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An inverse coefficient problem for quasilinear pseudo-parabolic of heat conduction of Poly(methyl methacrylate) (PMMA)

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Abstract. In this research, we consider a coefficient problem of an inverse problem of a quasilinear pseudo-parabolic equation with periodic boundary condition. It proved the existence, uniqueness and continuously dependence upon the data of the solution by iteration method.

1. Introduction

Consider the equation

$$u_t - u_{xx} - \varepsilon u_{xxt} - a(t)u = f(x,t,u),\ (x,t) \in\ \Gamma, \tag{1}$$

with the initial condition

$$u(x,0) = \varphi(x),\ \ x \in [0,\pi], \tag{2}$$

the periodic boundary condition

$$u(0,t) = u(\pi,t),\ u_x(0,t) = u_x(\pi,t),\ 0 \le t \le T, \tag{3}$$

and the overdetermination data

$$E(t) = \int_0^\pi xu(x,t)dx, 0 \le t \le T, \tag{4}$$

for a quasilinear parabolic equation with the nonlinear source term $f = f(x,t,u)$.

Here $\Gamma := \{0 < x < \pi,\ 0 < t < T\}$. The functions $\varphi(x)$ and $f(x,t,u)$ are given functions on $[0,\pi]$ and $\overline{\Gamma} \times (-\infty,\infty)$, respectively.

The inverse problem of determining unknown coefficient in a quasi-linear parabolic equation has generated an increasing amount of interest from engineers and scientist [1–11].

Definition 1.1. *The pair $\{a(t),u(x,t)\}$ from the class $C[0,T] \times (C^{2,1}(\Gamma) \cap C^{1,0}(\overline{\Gamma}))$ for which conditions (1)-(4) are satisfied is called the classical solution of the inverse problem (1)-(4).*

The paper organized as follows:

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2. Existence and Uniqueness of the Solution of the Inverse Problem

The main result on the existence and the uniqueness of the solution of the inverse problem (1)-(4) is presented as follows:

We have the following assumptions on the data of the problem (1)-(4).

(A1) $E(t) \in C^1[0,T]$.

(A2) $\varphi(x) \in C^2[0,\pi]$, $\varphi(0) = \varphi(\pi)$, $\varphi'(0) = \varphi'(\pi)$,

(A3) Let the function $f(x,t,u)$ is continuous with respect to all arguments in $\overline{\Gamma} \times (-\infty,\infty)$ and satisfies the following condition

(1)

$$\left| \frac{\partial^{(n)} f(x,t,u)}{\partial x^n} - \frac{\partial^{(n)} f(x,t,\tilde{u})}{\partial x^n} \right| \leq b(t,x)\,|u-\tilde{u}| \ ,n=0,1,2$$

where $b(x,t) \in L_2(\Gamma),\ b(x,t) \geq 0$,

(2) $f(x,t,u) \in C^2[0,\pi]$, $t \in [0,T]$,

(3) $f(x,t,u)|_{x=0} = f(x,t,u)|_{x=\pi}\,,\ f_x(0,t,u)|_{x=0} = f_x(\pi,t,u)|_{x=\pi}\,,$

By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of (1)-(3) for arbitrary $a(t) \in C[0,T]$:

$$u(x,t) = \frac{u_0(t)}{2} + \sum_{k=1}^{\infty} \left[u_{ck}(t)\cos 2kx + u_{sk}(t)\sin 2kx\right],$$

$$\begin{aligned}
u_0(t) &= \varphi_0 e^{-\int_0^t a(\tau)d\tau} + \frac{2}{\pi}\int_0^t\int_0^\pi f\left(\xi,\tau,\frac{u_0(\tau)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi + u_{sk}(\tau)\sin 2k\xi\right]\right)^{-\int_0^t a(\tau)d\tau} d\xi d\tau,\\
u_{ck}(t) &= \varphi_{ck}\, e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2} - -\int_0^t a(\tau)d\tau}\\
&+ \frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^t\int_0^\pi f\left(\xi,\tau,\frac{u_0(\tau)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi + u_{sk}(\tau)\sin 2k\xi\right]\right)\cos 2k\xi\ e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2} - -\int_\tau^t a(\tau)d\tau} d\xi d\tau,\\
u_{sk}(t) &= \varphi_{sk}\, e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2} - -\int_0^t a(\tau)d\tau}\\
&+ \frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^t\int_0^\pi f\left(\xi,\tau,\frac{u_0(\tau)}{2} + \sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi + u_{sk}(\tau)\sin 2k\xi\right]\right)\sin 2k\xi e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2} - -\int_\tau^t a(\tau)d\tau} d\xi d\tau.
\end{aligned}$$

$$\begin{aligned}
u(x,t) &= \varphi_0 e^{--\int_0^t a(\tau)d\tau} + \int_0^t f_0(\tau,u)\ d\tau\\
&+ \sum_{k=1}^{\infty}\cos 2kx\left[\varphi_{ck} e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2} - -\int_0^t a(\tau)d\tau} + \frac{1}{1+\varepsilon(2k)^2}\int_0^t f_{ck}(\tau,u) e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2} - -\int_\tau^t a(\tau)d\tau} d\tau\right]\\
&+ \sum_{k=1}^{\infty}\sin 2kx\left[\varphi_{sk} e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2} - -\int_0^t a(\tau)d\tau} + \frac{1}{1+\varepsilon(2k)^2}\int_0^t f_{sk}(\tau,u)\ e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2} - -\int_\tau^t a(\tau)d\tau} d\tau\right],
\end{aligned} \tag{5}$$

where $\varphi_0 = \frac{2}{\pi}\int_0^\pi \varphi(x)dx, \varphi_{ck} = \frac{2}{\pi}\int_0^\pi \varphi(x)\cos 2kxdx, \varphi_{sk} = \frac{2}{\pi}\int_0^\pi \varphi(x)\sin 2kxdx,$

$f_0(t,u)=\frac{2}{\pi}\int\limits_0^\pi f(x,t,u)dx,\ f_{ck}(t,u)=\frac{2}{\pi}\int\limits_0^\pi f(x,t,u)\cos 2kxdx,\ f_{sk}(t,u)=\frac{2}{\pi}\int\limits_0^\pi f(x,t,u)\sin 2kxdx(\ k=1,2,3,\dots.)$

Under the condition (A1)-(A3), differentiating (4), we obtain

$$E^{'}(t)=\int\limits_0^\pi xu_t(x,t)dx, 0\le t\le T. \tag{6}$$

(5) and (6) yield

$$\begin{aligned} a(t) &= \frac{1}{E(t)}\left[-E^{'}(t)+\frac{\pi^2}{2}f_0(t,u)\right] \\ & \frac{1}{E(t)}\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{-(2k)^2t}{1+\varepsilon(2k)^2}\left(\varphi_{sk}\, e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}--\int\limits_0^t a(\tau)d\tau}+\frac{1}{1+\varepsilon(2k)^2}\int\limits_0^t f_{ck}(\tau,u)\, e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}--\int\limits_\tau^t a(\tau)d\tau}d\tau\right) \\ & -\frac{1}{E(t)}\sum_{k=1}^{\infty}f_{sk}(t,u) \end{aligned} \tag{7}$$

Definition 2.1. *Denote the set*
$\{u(t)\}=\{u_0(t),u_{ck}(t),u_{sk}(t),k=1,...,n\}$, *of continuous on* $[0,T]$ *functions satisfying the condition* $\max\limits_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{ck}(t)|+\max\limits_{0\le t\le T}|u_{sk}(t)|\right)<\infty$, *by* **B**. *Let*
$\|u(t)\|_B=\max\limits_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{ck}(t)|+\max\limits_{0\le t\le T}|u_{sk}(t)|\right)$, *be the norm in* **B**.
It can be shown that **B** *is Banach space.*

Theorem 2.2. *Let the assumptions (A1)-(A3) be satisfied. Then the inverse problem (1)-(4) has a unique solution.*

Proof. Iterations for the Fourier coefficients of (5) are defined as follows:

$$\begin{aligned} u_0^{(N+1)}(t) &= u_0^{(0)}(t)+\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi f\left(\xi,\tau,u^{(N)}(\xi,\tau)\right)\, e^{--\int\limits_\tau^t a^{(N)}(\tau)d\tau}d\xi d\tau, \\ u_{ck}^{(N+1)}(t) &= u_{ck}^{(0)}(t)+\frac{2}{\pi(1+\varepsilon(2k)^2)}\int\limits_0^t\int\limits_0^\pi f(\xi,\tau,u^{(N)}(\xi,\tau))\cos 2k\xi\, e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}--\int\limits_\tau^t a^{(N)}(\tau)d\tau}d\xi d\tau, \\ u_{sk}^{(N+1)}(t) &= u_{sk}^{(0)}(t)+\frac{2}{\pi(1+\varepsilon(2k)^2)}\int\limits_0^t\int\limits_0^\pi f(\xi,\tau,u^{(N)}(\xi,\tau))\sin 2k\xi e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}--\int\limits_\tau^t a^{(N)}(\tau)d\tau}d\xi d\tau, \end{aligned} \tag{8}$$

$$u_0^{(0)}(t)=\varphi_0 e^{--\int\limits_\tau^t a^{(0)}(\tau)d\tau}, u_{ck}^{(0)}(t)=\varphi_{ck}e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}--\int\limits_0^t a^{(0)}(\tau)d\tau}, u_{sk}^{(0)}(t)=\varphi_{sk}e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}--\int\limits_0^t a^{(0)}(\tau)d\tau}.$$

Applying Cauchy inequality, Hölder inequality, Bessel inequality and using Lipschitzs condition and taking the maximum of both side, we have:

$$\begin{aligned}\left\|u^{(1)}(t)\right\|_{\mathbf{B}} &= \max_{0\le t\le T}\left\|u_0^{(1)}(t)\right\|_B+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left\|u_{ck}^{(1)}(t)\right\|_B+\max_{0\le t\le T}\left\|u_{sk}^{(1)}(t)\right\|_B\right)\\ &\le \frac{\|\varphi_0\|}{2}+\sum_{k=1}^{\infty}\left(\|\varphi_{ck}\|+\|\varphi_{sk}\|\right)\\ &\quad+\left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B\\ &\quad+\left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\|f(x,t,0)\|_{L_2(\Gamma)}.\end{aligned}$$

From the conditions of the theorem $u^{(1)}(t)\in\mathbf{B}$.
Same estimations for the step N,

$$\begin{aligned}\left\|u^{(N+1)}(t)\right\|_{B} &= \max_{0\le t\le T}\left\|u_0^{(N)}(t)\right\|_B+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left\|u_{ck}^{(N)}(t)\right\|_B+\max_{0\le t\le T}\left\|u_{sk}^{(N)}(t)\right\|_B\right)\\ &\le \frac{\|\varphi_0\|}{2}+\sum_{k=1}^{\infty}\left(\|\varphi_{ck}\|+\|\varphi_{sk}\|\right)\\ &\quad+\left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(N)}(t)\right\|_B\\ &\quad+\left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\|f(x,t,0)\|_{L_2(\Gamma)}.\end{aligned}$$

Since $u^{(N)}(t)\in\mathbf{B}$ and from the conditions of the theorem, we have $u^{(N+1)}(t)\in\mathbf{B}$,

$$\{u(t)\}=\{u_0(t),u_{ck}(t),u_{sk}(t),\ k=1,2,...\}\in\mathbf{B}.$$

By same estimations,

$$\begin{aligned}\left\|a^{(1)}(t)\right\|_{C[0,T]} &\le \left\|\frac{E'(t)}{E(t)}\right\|+\frac{\pi^2}{4\sqrt{6}E(t)}\sum_{k=1}^{\infty}\left\|\varphi_{ck}'''\right\|\\ &\quad+\frac{\pi}{\|E(t)\|}\left(\frac{4\sqrt{6}+2+\sqrt{2}}{4\sqrt{6}}\right)\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B\\ &\quad+\frac{\pi}{\|E(t)\|}\left(\frac{4\sqrt{6}+2+\sqrt{2}}{4\sqrt{6}}\right)M\end{aligned}$$

Same estimations for the step N,

$$\begin{aligned}\left\|a^{(N+1)}(t)\right\|_{C[0,T]} &\le \left\|\frac{E'(t)}{E(t)}\right\|+\frac{\pi^2}{4\sqrt{6}E(t)}\sum_{k=1}^{\infty}\left\|\varphi_{ck}'''\right\|\\ &\quad+\frac{\pi}{\|E(t)\|}\left(\frac{4\sqrt{6}+2+\sqrt{2}}{4\sqrt{6}}\right)\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(N)}(t)\right\|_B\\ &\quad+\frac{\pi}{\|E(t)\|}\left(\frac{4\sqrt{6}+2+\sqrt{2}}{4\sqrt{6}}\right)M\end{aligned}$$

Now we prove that the iterations $u^{(N+1)}(t), a^{(N+1)}$ converge **B** and $C[0,T]$, respectively.($as\ N \to \infty$)

$$u^{(1)}(t)-u^{(0)}(t)=\frac{(u_0^{(1)}(t)-u_0^{(0)}(t))}{2}+\sum_{k=1}^{\infty}[(u_{ck}^{(1)}(t)-u_{ck}^{(0)}(t))+(u_{sk}^{(1)}(t)-u_{sk}^{(0)}(t))]$$

Applying Cauchy inequality, Bessel inequality, Hölder inequality, Lipschitzs condition in the last equation, taking maximum of both side of the last inequality :

$$\begin{aligned}\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_B &\le (\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\,\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B \\ &+(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\,\|f(x,t,0)\|_{L_2(\Gamma)}.\end{aligned}$$

$$A=(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})(\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}}+\|f(x,t,0)\|_{L_2(\Gamma)}).$$

Applying Cauchy inequality, Hölder Inequality, Lipschitzs condition and Bessel inequality to the last equation and taking maximum of both side of the last inequality, we obtain

$$\begin{aligned}\left\|a^{(1)}(t)-a^{(0)}(t)\right\|_{C[0,T]} &\le \frac{\pi}{\|E(t)\|}\left(\frac{4\sqrt{6}+2+\sqrt{2}}{4\sqrt{6}}\right)\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_B \\ &+\left(\frac{\pi TM}{\left\|E(t)4\sqrt{3}\right\|}+\frac{\pi^2 T}{\left\|E(t)4\sqrt{6}\right\|}\sum_{k=1}^{\infty}\left|\varphi_{ck}'''\right|\right)\left\|a^{(1)}(t)-a^{(0)}(t)\right\|_{C[0,T]}\end{aligned}$$

where

$$\begin{aligned}B &= \frac{\pi}{\|E(t)\|}\left(\frac{4\sqrt{6}+2+\sqrt{2}}{4\sqrt{6}}\right)\\ C &= \left(\frac{\pi TM}{\left\|E(t)4\sqrt{3}\right\|}+\frac{\pi^2 T}{\left\|E(t)4\sqrt{6}\right\|}\sum_{k=1}^{\infty}\left|\varphi_{ck}'''\right|\right)\end{aligned}$$

$$\left\|a^{(1)}(t)-a^{(0)}(t)\right\|_{C[0,T]}\le\frac{B}{1-C}\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_B$$

$$\begin{aligned}\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_B &\le (\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\,\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(1)}-u^{(0)}\right\|_B \\ &+(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\frac{BT}{1-C}M\,\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_B\end{aligned}$$

$$\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_B\le\left\{(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}})\left(1+\frac{BT}{1-C}\right)\right\}A\,\|b(x,t)\|_{L_2(\Gamma)},$$

For the step N :

$$\left\|a^{(N+1)}(t)-a^{(N)}(t)\right\|_{C[0,T]}\le\frac{B}{1-C}\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B$$

$$\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B \leq \left\{ \left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\left(1+\frac{BT}{1-C}\right)\right\}^N \frac{A}{\sqrt{N!}}\|b(x,t)\|^N_{L_2(\Gamma)}.$$

By the Weierstrass M test we deduce from (9) that the series $\sum_{N=0}^{\infty}\left|u^{(N+1)}(t)-u^{(N)}(t)\right|$ is uniformly convergent to an element of B. However,the general term of the sequence $\left\{u^{(N+1)}(t)\right\}$ may be written as

$$u^{(N+1)}(t)=u^{(0)}(t)+\sum_{n=0}^{N}\left|u^{(n+1)}(t)-u^{(n)}(t)\right|,$$

so the sequence $\left\{u^{(N+1)}(t)\right\}$ is uniformly convergent to an element of **B** because the sum on the right is the N th partial sum of the aforementioned uniformly convergent series. So $u^{(N+1)} \to u^{(N)}$, $N\to\infty$, then $a^{(N+1)}\to a^{(N)}$, $N\to\infty$.

Therefore $u^{(N+1)}(t)$ and $a^{(N+1)}(t)$ converge in **B** and $C[0,T]$, respectively.

Now let us show that there exists u and a such that

$$\lim_{N\to\infty}u^{(N+1)}(t)=u(t),\ \lim_{N\to\infty}a^{(N+1)}(t)=a(t).$$

$$\begin{aligned}\left\|u-u^{(N+1)}\right\|_B \leq\ & \left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\|b(x,t)\|_{L_2(\Gamma)}\left\|u(t)-u^{(N+1)}(t)\right\|_B \\ &+\left\{ \left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\left(1+\frac{BT}{1-C}\right)\right\}^N \frac{A}{\sqrt{N!}}\|b(x,t)\|_{L_2(\Gamma)} \\ &+\left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)M\left\|a(\tau)-a^{(N)}(\tau)\right\|_{C[0,T]},\end{aligned} \tag{9}$$

$$\left\|a(\tau)-a^{(N+1)}(\tau)\right\|_{C[0,T]} \leq \frac{B}{1-C}\|b(x,t)\|_{L_2(\Gamma)}\left\|u(t)-u^{(N+1)}(t)\right\|_B. \tag{10}$$

Let us consider (10) in (9) and apply Gronwall's inequality to (9) and taking maximum of both side of the last inequality, we have

$$\begin{aligned}\left\|u(t)-u^{(N+1)}(t)\right\|^2_{\mathbf{B}} \leq\ & \\ & 2\left[\frac{A}{\sqrt{N!}}\left(\left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\left(1+\frac{BT}{1-C}\right)\right)^{N+1}\|b(x,t)\|_{L_2(\Gamma)}\right]^2 \\ &\times\exp 2\left(1+\left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\left(1+\frac{BT}{1-C}\right)\right)^2\|b(x,t))\|^2_{L_2(\Gamma)}.\end{aligned}$$

We obtain $u^{(N+1)}\to u$, $a^{(N+1)}\to a$, $N\to\infty$.

For the uniqueness, we assume that the problem (1)-(4) has two solution pair (a,u), (b,v). Applying Cauchy inequality, Hölder Inequality, Lipschitzs condition and Bessel inequality to $|u(t)-v(t)|$ and $|a(t)-b(t)|$, we obtain

$$\begin{aligned}\|u(t)-v(t)\|_B \leq\ & \left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)M\|a(t)-b(t)\|_{C[0,T]} \\ &+\left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left|u(\tau)-v(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}},\end{aligned}$$

$$\|a(t)-b(t)\|_{C[0,T]} \leq \frac{B}{1-C}\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\,|u(\tau)-v(\tau)|^2\,d\xi d\tau\right)^{\frac{1}{2}},$$

$$\|u(t)-v(t)\|_B \leq \left[\left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\left(1+\frac{B}{1-C}\right)\right]\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\,|u(\tau)-v(\tau)|^2\,d\xi d\tau\right)^{\frac{1}{2}}, \tag{11}$$

applying Gronwall's inequality to (11) we have
$u(t)=v(t)$. Hence $a(t)=b(t)$. □

This completes the proof of Theorem 2.2.

3. Continuous Dependence of (a,u) upon the data

Theorem 3.1. *Under assumption (A1)-(A3) the solution (r,u) of the problem (1)-(4) depends continuously upon the data* φ, E.

Proof. Let $\Phi=\{\varphi,\ a,\ f\}$ and $\overline{\Phi}=\{\overline{\varphi},\ \overline{a},\ f\}$ be two sets of the data, which satisfy the assumptions $(A_1)-(A_3)$. Suppose that there exist positive constants M_i, $i=1,2$ such that

$$\|a\|_{C^1[0,T]} \leq M_1, \|\overline{a}\|_{C^1[0,T]} \leq M_1,\ \|\varphi\|_{C^3[0,\pi]} \leq M_2, \|\overline{\varphi}\|_{C^3[0,\pi]} \leq M_2.$$

Let us denote $\|\Phi\| = (\|a\|_{C^1[0,T]}+\|\varphi\|_{C^3[0,\pi]}+\|f\|_{C^{3,0}(\overline{D})})$.
By using same estimations to $u-\overline{u}$, we obtain

$$\begin{aligned}\|u-\overline{u}\| \quad &\leq \quad M_3\left\|\Phi-\overline{\Phi}\right\| \\ &\quad +M_4\left(\int_0^t\int_0^\pi r^2(\tau)b^2(\xi,\tau)\,\|u(\tau)-\overline{u}(\tau)\|^2\,d\xi d\tau\right)^{\frac{1}{2}}\end{aligned} \tag{12}$$

applying Gronwall's inequality to the last equation, we obtain

$$\begin{aligned}\|u-\overline{u}\|^2 \quad &\leq \quad 2M_3^2\left\|\Phi-\overline{\Phi}\right\|^2 \\ &\quad \times\exp\left(2M_4^2\int_0^t\int_0^\pi r^2(\tau)b^2(\xi,\tau)d\xi d\tau\right).\end{aligned}$$

For $\Phi\to\overline{\Phi}$ then $u\to\overline{u}$. Hence $a\to\overline{a}$. □

4. Numerical Procedure for the nonlinear problem (1)-(4)

We construct an iteration algorithm for the linearization of the problem (1)-(4):

$$\frac{\partial u^{(n)}}{\partial t}-\frac{\partial^2 u^{(n)}}{\partial x^2}-\varepsilon\frac{\partial^3 u^{(n)}}{\partial x^2\partial t}-a(t)u \;=\; f(x,t,u^{(n-1)}), \quad (x,t)\in D \tag{13}$$

$$u^{(n)}(0,t) \;=\; u^{(n)}(\pi,t), \quad t\in[0,T] \tag{14}$$

$$u_x^{(n)}(0,t) \;=\; u_x^{(n)}(\pi,t)=0,\ t\in[0,T] \tag{15}$$

$$u^{(n)}(x,0) \;=\; \varphi(x)\,, \quad x\in[0,\pi]\,. \tag{16}$$

Let $u^{(n)}(x,t) = v(x,t)$ and $f(x,t,u^{(n-1)}) = \widetilde{f}(x,t)$. Then the problem (13)-(16) can be written as a linear problem:

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + \varepsilon \frac{\partial^3 v}{\partial x^2 \partial t} + r(t)\widetilde{f}(x,t) \quad (x,t) \in D & (17)\\
v(0,t) &= v(\pi,t), \qquad t \in [0,T] & (18)\\
v_x(0,t) &= v_x(\pi,t), \qquad t \in [0,T] & (19)\\
v(x,0) &= \varphi(x), \qquad x \in [0,\pi] \ . & (20)
\end{aligned}$$

After linearization, we use the finite difference method to solve (17)-(20).

We subdivide the intervals $[0,\pi]$ and $[0,T]$ into subintervals N_x and N_t of equal lengths $h = \frac{\pi}{N_x}$ and $\tau = \frac{T}{N_t}$, respectively. We choose the implicit scheme which is absolutely stable and has a second-order accuracy in h and a first-order accuracy in τ. The implicit scheme for (17)-(20) is as follows:

$$\frac{1}{\tau}\left(v_i^{j+1} - v_i^j\right) = \frac{1}{2h^2}\left(v_{i-1}^j - 2v_i^j + v_{i+1}^j\right) + \varepsilon \frac{1}{2h^2\tau}\left[(v_{i-1}^{j+1} - 2v_i^{j+1} + v_{i+1}^{j+1}) - (v_{i-1}^j - 2v_i^j + v_{i+1}^j)\right] - a^j v_i^{j+1} = \widetilde{f}_i^j, \tag{21}$$

$$v_i^0 = \phi_i, \tag{22}$$

$$v_0^j = v_{N_x+1}^j, \tag{23}$$

$$\frac{v_1^j + v_{N_x}^j}{2} = v_{N_x+1}^j, \tag{24}$$

where $1 \le i \le N_x$ and $0 \le j \le N_t$ are the indices for the spatial and time steps respectively, $v_i^j = v(x_i,t_j)$, $\phi_i = \varphi(x_i)$, $\widetilde{f}_i^j = \widetilde{f}(x_i,t_j)$, $x_i = ih$, $t_j = j\tau$. At the level $t = 0$, adjustment should be made according to the initial condition and the compatibility requirements.

Now, let us construct the predicting-correcting mechanism. First, integrating the equation (1) with respect to x from 0 to π and using (3) and (4), we obtain

$$a(t) = \frac{-E'(t) + \int\limits_0^\pi x\widetilde{f}(x,t)dx + v_t(\pi,t)}{E(t)}. \tag{25}$$

The finite difference approximation of (25) is

$$a^j = \frac{-\left(E^{j+1} - E^j\right)/\tau + (f_{in})^j + \left(v_{N_x}^{j+1} - v_{N_x}^j\right)/\tau}{E^j},$$

where $E^j = E(t_j)$, $j = 0,1,...,N_t$.

For $j = 0$,

We denote the values of a^j, v_i^j at the s-th iteration step .and the values of ϕ_i provide us to start our computation. We denote the values of p^j, v_i^j at the s-th iteration step a$r^{j(s)}$, $v_i^{j(s)}$, respectively. In numerical computation, since the time step is very small, we can take $a^{j+1(0)} = a^j$, $v_i^{j+1(0)} = v_i^j$, $j = 0,1,2,....N_t$, $i = 1,2,...,N_x$. At each $(s+1)$-th iteration step we first determine $a^{j+1(s+1)}$ from the formula

$$a^{j+1(s+1)} = \frac{-\left(E^{j+1(s+1)} - E^{j(s+1)}\right)/\tau + (f_{in})^{j(s+1)} + \left(v_{N_x}^{j+1(s+1)} - v_{N_x}^{j(s+1)}\right)/\tau}{E^{j(s+1)}}.$$

Then from (21)-(24) we obtain

$$\frac{1}{\tau}\left(v_i^{j+1(s+1)}-v_i^{j+1(s)}\right) = \frac{1}{h^2}\left(v_{i-1}^{j+1(s+1)}-2v_i^{j+1(s+1)}+v_{i+1}^{j+1(s+1)}\right) \tag{26}$$

$$+\varepsilon\frac{1}{2h^2\tau}\left[\left(v_{i-1}^{j+1(s+1)}-2v_i^{j+1(s+1)}+v_{i+1}^{j+1(s+1)}\right)-\left(v_{i-1}^{j+1(s)}-2v_i^{j+1(s)}+v_{i+1}^{j+1(s)}\right)\right]$$

$$+a^j v_i^{j+1(s+1)} = \widetilde{f}_i^{j+1}, \tag{27}$$

$$v_0^{j(s)} = v_{N_x+1}^{j(s)}, \tag{28}$$

$$\frac{v_1^{j(s)}+v_{N_x}^{j(s)}}{2} = v_{N_x+1}^{j(s)}, \tag{29}$$

The system of equations (26)-(29) can be solved by the Gauss elimination method and $v_i^{j+1(s+1)}$ is determined. If the difference of values between two iterations reaches the prescribed tolerance, the iteration is stopped and we accept the corresponding values $a^{j+1(s+1)}$, $v_i^{j+1(s+1)} (i=1,2,...,N_x)$ as a^{j+1}, $v_i^{j+1} (i=1,2,...,N_x)$, on the $(j+1)$-th time step, respectively. In virtue of this iteration, we can move from level j to level $j+1$.

5. Conclusions

The inverse problem regarding the simultaneously identification of the time-dependent source and the temperature distribution in one-dimensional quasilinear pseudo parabolic equation with periodic boundary and integral overdetermination conditions has been considered. This inverse problem has been investigated from both theoretical and numerical points of view. In the theoretical part of the article, the conditions for the existence, uniqueness and continuous dependence upon the data of the problem have been established. The problem is solved implicit difference scheme and an example is given.

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