}{\partial v}\right|_{v_0} \sin\theta^*(s) - \left.\frac{\partial z(s,v)}{\partial v}\right|_{v_0} \cos^*\theta(s) \neq 0 \\ \left.\frac{\partial y(s,v)}{\partial v}\right|_{v_0} \cos\theta^*(s) - \left.\frac{\partial z(s,v)}{\partial v}\right|_{v_0} \sin\theta^*(s) = 0 \end{aligned} . \quad \textbf{(3.9)}$$

From here we can derive the equations below with an arbitrary nonzero valued function $\beta(s) \neq 0$

$$\begin{cases} y(s,v_0) = (v-v_0)\beta(s)\sin\theta^*(s) \\ z(s,v_0) = (v-v_0)\beta(s)\cos\theta^*(s) \end{cases} \quad \textbf{(3.10)}$$

4. Ruled Surfaces with Common Mannheim D- Geodesic Curve

In this part of the study we derive the formulations of a ruled surface family whose basis curve is Mannheim D- and it is geodesic on the surface as well.

Theorem 4.1: Given an arc-length curve, $\alpha^*(s)$ there exists a ruled surface family possessing $\alpha^*(s)$ as a common Mannheim D- geodesic.

Proof: Choosing marching-scale functions as

$$\begin{aligned} x(s,v) &= (v-v_0)f(s), \\ y(s,v) &= (v-v_0)\beta(s)\sin\theta^*, \\ z(s,v) &= (v-v_0)\beta(s)\cos\theta^* \end{aligned} \quad \textbf{(4.1)}$$

equation (3.10) takes the following form of a ruled surface

$$\varphi(s,v) = \alpha^*(s) + (v-v_0)\left(f(s)T^*(s) + \beta(s)B^*(s)\right) \quad \textbf{(4.2)}$$

which clearly satisfies the conditions given in equation (3.4).

Corollary 4.2. Ruled surface given in (4.2) is developable, if and only if

$$f(s) = \frac{\tau^*(s)}{\kappa^*(s)}\beta(s).. \quad \textbf{(4.3)}$$

Proof: We remind that $\varphi(s,v)$ is developable, iff $\det(\alpha', R, R') = 0$, where $R(s) = f(s)T^*(s) + \beta(s)B^*(s)$. If necessary, calculations are made we simply get

$$\beta(s)\left(\tau^*(s)\beta(s) - f(s)\kappa^*(s)\right) = 0. \quad \textbf{(4.4)}$$

Since $\beta(s) \neq 0$, we have $f(s) = \dfrac{\tau^*(s)}{\kappa^*(s)}\beta(s)$, which completes the proof

Example 1. Let $\alpha(s) = \dfrac{1}{\sqrt{2}}(-\cos s, -\sin s, s)$ be a unit speed curve and take the surface as $\varphi(s,v) = \dfrac{1}{\sqrt{2}}(-\cos s, -\sin s, s+v)$. Then we can simply calculate the frame elements as

$$\begin{cases} T(s)=\frac{1}{\sqrt{2}}(\sin s,-\cos s,1) \\ g(s)=\frac{1}{\sqrt{2}}(-\sin s,\cos s,1) \\ n(s)=(-\cos s,-\sin s,0) \\ \kappa=\frac{1}{\sqrt{2}}, \quad \tau=\frac{1}{\sqrt{2}} \end{cases}$$

Now by recalling the equation (3.2) and taking $\lambda=\frac{1}{\sqrt{2}}$ we get

$$\alpha^*(s)=(-\frac{1}{2}\sqrt{2}\cos s-\frac{1}{2}\sin s,-\frac{1}{2}\sqrt{2}\sin s+\frac{1}{2}\cos s,\frac{1}{2}\sqrt{2}s+\frac{1}{2})$$

The Frenet apparatus of the latter curve are

$$\begin{cases} T^*(s)=(-\frac{1}{5}\sqrt{5}\left(-\sqrt{2}\sin(s)+\cos(s)\right),-\frac{1}{5}\sqrt{5}\left(\sqrt{2}\cos(s)+\sin(s)\right),\frac{1}{5}\sqrt{10}) \\ N^*(s)=(\frac{1}{3}\sqrt{3}\left(\sqrt{2}\cos(s)+\sin(s)\right),-\frac{1}{3}\sqrt{3}\left(-\sqrt{2}\sin(s)+\cos(s)\right),0) \\ B^*(s)=(\frac{1}{15}\sqrt{15}\left(\sqrt{2}\cos(s)-2\sin(s)\right),\frac{1}{15}\sqrt{15}\left(\sqrt{2}\sin(s)+2\cos(s)\right),\frac{1}{5}\sqrt{15}) \\ \kappa^*=\frac{2\sqrt{3}}{5}, \quad \tau^*=\frac{2\sqrt{2}}{5} \end{cases}$$

From here to the end the parameter range will be set $-\pi \le s \le \pi,\ -0.75 \le v \le 0.75$ to draw the corresponding figures.

a) Now if we take $x(s,v)=0,\ y(s,v)=vs\sin\theta^*,\ z(s,v)=vs\cos\theta^*$ and $\beta(s)=s,\ v_0=0$ then the conditions given equation (3.4) are satisfied. Thereby, we obtain one member of the surface family with a common Mannheim D- isogeodesic curve as

$$\begin{aligned}\varphi_1(s,v)=(&-\frac{1}{2}\sqrt{2}\cos(s)-\frac{1}{2}\sin(s)+\frac{1}{15}vs\sqrt{15}\left(\sqrt{2}\cos(s)-2\sin(s)\right),\\ &-\frac{1}{2}\sqrt{2}\sin(s)+\frac{1}{2}\cos(s)+\frac{1}{15}vs\sqrt{15}\left(\sqrt{2}\sin(s)+2\cos(s)\right),\\ &\frac{1}{2}\sqrt{2}s+\frac{1}{2}+\frac{1}{5}\sqrt{15}vs)\end{aligned}$$

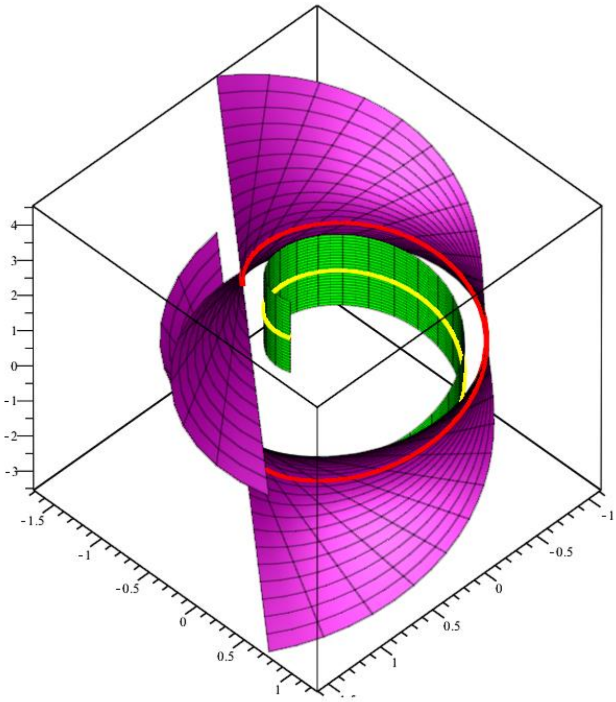

Figure 1: The surface, $\varphi(s,v)$ (in green) and the curve, $\alpha(s)$ (in yellow) together with the surface, $\varphi_1(s,v)$ (in purple) and isogeodesic Mannheim D- partner curve, $\alpha^*(s)$ (in red).

b) If we take this time the marching scales as $x(s,v)=sv$, $y(s,v)=vs^2\sin\theta^*$, $z(s,v)=vs^2\cos\theta^*$ and $\beta(s)=s^2$, $v_0=0$ the conditions given in (3.4) still hold. Thus, another member of the surface family can be given in parametric form as

$$\varphi_2(s,v)=\Big(-\frac{\sqrt{2}\cos(s)}{2}-\frac{\sin(s)}{2}-\frac{sv\sqrt{5}\left(-\sqrt{2}\sin(s)+\cos(s)\right)}{5}+\frac{vs^2\sqrt{15}\left(\sqrt{2}\cos(s)-2\sin(s)\right)}{15},$$
$$-\frac{\sqrt{2}\sin(s)}{2}+\frac{\cos(s)}{2}-\frac{sv\sqrt{5}\left(\sqrt{2}\cos(s)+\sin(s)\right)}{5}+\frac{vs^2\sqrt{15}\left(\sqrt{2}\sin(s)+2\cos(s)\right)}{15},$$
$$\frac{\sqrt{2}s}{2}+\frac{1}{2}+\frac{\sqrt{2}\sqrt{5}sv}{5}+\frac{vs^2\sqrt{15}}{5}\Big)$$

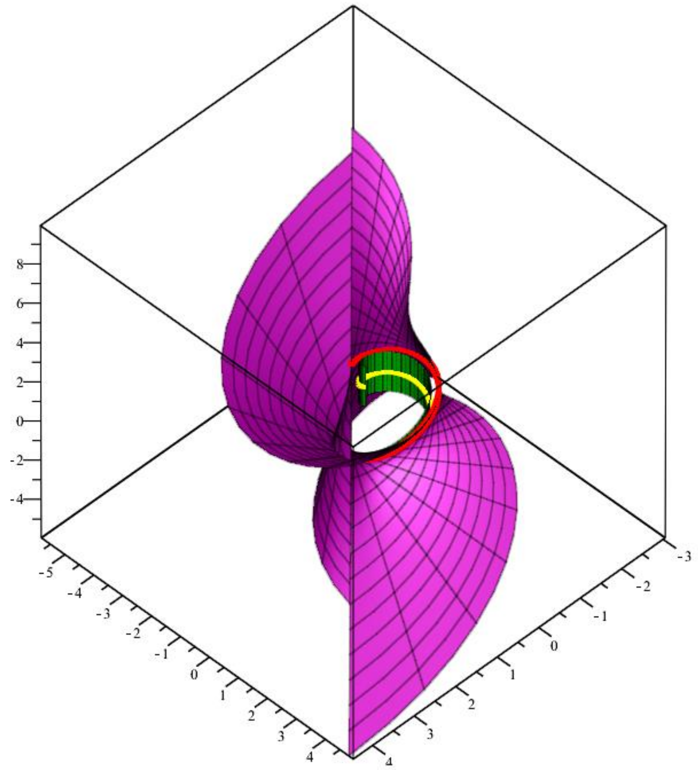

Figure 2: The surface, $\varphi(s,v)$ (in green) and the curve, $\alpha(s)$ (in yellow) together with the surface, $\varphi_2(s,v)$ (in purple) and isogeodesic Mannheim D- partner curve, $\alpha^*(s)$ (in red).

c) When chosen $x(s,v)=\sin v$, $y(s,v)=vs^3\sin\theta^*$, $z(s,v)=vs^3\cos\theta^*$ and $\beta(s)=s^3$, $v_0=0$ we have another member of the surface family with a common Mannheim D- isogeodesic curve as

$$\varphi_3(s,v)=\Big(-\frac{\sqrt{2}\cos(s)}{2}-\frac{\sin(s)}{2}-\frac{\sin(v)\sqrt{5}\left(-\sqrt{2}\sin(s)+\cos(s)\right)}{5}+\frac{vs^3\sqrt{15}\left(\sqrt{2}\cos(s)-2\sin(s)\right)}{15},$$
$$-\frac{\sqrt{2}\sin(s)}{2}+\frac{\cos(s)}{2}-\frac{\sin(v)\sqrt{5}\left(\sqrt{2}\cos(s)+\sin(s)\right)}{5}+\frac{vs^3\sqrt{15}\left(\sqrt{2}\sin(s)+2\cos(s)\right)}{15},$$
$$\frac{\sqrt{2}s}{2}+\frac{1}{2}+\frac{\sin(v)\sqrt{5}\sqrt{2}}{5}+\frac{vs^3\sqrt{15}}{5}\Big)$$

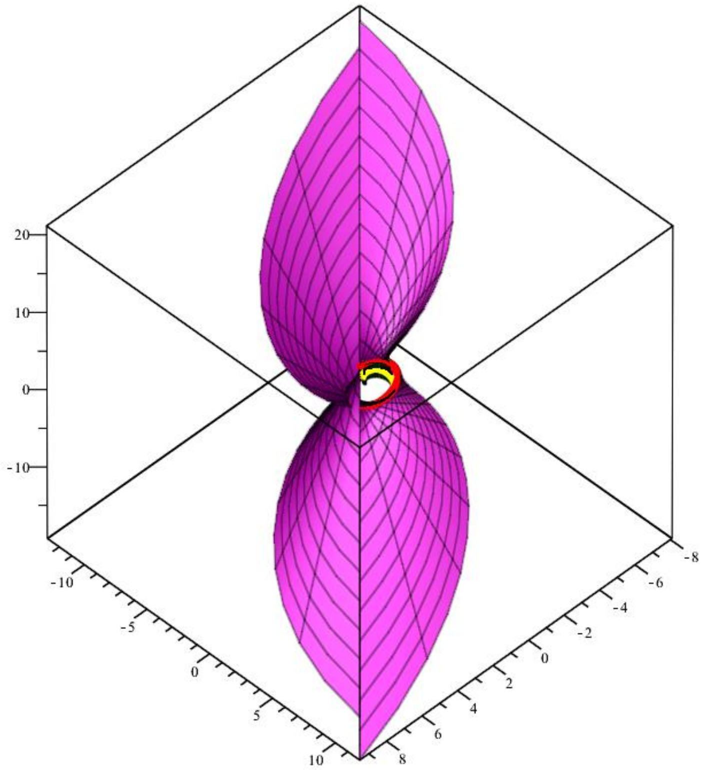

Figure 3: The surface, $\varphi(s,v)$ (in green) and the curve, $\alpha(s)$ (in yellow) together with the surface, $\varphi_3(s,v)$ (in purple) and isogeodesic Mannheim D- partner curve, $\alpha^*(s)$ (in red).

d) For a developable ruled surface, we pick $f(s)=\frac{\sqrt{6}}{2}s$, $\beta(s)=s$ and $v_0=0$ satisfying the given condition in (4.2), we form the following parametrization

$$\begin{aligned}\varphi_4(s,v)=(&\frac{1}{15}\sqrt{15}\sin(s)vs-\frac{1}{30}\sqrt{30}\cos(s)vs-\frac{1}{2}\sqrt{2}\cos(s)-\frac{1}{2}\sin(s),\\ &-\frac{1}{15}\sqrt{15}\cos(s)vs-\frac{1}{30}\sqrt{30}\sin(s)vs-\frac{1}{2}\sqrt{2}\sin(s)+\frac{1}{2}\cos(s),\\ &\frac{1}{2}\sqrt{2}s+\frac{1}{2}+\frac{2}{5}\sqrt{15}vs)\end{aligned}$$

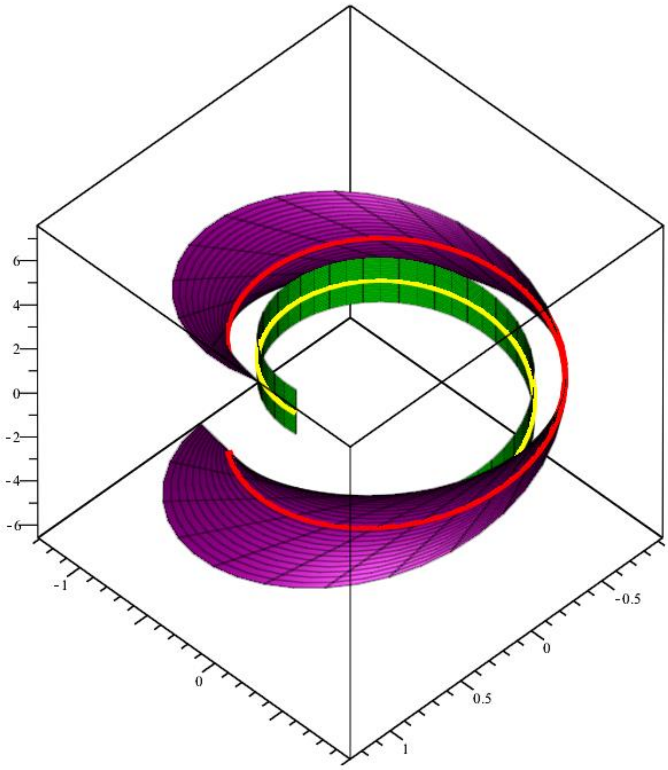

Figure 4: The surface, $\varphi(s,v)$ (in green) and the curve, $\alpha(s)$ (in yellow) together with the developable ruled surface, $\varphi_4(s,v)$ (in purple) and isogeodesic Mannheim D- partner curve, $\alpha^*(s)$ (in red).

e) For a nondevelopable ruled surface we pick now $f(s)=s$, $\beta(s)=1$ and $v_0=0$ then we obtain the following nondevelopable ruled surface with a common Mannheim D-isogeodesic curve as

$$\varphi_5(s,v)=\Big(\frac{\sqrt{10}\sin(s)sv}{5}+\frac{\sqrt{30}\cos(s)v}{15}-\frac{\sqrt{5}\cos(s)sv}{5}-\frac{2\sqrt{15}\sin(s)v}{15}-\frac{\sqrt{2}\cos(s)}{2}-\frac{\sin(s)}{2},$$
$$-\frac{\sqrt{10}\cos(s)sv}{5}+\frac{\sqrt{30}\sin(s)v}{15}-\frac{\sqrt{5}\sin(s)sv}{5}+\frac{2\sqrt{15}\cos(s)v}{15}-\frac{\sqrt{2}\sin(s)}{2}+\frac{\cos(s)}{2},$$
$$\frac{\sqrt{2}s}{2}+\frac{1}{2}+\frac{\sqrt{10}sv}{5}+\frac{\sqrt{15}v}{5}\Big)$$

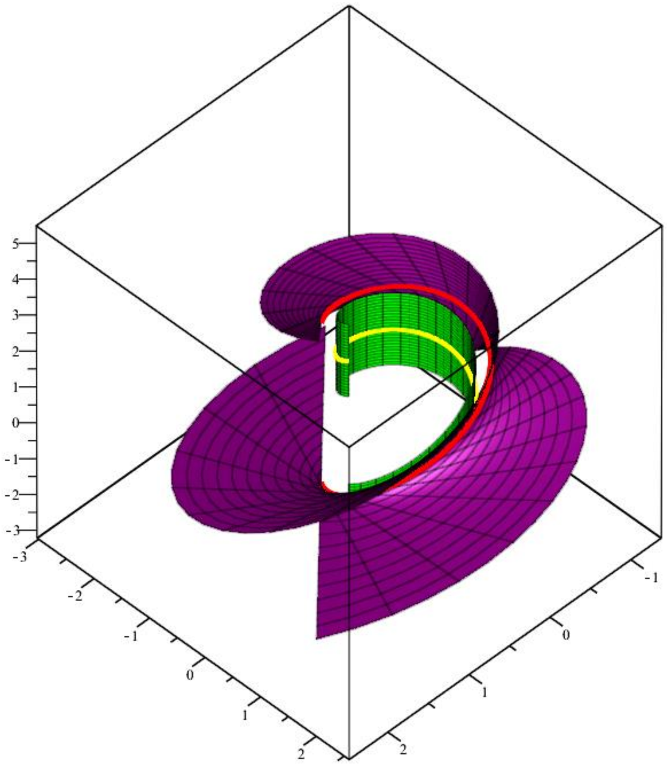

Figure 5: The surface, $\varphi(s,v)$ (in green) and the curve, $\alpha(s)$ (in yellow) together with the nondevelopable ruled surface, $\varphi_5(s,v)$ (in purple) and isogeodesic Mannheim D- partner curve, $\alpha^*(s)$ (in red).

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