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The Signatures and Boundary Components of The Groups $\hat{\Gamma}_{0,n}(N)$

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Abstract. In this paper, we established the group $\hat{\Gamma}_{0,n}(N)$ by group $\Gamma_{0,n}(N)$ extending with reflection. Then, we obtain boundary components in signature of the group and we get some calculation for link periods $2, 3, \infty$. And then, we constitute chain of reflections with fixed points via Extended Hoore-Uzzell Theorem in the group. Finally, The number of boundary components in the signature of some groups $\hat{\Gamma}_{0,p}(p)$ and $\hat{\Gamma}_{0,p}(p^2)$, p is a prime number, and the number of link periods was found.

1. Introduction and Preliminaries

Modular group and its congruence subgroups have an important role on discrete group theory. Many authors studied at this area such as Akbaş [1], Beşenk [3], Jones [6], Kader [7], Tekcan [10], etc.

Non-euclidean crystallographic groups (written NEC group) have an important role on discrete group theory and firstly defined by Wilkie [11]. And then Bujalance [4], Jones [6], Macbeath [8], etc. studied. So in this paper, we research signatures and boundary components of a special groups. And now we give some basic definitions and theorems for understanding our paper.

Definition 1.1. *[5] Let*

$$T(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R}, \quad \Delta = ad-bc > 0; \tag{1}$$

then dividing the numerator and denominator by $\sqrt{\Delta}$ *we obtain*

$$T(z) = \frac{\left(a/\sqrt{\Delta}\right)z + \left(b/\sqrt{\Delta}\right)}{\left(c/\sqrt{\Delta}\right)z + \left(d/\sqrt{\Delta}\right)}$$

and as $\left(a/\sqrt{\Delta}\right)\left(d/\sqrt{\Delta}\right) - \left(b/\sqrt{\Delta}\right)\left(c/\sqrt{\Delta}\right) = 1$, *this shows that* $T \in PSL(2,\mathbb{R})$. *We can show the elements of* $PSL(2,\mathbb{R})$ *as follows,*

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}, a,b,c,d \in \mathbb{R} \quad and \quad ad-bc = 1.$$

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Remark 1.2. *This set is a group of all linear fractional transformations. It is the automorphism group of the upper half plane* $\mathbb{H} := \{z \in \mathbb{C} : Im(z) > 0\}$.

Definition 1.3. *[5] The modular group* $\Gamma = PSL(2,\mathbb{Z})$ *is the subgroup of* $PSL(2,\mathbb{R})$.

Definition 1.4. *[11] The group G consist of all transformations of one or other of the two forms:*

$$w = \frac{az+b}{cz+d}, \quad ad-bc=1 \quad a,b,c,d \in \mathbb{R}, \tag{2}$$

$$w = \frac{a\bar{z}+b}{c\bar{z}+d}, \quad ad-bc=-1 \quad a,b,c,d \in \mathbb{R}. \tag{3}$$

Those of the form (2) *preserve orientation, and form a subgroup* $LF(2,\mathbb{R})$ *of index2-the hyperbolic group; Those of the form* (3) *do not preserve orientation. G maps* $\mathbb{H}$ *into itself. The topology on G comes from the numbers* $a,b,c,d \in \mathbb{R}$.

Definition 1.5. *[11] Firstly, we assume that* $T \in PSL(2,\mathbb{R}) \setminus I$ *and* $T(z) = \frac{az+b}{cz+d}$. *Then*

1. *Hyperbolic if* $|a+d| > 2$ *with two fixed points on the real axis,*
2. *Elliptic if* $|a+d| < 2$ *with one fixed point in* $\mathbb{H}$,
3. *Parabolic if* $|a+d| = 2$ *with one fixed point multiplicity two on the real axis.*

Secondly, we assume that $S \in \overline{PSL}(2,\mathbb{Z})$ *and* $S(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$. *Then*

1. *Glide reflection if* $a+d \neq$ *with two fixed points on the real axis.*
2. *Reflection if* $a+d=0$ *with hyperbolic line perpendicular to* $\mathbb{R}$.

Definition 1.6. *[11] A non-Euclidean crystallographic (written N. E. C.) group is a discrete subgroup of G.*

Theorem 1.7. *[5] Finite-order elements different from the unit of G are either elliptic or reflection transformations.*

Definition 1.8. *[9] We suppose that* Λ *is a NEC group and* $x \in \mathbb{R} \cup \{\infty\}$. *In this case, if there is a parabolic element* $g \in \Lambda$ *such that* $g(x) = x$, *then x is called "cusp point (cusp representative)". Hence, the expression of* Λx *which it is orbit* Λ *of x is called cusp and denoted by* $[x]$. *Moreover, if there is a reflection* $S \in \Lambda$ *such that* $S([x]) = [x]$, *then* $[x]$ *is called "real cusp".*

Remark 1.9. *Throughout this article we will study at finite generated NEC group* Λ *provided that the orbital space* $\mathbb{H}^*/\Lambda$ *is compact. Here,* $\mathbb{H}^* = \mathbb{H} \cup \mathcal{B}$, *and* $\mathcal{B} := \left\{[x] : x \in \mathbb{R}_\infty\right\}$.

Remark 1.10. *We can write the following table for generators and relations of NEC group* Λ *[8],[11]*

Table 2.1 : *Generators and relations of NEC group* Λ

Generators	x_i ; $i = 1,\dots,r$	
	e_i ; $i = 1,\dots,k$	
	c_{ij} ; $i = 1,\dots,k$ *and* $j = 0,1,\dots,s_i$	
	a_i, b_i ; $i = 1,\dots,g$	(*I. kind*)
	d_i ; $i = 1,\dots,g$	(*II. kind*)
Relations	$x_i^{m_i} = 1$; $i = 1,\dots,r$	
	$c_{is_i} = e_i^{-1} c_{i0} e_i$; $i = 1,\dots,k$	
	$c_{i,j-1}^2 = c_{ij}^2 = (c_{i,j-1}c_{ij})^{n_{ij}} = 1$	
	$x_1 \dots x_r e_1 \dots e_k a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1$	(*I. kind*)
	$x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_g^2 = 1$	(*II. kind*)

Here, let $\mathbb{N}_2 := \{2,3,\dots\}$. *If* $m_i \in \mathbb{N}_2$, *then* x_i *is an elliptic element. If* $m_i = \infty$, *then* x_i *is a parabolic element. If* $n_{ij} \in \mathbb{N}_2$, *then the combination of the two reflections is an elliptical element. And if* $n_{ij} = \infty$, *this combination is either a parabolic element or a hyperbolic element. It is clear that the numbers* $m_i, n_{ij} \in \mathbb{N}_2 \cup \{\infty\}$ *are the order of the direction-protecting elements of* Λ.

Definition 1.11. *[4] The representation*

$$\sigma(\Lambda) = (g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\})$$

is called a NEC signature of Λ for NEC group Λ given at Table 2.1. *We say shortly $\sigma(\Lambda)$ or signature of Λ. Moreover, it is called some notions at the signature $\sigma(\Lambda)$ as follow:*

(1.) *Number $g \in \mathbb{N}$ in the signature is called genus of orbit space's $\mathbb{H}^*/\Lambda$. And it is topologically invariant of surface.*

(2.) *If orbit space $\mathbb{H}^*/\Lambda$ can be directable, then $sgn\sigma(\Lambda) = '' + ''$ or indirectable, then $sgn\sigma(\Lambda) = '' - ''$.*

(3.) *For $i = 1, 2, \cdots, r$, the numbers $m_i \in \mathbb{N}_2$ is called natural period of Λ.*

(4.) *For $i = 1, 2, \cdots, r$, the numbers $m_i \in \mathbb{N}_2 \cup \{\infty\}$ is called special period of Λ.*

(5.) *The set $C = \{C_1, C_2, \cdots, C_k\}$ is called boundary component of Λ.*

(6.) *For $i = 1, 2, \cdots, k$, the notion $C_i = (n_{i_1}, n_{i_2}, \ldots, n_{is_i})$ are called i-th boundary component of signature or i-th periodic-cycles.*

(7.) *For $i = 1, 2, \cdots, k$, the numbers $n_{i_1}, n_{i_2}, \cdots, n_{is_i} \in \mathbb{N}_2 \cup \{\infty\}$ are called period of i-th boundary component or link period of Λ.*

Theorem 1.12. *[5] (Extended Hoare-Uzzell Theorem) Let G be a NEC group with signature*

$$\sigma(G) = \left(g; \mp; [m_1, \cdots, m_r]; \{(n_{11}, \cdots, n_{1s_1}), \cdots, (n_{k1}, \cdots, n_{ks_k})\}\right)$$

and H a subgroup of finite index. Each fixed point of a reflection c_i of the permutation representation of G on the H-cosets gives a reflection in H.

Let c_i, c_{i+1} be two reflections, with $c_i c_{i+1}$ having order $n_i \le \infty$. Let $y_i = c_i c_{i+1}$ have an orbit (cycle) of length r_i. Then: either

- ***a)*** *this orbit contains no fixed points of c_i or c_{i+1} in which case there exists another orbit of the same length, and these two together induce an ordinary period n_i/r_i.*

or

- ***b)*** *this orbit contains two fixed points of c_i and c_{i+1} (one fixed by each if r_i is odd, two by one and one by the other if r_i is even): and there is a relation between two induced reflections as, $c_i \backsim^{n_i/r_i} c_{i+1}$. Combining these relations makes up period cycles with link periods n_i/r_i.*

Lemma 1.13. *[6] Let T, K be $\in \hat{\Gamma}_0(N)$*

$$T = \begin{pmatrix} r & -k \\ s & -t \end{pmatrix} \text{ and } K = \begin{pmatrix} x & -m \\ y & -n \end{pmatrix} \in \hat{\Gamma}$$

then,

$$\frac{r}{s} \approx \frac{x}{y} \iff ry - sx \equiv 0 \ mod \ N \ \ (ry - sx = \mp N).$$

Here the relation $'' \approx ''$ is on $\hat{\mathbb{Q}}$ that $\hat{\Gamma}_0(N)$ is a reduced $\hat{\Gamma}$ invariant equivalence relation,

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) \ : \ c \equiv 0 \ mod \ N \right\}, \ \hat{\Gamma}_0(N) := \left\langle \Gamma_0(N), z \to -\overline{z} \right\rangle,$$

$X_0(N) = \mathbb{H}^*/\Gamma_0(N)$ *and* $\hat{X}_0(N) = \mathbb{H}^*/\hat{\Gamma}_0(N)$.

Theorem 1.14. *[1] Let the numbers $N \in \mathbb{Z}^+$ and r are divisor number of N. We can write the followings for the group $\hat{\Gamma}_0(N)$:*

I. case: *If N is odd, then the number of boundary component of $X_0(N)$ is 2^{r-1} and there are 2 cusps in each boundary component.*

II. case: ***a)*** *Let $2 \| N$.*

i) *If* $N = 2$*, then there is only one boundary component. And there are* 2 *cusps belonging to it.*

ii) *If* $N = 2m$*,* $m > 1$*, then there are* 2^{r-2} *boundary component. And there are* 4 *cusps belonging to each boundary components.*

b) *Let* $2^2 \| N$*.*

i) *If* $N = 4$*, then there is only one boundary component. And there are* 3 *cusps belonging to it.*

ii) *If* $N > 4$*, then there are* 2^{r-2} *boundary component. And there are* 6 *cusps belonging to each boundary components.*

c) *If* $2^3 | N$*, then the number of boundary component are* 2^{r-1}*. And there are* 4 *cusps in each boundary component.*

2. Main Results

2.1. Signature of the Extended Congruence Subgroup

Let we consider the following extended congruence subgroup for $N \in \mathbb{Z}^+$

$$\hat{\Gamma}_0(N) = \left\langle \Gamma_0(N), z \to -\bar{z} \right\rangle = \Gamma_0(N) \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_0(N).$$

Thus, $\hat{\Gamma}_\infty < \hat{\Gamma}_0(N) < \hat{\Gamma}$. If we take $u = \dfrac{r}{s}, v = \dfrac{x}{y} \in \hat{\mathbb{Q}}$, then there are $T, K \in \hat{\Gamma}$ such that $T(\infty) = u$ and $K(\infty) = v$

$$T = \begin{pmatrix} r & -k \\ s & -t \end{pmatrix} \text{ and } K = \begin{pmatrix} x & -m \\ y & -n \end{pmatrix}.$$

Now we consider the special subgroup of $\hat{\Gamma}_0(N)$ for $N \in \mathbb{Z}^+$, namely,

$$\hat{\Gamma}_{0,n}(N) = \left\langle \Gamma_{0,n}(N), z \to -\bar{z} \right\rangle = \Gamma_{0,n}(N) \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_{0,n}(N).$$

Let we calculate in the signature of the group

$$\hat{\Gamma}_{0,n}(N) = \left\{ \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \hat{\Gamma}_0(N) \,:\, a \equiv \mp d \ mod\ n \right\}.$$

And also let we determine the orbit space $Y_0(N) = \mathbb{H}^* \diagup \Gamma_{0,n}(N)$ and $\hat{Y}_0(N) = \mathbb{H}^* \diagup \hat{\Gamma}_{0,n}(N)$ for $\Gamma_{0,n}(N)$ and $\hat{\Gamma}_{0,n}(N)$, respectively.

Theorem 2.1. *Let* $\hat{\Gamma}$ *be an extended modular group and*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \hat{\Gamma}, \qquad c_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, c_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c_3 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Then,

a.) c_1 *leaves fixed to* $\hat{\Gamma}_{0,n}(N) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff N | 2cd$ *and* $(ad + bc)^2 \equiv 1 \ mod\ n$*,*

b.) c_2 *leaves fixed to* $\hat{\Gamma}_{0,n}(N) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff N | d^2 - c^2$ *and* $(bd - ac)^2 \equiv 1 \ mod\ n$*,*

c.) c_3 *leaves fixed to* $\hat{\Gamma}_{0,n}(N) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff N | 2cd - c^2$ *and* $(ad - ac + bc)^2 \equiv 1 \ mod\ n$*.*

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \hat{\Gamma}$ and $\hat{\Gamma} = PSL(2, \mathbb{Z}) \cup \overline{PSL}(2, \mathbb{Z})$.

a)

$$\hat{\Gamma}_{0,n}(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix}c_1 = \hat{\Gamma}_{0,n}(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff \begin{pmatrix} ad+bc & -2ab \\ 2cd & -bc-ad \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff N|2cd \text{ and } (ad+bc)^2 \equiv 1 \bmod n.$$

b)

$$\hat{\Gamma}_{0,n}(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix}c_2 = \hat{\Gamma}_{0,n}(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff \begin{pmatrix} b & a \\ d & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff \begin{pmatrix} bd-ac & a^2-b^2 \\ d^2-c^2 & ac-bd \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff N|d^2-c^2 \text{ and } (bd-ac)^2 \equiv 1 \bmod n.$$

c)

$$\hat{\Gamma}_{0,n}(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix}c_3 = \hat{\Gamma}_{0,n}(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff \begin{pmatrix} a & a-b \\ c & c-d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff \begin{pmatrix} ad-ac+bc & a^2-2ab \\ 2cd-c^2 & -bc+ac-ad \end{pmatrix} \in \hat{\Gamma}_{0,n}(N)$$
$$\iff N|2cd-c^2 \text{ and } (ad-ac+bc)^2 \equiv 1 \bmod n.$$

So, the proof is completed.

Lemma 2.2. *Elliptic and parabolic elements generated with reflections of* c_1, c_2, c_3 *in* $\hat{\Gamma}$ *are determined as follows:*

a.) $T_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, T_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $T_1^2 = T_2^3 = T_3^\infty = I.$

b.) $T_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T_5 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, T_6 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ *and* $T_4^2 = T_5^3 = T_6^\infty = I.$

Proof. We know

$$c_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, c_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c_3 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, (c_1c_2)^2 = (c_2c_3)^3 = (c_1c_3)^\infty = I.$$

Then,

$$\textbf{a)}\quad T_1 = c_1c_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$T_2 = c_2c_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$
$$T_3 = c_1c_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In this case, we obtain the relation $T_1^2 = T_2^3 = T_3^\infty = I$. Then,

$$\begin{aligned}
\textbf{b)}\ T_4 &= c_2c_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
T_5 &= c_3c_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \\
T_6 &= c_3c_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

So, we have $T_4^2 = T_5^3 = T_6^\infty = I$.

Remark 2.3. *The combinations of these transformations can also be used.*

$$(c_2c_3)^2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \textit{and } (c_3c_1)^k = \begin{pmatrix} 1 & -k \\ -1 & 1 \end{pmatrix}$$

Lemma 2.4. *[1] $ad \equiv 1 \ mod \ s$ provides $a \equiv d \ mod \ s$ if and only if s is the integer divisor of* 24*.*

Proof. "$\Longrightarrow$": Let $ad \equiv 1 \ mod \ s$ provides the congruence $a \equiv d \ mod \ s$ and $U_s := \{a \in \mathbb{Z}_s \mid (a,s) = 1\}$. Here, $a^2 \equiv 1 \ mod \ s$ reduces to finding s for each $a \in U_s$ that satisfies the congruence. In this case, we assume that $s = 2^\alpha.3^\beta q_1^{\alpha_1} \ldots q_k^{\alpha_k}, (q_i \in \mathbb{P}, q_i \neq 2, q_i \neq 3)$. So, we have $U_s \cong U_{2^\alpha} \times U_{3^\beta} \times U_{q_1^{\alpha_1}} \times \ldots \times U_{q_k^{\alpha_k}}$. If p is odd prime number and $n \geq 1$, then U_{p^n} is cyclic. The order of these groups are $\varphi(3^\beta), \varphi(q_1^{\alpha_1}), \ldots, \varphi(q_k^{\alpha_k})$, respectively. Here φ is an Euler function. Because each of these groups has two members with an order of 2. So β should be 1, and $q_i^{\alpha_i}$ does not exist. Thus, it is determined as $s = 2^\alpha 3^\beta$, either $\beta = 0$ or $\beta = 1$. On the other hand, if $\alpha \geq 3$, then $U_{2^\alpha} := \{\mp 5^t \ : \ 0 \leq t \leq 2^{\alpha-2}\}$. Here, mth order of 5 is exactly $2^{\alpha-2}$. If $\alpha > 3$, then m will be at least 4. But it is a contradiction because each elements of U_{2^α} have got 2nd order. So it should be $\alpha \leq 3$. Consequently, we obtain $s|24$.

"$\Longleftarrow$": Let $ad \equiv 1 \ mod \ s$ and $s|24$. In this case, due to $\varphi(24) = 8$ we determine the integer a and d such that $a, d \in \{1,5,7,11,13,17,19,23\}$. That is, the counting number less than 24 and prime between 24 is 8, and let's make the selection according to the cluster above. In this case, we get $a^2 \equiv d^2 \equiv 1 \ mod \ s$. Thus, we obtain $a \equiv d \ mod \ s$.

$$\begin{aligned}
&\alpha = 1 \Longrightarrow U_{2^1} := \{a \in \mathbb{Z}_2 \ : \ (a,2) = 1\} = \{1\} \text{ and } a^2 \equiv 1 \ mod \ 2, \\
&\alpha = 2 \Longrightarrow U_{2^2} := \{a \in \mathbb{Z}_4 \ : \ (a,4) = 1\} = \{1,3\} \text{ and } a^2 \equiv 1 \ mod \ 4, \\
&\alpha = 3 \Longrightarrow U_{2^3} := \{a \in \mathbb{Z}_8 \ : \ (a,8) = 1\} = \{1,3,5,7\} \text{ and } a^2 \equiv 1 \ mod \ 8, \\
&\alpha = 4 \Longrightarrow U_{2^4} := \{a \in \mathbb{Z}_{16} \ : \ (a,16) = 1\} = \{1,3,5,7,9,11,13,15\} \text{ and } a^2 \equiv 1 \ mod \ 16.
\end{aligned}$$

Now, the order U_{16} is 4, but it does not. Namely, counting number α and β exist such that $0 \leq \beta \leq 1$ for $s = 2^\alpha 3^\beta$.

Theorem 2.5. *Let $n, N \in \mathbb{Z}^+$ and $n|N$. Then,*
a) $n|24 \Longleftrightarrow \Gamma_{0,n}(N) = \Gamma_0(N)$,
b) $n|24 \Longleftrightarrow \hat{\Gamma}_{0,n}(N) = \hat{\Gamma}_0(N)$.

Proof. **a)** "$\Longrightarrow$:" Let $n|24$. Thus, $\exists k \in \mathbb{Z}$ such that $24 = nk$. It is clear that $\Gamma_{0,n}(N) \subset \Gamma_0(N)$ from $\Gamma_{0,n}(N) \leq \Gamma_0(N)$. Now let we show $\Gamma_0(N) \subset \Gamma_{0,n}(N)$.

We take $T = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$. In this case, we have $detT = ad - bcN = 1$ and $ad \equiv 1 \ mod \ n$. We obtain $a \equiv d \ mod \ n$ from Lemma 2.4 for $n|24$ and $ad \equiv 1 \ mod \ n$. That is, $a^2 \equiv 1 \ mod \ n$ and thus $T \in \Gamma_{0,n}(N)$.

$''\Longleftarrow''$ Let $\Gamma_{0,n}(N) = \Gamma_0(N)$. We take $\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_{0,n}(N) = \Gamma_0(N)$. From this $ad - bcN = 1$ and we obtain $ad \equiv 1 \ mod \ N$. Thus, $ad \equiv 1 \ mod \ n$ from $n|N$. Furthermore, it should be $a \equiv d \ mod \ n$ from $T \in \Gamma_{0,n}(N)$ and $n|24$ from Lemma 2.4.

b) The proof is clear according to case of a) from $\hat{\Gamma}_{0,n}(N) = \Gamma_{0,n}(N) \cup R\Gamma_{0,n}(N)$ and $R(z) = -\bar{z}$ for $\Gamma_{0,n}(N)$. Now we prove for $R\Gamma_{0,n}(N)$.

$''\Longrightarrow''$: Let $n|24$, and $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in R\Gamma_{0,n}(N)$. Thus, $\begin{pmatrix} a & b \\ -cN & -d \end{pmatrix} \in R\Gamma_{0,n}(N)$ and $-ad + bcN = -1$. If we use $-ad \equiv -1 \ mod \ n$ and $n|24$ with Lemma 2.4, then $a \equiv d \ mod \ n$.

$''\Longleftarrow''$ Let $\hat{\Gamma}_{0,n}(N) = \hat{\Gamma}_0(N)$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in R\Gamma_{0,n}(N)$. In this case, $-ad + bcN = -1$ and $a \equiv d \ mod \ n$. So, we also obtain $-ad \equiv -1 \ mod \ n$ and $a \equiv d \ mod \ n$. And we have the same result $n|24$ from Lemma 2.4.

2.2. Boundary Components in the Signature

Theorem 2.6. *Let $p \in \mathbb{P}$. Then, it can be given for the boundary components in the signature of the group $\hat{\Gamma}_{0,p}(p)$ as follows:*
a) *If $p = 2$, then the group's signature has one boundary component and there is one 2 valued link period and two cusp in this component.*
b) *If $p = 3$, then the group's signature has one boundary component and there is one 3 valued link period and two cusp in this component.*
c) *If $p = 5$, then the group's signature has one boundary component and there are two cusp in this component.*

Proof. **a)** Let $N = p = 2$. Then from Theorem 2.5, we have $\hat{\Gamma}_{0,2}(2) = \hat{\Gamma}_0(2)$, and instead of the second terms of Theorem 2.1, only the first conditions can be examined.

$$c_1 \quad \text{reflection leaves fixed to the elements} \quad \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix},$$
$$c_2 \quad \text{reflection leaves fixed to the elements} \quad \begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix},$$
$$c_3 \quad \text{reflection leaves fixed to the elements} \quad \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.$$

The chain $\mathfrak{T}_1$ is below from Theorem 1.14 and Lemma 2.2 for boundary components;

$$^{c_1}\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix} \overset{1}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \overset{\infty}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix} \overset{2}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix}$$
$$\overset{1}{\sim} {}^{c_3}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \overset{\infty}{\sim} {}^{c_3}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}.$$

So, there is a boundary component in the group's signature. There is a 2-valued link period in the signature. And there are also two cusps in it.

b) Let $N = p = 3$. From Theorem 2.5 we have $\hat{\Gamma}_{0,3}(3) = \hat{\Gamma}_0(3)$. And thus instead of the second terms of Theorem 2.1, only the first conditions can be examined.

$$c_1 \text{ reflection leaves fixed to the elements } \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix},$$

$$c_2 \text{ reflection leaves fixed to the elements } \begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix},$$

c_3 reflection leaves fixed to the elements $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}$.

The chain $\mathfrak{T}_2$ is below from Theorem 1.14 and Lemma 2.2 for boundary components;

$$^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \underset{\sim}{1} \, ^{c_1}\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix} \underset{\sim}{\infty} \, ^{c_3}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix} \underset{\sim}{3} \, ^{c_2}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}$$

$$\underset{\sim}{1} \, ^{c_2}\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix} \underset{\sim}{1} \, ^{c_3}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \underset{\sim}{\infty} \, ^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.$$

So, there is a boundary component in the group's signature. There is a 3-valued link period in the boundary component. And there are also two cusps in the boundary component.

c) Let we research the group $\hat{\Gamma}_{0,5}(5)$ for $N = p = 5$.

i) The reflection c_1 leaves fixed to $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ 5c & d \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ c & 5d \end{pmatrix}$. Here the condition of Theorem 2.1-*a*) satisfies. Indeed, we have $N|5cd$ and $(ad + 5bc)^2 \equiv 1 \ mod\ 5$ due to $ad - 5bc = \pm 1$. And then we get $(5ad + bc)^2 \equiv 1 \ mod\ 5$.

$$(ad)^2 \equiv 1 \ mod\ 5 \Longrightarrow ad \equiv \pm 1 \ mod\ 5 \Longrightarrow \begin{cases} a = 1 \text{ and } d = 1;4 \\ a = 2 \text{ and } d = 2;3 \\ a = 3 \text{ and } d = 2;3 \\ a = 4 \text{ and } d = 1;4 \end{cases}$$

So, $a \equiv -d \ mod\ 5$. Similarly, the same situation occurs with $(bc)^2 \equiv 1 \ mod\ 5$. Thus, the reflection c_1 leaves fixed to $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} \pm 1 & k \\ 0 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} k & \pm 1 \\ 1 & 0 \end{pmatrix}$. So, we have

$$\begin{pmatrix} a & b \\ 5c & d \end{pmatrix}\begin{pmatrix} \mp 1 & k \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 5c & d \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} a & -ak \mp b \\ 5c & -5kc \mp d \end{pmatrix} \in \hat{\Gamma}_{0,5}(5)$$

and

$$\begin{pmatrix} a & b \\ c & 5d \end{pmatrix}\begin{pmatrix} k & \mp 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & 5d \end{pmatrix}\begin{pmatrix} 0 & \pm 1 \\ -1 & k \end{pmatrix} = \begin{pmatrix} -b & \mp a + bk \\ -5d & \mp c + 5kd \end{pmatrix} \in \hat{\Gamma}_{0,5}(5).$$

In this case, the reflection c_1 leaves fixed to $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ 5c & d \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ c & 5d \end{pmatrix}$. Moreover, these elements $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} \pm 1 & k \\ 0 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} k & \pm 1 \\ 1 & 0 \end{pmatrix}$ are in the same coset class. Thus, the reflection c_1 without breaking generality leaves fixed to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}$.

ii) From Theorem 2.1, the reflection c_2 leaves fixed to

$$\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longleftrightarrow \begin{cases} 5|\, d^2 - c^2 \\ (bc - ad)^2 \equiv 1 \ mod\ 5. \end{cases}$$

From this, we have $5|(d - c)(d + c)$. And $5|d - c$ or $5|d + c$. Therefore $d - c \equiv 0 \ mod\ 5$ or $d + c \equiv 0 \ mod\ 5$. According to this, we can take either $c = d = 1$ or $c = -1, d = 1$.

The reflection c_2 leaves fixed to $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}$. So,

$$\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ -1 & k \end{pmatrix} = \begin{pmatrix} a - b & -at + bk \\ 0 & k - t \end{pmatrix} \in \hat{\Gamma}_{0,5}(5)$$

and

$$\begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ 1 & k \end{pmatrix} = \begin{pmatrix} a+b & -at+bk \\ 0 & t+k \end{pmatrix} \in \hat{\Gamma}_{0,5}(5).$$

Hence the reflection c_2 leaves fixed to $\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix}$.

iii) From Theorem 2.1, the reflection c_3 leaves fixed to

$$\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longleftrightarrow \begin{cases} 5|2cd - c^2 \\ (ad - ac + bc)^2 \equiv 1 \ mod \ 5. \end{cases}$$

Here, there are two important conditions. Hence, it can be taken either $c = 0, d = 1$ or $c = 2, d = 1$.

The reflection c_3 leaves fixed to $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(5)\begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix}$. In this case, we have

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ 0 & k \end{pmatrix} = \begin{pmatrix} a & -at+bk \\ 0 & k \end{pmatrix} \in \hat{\Gamma}_{0,5}(5)$$

and

$$\begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ -2 & k \end{pmatrix} = \begin{pmatrix} a-2b & -at+bk \\ 0 & -2t+k \end{pmatrix} \in \hat{\Gamma}_{0,5}(5).$$

So, the reflection c_3 leaves fixed to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}$. The chain $\mathfrak{T}_3$ is below from the conditions *i)*, *ii)*, *iii)* with Theorem 1.14 and Lemma 2.2;

$$^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix} \overset{\infty}{\sim} {}^{c_3}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}$$

$$\overset{1}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix} \overset{\infty}{\sim} {}^{c_3}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.$$

Hence, there is a boundary component in the signature. There are two ∞-valued link period in the boundary component.

Corollary 2.7. *We obtain the following results:*

a) *For the signature of* $\hat{\Gamma}_{0,1}(1) = \hat{\Gamma}_0(1)$; $C = \{(2, 3, \infty)\}$,

b) *For the signature of* $\hat{\Gamma}_{0,2}(2)$; $C = \{(\infty, 2, \infty)\}$,

c) *For the signature of* $\hat{\Gamma}_{0,3}(3)$; $C = \{(\infty, 3, \infty)\}$,

d) *For the signature of* $\hat{\Gamma}_{0,5}(5)$; $C = \{(\infty, \infty)\}$.

Theorem 2.8. *Let* $p \in \mathbb{P}$. *Then we can give the follows for the signature of the group* $\hat{\Gamma}_{0,p}(p^2)$ *in the boundary component,*

a) *If* $p = 2$, *then there is a boundary component in the signature and there are* 3 *cusp in the boundary component.*

b) *If* $p = 3$, *then there is a boundary component in the signature and there are* 2 *cusp in the boundary component.*

c) *If* $p = 5$, *then there is a boundary component in the signature and there are* 2 *cusp in the boundary component.*

Proof. **a)** Let $n = p = 2$ and $N = 2^2$. Then $\hat{\Gamma}_{0,2}(4) = \hat{\Gamma}_0(4)$ from Theorem 2.5, and hence instead of the second terms of Theorem 2.1, only the first conditions can be examined.

The reflection c_1 leaves fixed to the elements $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ 1 & 2 \end{pmatrix}$,

The reflection c_2 leaves fixed to the elements $\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix}$,

The reflection c_3 leaves fixed to the elements $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}$.

So, the chain $\mathfrak{T}_4$ is below from Theorem 1.14 and Lemma 2.2

$$\overset{c_1}{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}} \overset{1}{\sim} \overset{c_1}{\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}} \overset{\infty}{\sim} \overset{c_1}{\begin{pmatrix} * & * \\ 1 & 2 \end{pmatrix}} \overset{1}{\sim} \overset{c_1}{\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}}$$

$$\overset{\infty}{\sim} \overset{c_3}{\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}} \overset{1}{\sim} \overset{c_2}{\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix}} \overset{1}{\sim} \overset{c_2}{\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix}}$$

$$\overset{1}{\sim} \overset{c_3}{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}} \overset{\infty}{\sim} \overset{c_1}{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}}.$$

Hence, there is a boundary component in the group's signature, and there are 3 cusps in the boundary component.

b) Let $n = p = 3$ and $N = 3^2$. we have $\hat{\Gamma}_{0,3}(9) = \hat{\Gamma}_0(9)$ from Theorem 2.5, and instead of the second terms of Theorem 2.1, only the first conditions can be examined.

$$\text{The reflection } c_1 \text{ leaves fixed to the elements } \begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix},$$

$$\text{the reflection } c_2 \text{ leaves fixed to the elements } \begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix},\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix},$$

$$\text{the reflection } c_3 \text{ leaves fixed to the elements } \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix},\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}.$$

The chain $\mathfrak{T}_5$ is below from Theorem 1.14 and Lemma 2.2

$$\overset{c_1}{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}} \overset{1}{\sim} \overset{c_1}{\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}} \overset{\infty}{\sim} \overset{c_3}{\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}} \overset{1}{\sim} \overset{c_2}{\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}}$$

$$\overset{1}{\sim} \overset{c_2}{\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix}} \overset{\infty}{\sim} \overset{c_1}{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}}.$$

Hence there is a boundary component, and there are 2 cusps in the boundary component.

c) Let $n = p = 5$ and $N = 5^2$. Now we research the group $\hat{\Gamma}_{0,5}(25)$.

i) According to Theorem 2.1,

$$\text{The reflection } c_1 \text{ leaves fixed to } \hat{\Gamma}_{0,5}(5^2)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longleftrightarrow \begin{cases} 25|2cd \\ (ad+bc)^2 \equiv 1 \ mod\ 5. \end{cases}$$

In this case, the reflection c_1 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ 25c & d \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ c & 25d \end{pmatrix}$. Here, it satisfies Theorem 2.1-a). Indeed, firstly we have $N|25cd$ and $(ad + 25bc)^2 \equiv 1 \ mod\ 5$ from $N = 25$ and $ad - 25bc = \pm 1$. Secondly, we have $N|25cd$ and $(25ad + bc)^2 \equiv 1 \ mod\ 5$ from $N = 25$ and $25ad - bc = \pm 1$. Hence the reflection c_1 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} \mp 1 & k \\ 0 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} k & \mp 1 \\ 1 & 0 \end{pmatrix}$. In this case, we obtain

$$\begin{pmatrix} a & b \\ 25c & d \end{pmatrix}\begin{pmatrix} \mp 1 & k \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 25c & d \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & \mp 1 \end{pmatrix} = \begin{pmatrix} a & -ak \mp b \\ 25c & -25kc \mp d \end{pmatrix} \in \hat{\Gamma}_{0,5}(25)$$

and

$$\begin{pmatrix} a & b \\ c & 25d \end{pmatrix}\begin{pmatrix} k & \mp 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & 25d \end{pmatrix}\begin{pmatrix} 0 & \mp 1 \\ -1 & k \end{pmatrix} = \begin{pmatrix} -b & \mp a + bk \\ -25d & \mp c + 25kd \end{pmatrix} \in \hat{\Gamma}_{0,5}(25).$$

From this, the reflection c_1 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ 25c & d \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ c & 25d \end{pmatrix}$. So, these elements and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} \mp 1 & k \\ 0 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} k & \mp 1 \\ 1 & 0 \end{pmatrix}$ elements are in the same coset class. Therefore, the reflection c_1 leaves fixed to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}$.

ii) According to Theorem 2.1, the reflection c_2 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longleftrightarrow \begin{cases} 25|d^2 - c^2 \\ (bd - ac)^2 \equiv 1 \ mod\ 5. \end{cases}$
From this, $25|d^2 - c^2 \Longrightarrow 5|(d-c)(d+c) \Longrightarrow$ if and only if $5|d-c$ or only $5|d+c$. So, we obtain $d - c \equiv 0 \ mod\ 5^2$ or $d + c \equiv 0 \ mod\ 5^2$. Hence we can take either $c = d = 1$ or $c = -1, d = 1$.
The reflection c_2 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$. Because of $25|1^2 - 1^2$ and $(a1 - b1)^2 \equiv 1 \ mod\ 5$, it satisfies Theorem 2.1. Then, the reflection c_2 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}$. In this case, we have

$$\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ -1 & k \end{pmatrix} = \begin{pmatrix} a-b & -at+bk \\ 0 & k-t \end{pmatrix} \in \hat{\Gamma}_{0,5}(25)$$

and

$$\begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -t \\ -1 & k \end{pmatrix} = \begin{pmatrix} -a-b & -at+bk \\ 0 & k+t \end{pmatrix} \in \hat{\Gamma}_{0,5}(25).$$

Hence, the reflection c_2 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}$. These elements $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} k & t \\ 1 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} k & t \\ -1 & 1 \end{pmatrix}$ are in the same coset. Thus, the reflection c_2 leaves fixed to $\begin{pmatrix} * & * \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix}$.

iii) According to Theorem 2.1 the reflection c_3 leaves fixed to

$$\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longleftrightarrow \begin{cases} 25|2cd - c^2 \\ (ad - ac + bc)^2 \equiv 1 \ mod\ 5. \end{cases}$$

In this case, there are either $c = 0, d = 1$ or $c = 2, d = 1$.

The reflection c_3 leaves fixed to $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and $\hat{\Gamma}_{0,5}(25)\begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix}$. These elements satisfy the condition of Theorem 2.1-c). Thereby, we get

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ 0 & k \end{pmatrix} = \begin{pmatrix} a & -at+bk \\ 0 & k \end{pmatrix} \in \hat{\Gamma}_{0,5}(25)$$

and

$$\begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix}\begin{pmatrix} k & t \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ -2 & k \end{pmatrix} = \begin{pmatrix} a-2b & -at+bk \\ 0 & -2t+k \end{pmatrix} \in \hat{\Gamma}_{0,5}(25).$$

And these elements are also in the same coset. From this the reflection c_3 leaves fixed to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix}$. Hence, the chain $\mathfrak{T}_6$ is below from Theorem 1.14 and Lemma 2.2

$$^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix} \overset{\infty}{\sim} {}^{c_3}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ 2 & 1 \end{pmatrix} \overset{1}{\sim} {}^{c_2}\begin{pmatrix} * & * \\ -1 & 1 \end{pmatrix} \overset{\infty}{\sim} {}^{c_1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.$$

Consequently, there is a boundary component in the group' s signature, and there are 2 cusps in the boundary component.

Corollary 2.9. *We obtain the following results:*

a) *For the signature of* $\hat{\Gamma}_{0,2}(4)$; $C = \{(\infty, \infty, \infty)\}$,

b) *For the signature of* $\hat{\Gamma}_{0,3}(9)$; $C = \{(\infty, \infty)\}$,

c) *For the signature of* $\hat{\Gamma}_{0,5}(25)$; $C = \{(\infty, \infty)\}$.

Corollary 2.10. *There are not 2 and 3-valued link periods in the signature of the group* $\hat{\Gamma}_{0,5}(5^{\alpha})$ *for* $\alpha \in \mathbb{Z}$ *and* $\alpha \geq 1$. *Then there is only one boundary component and there are two cusps in the group' s signature. Namely, the set of boundary component is* $C = \{(\infty, \infty)\}$.

3. Conclusions

Considering the investigations done so far, we can get more general results as in the Table 3.1 by using Theorem 2.5 as we did before, based on Theorem 1.14

It should be noted that there are no 2 and 3-valued link periods except the groups $\hat{\Gamma}, \hat{\Gamma}_{0,2}(2), \hat{\Gamma}_{0,3}(3)$. In all other cases there is a ∞-valued link period. These ∞-valued link periods appear to be associated with parabolic transformations and even with fixed points they left constant.

Table 3.1 : Boundary components of the signatures of the some groups $\hat{\Gamma}_{0,n}(N)$

The Group Name	The set of boundary component in the signature
$\hat{\Gamma}_{0,4}(4)$	$\{(\infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,4}(8)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,4}(16)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,4}(24)$	$\{(\infty, \infty, \infty, \infty), (\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,2}(6)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,6}(6)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,6}(12)$	$\{(\infty, \infty, \infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,6}(18)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,6}(24)$	$\{(\infty, \infty, \infty, \infty), (\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,8}(8)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,8}(16)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,8}(24)$	$\{(\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,12}(12)$	$\{(\infty, \infty, \infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,12}(24)$	$\{(\infty, \infty, \infty, \infty), (\infty, \infty, \infty, \infty)\}$
$\hat{\Gamma}_{0,24}(24)$	$\{(\infty, \infty, \infty, \infty), (\infty, \infty, \infty, \infty)\}$

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