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A NEW SUBCLASS OF MEROMORPHIC FUNCTIONS DEFINED BY RAPID OPERATOR

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Abstract. We present and investigate a new subclass of meromorphic univalent functions described by the Rapid operator in this study. Coefficient inequalities is discussed, as well as distortion properties, closure theorems, Hadamard product. After this, integral transforms for the class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ are obtained.

1. Introduction

Let Σ stands for the function class of the form

$$\aleph(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} a_\ell \hbar^\ell, \ \ell \in N = \{1, 2, 3, \cdots\} \tag{1}$$

analytic in the punctured unit disc $\Upsilon^* = \{\hbar \in C : 0 < |\hbar| < 1\} = \Upsilon \setminus \{0\}$.

A function $\aleph \in \Sigma$ given by (1) is said to be meromorphically starlike of order ϱ if it satisfies the following:

$$\Re\left\{-\left(\frac{\hbar\aleph'(\hbar)}{\aleph(\hbar)}\right)\right\} > \varrho, \ (\hbar \in \Upsilon)$$

for some $\varrho(0 \leq \varrho < 1)$. We say that $\aleph$ is in the class $\Sigma^*(\varrho)$ of such functions.

Similarly a function $\aleph \in \Sigma$ given by (1) is said to be meromorphically convex of order ϱ if it satisfies the following:

$$\Re\left\{-\left(1 + \frac{\hbar\aleph''(\hbar)}{\aleph'(\hbar)}\right)\right\} > \varrho, \ (\hbar \in \Upsilon)$$

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for some $\varrho(0 \leq \varrho < 1)$. We say that $\aleph$ is in the class $\Sigma_\ell(\varrho)$ of such functions.

Akgul [1,2], Miller [8], Pommerenke [9], Royster [10], Aydogan and Sakar [4,5,11] and Venkateswarlu et al. [14, 15, 16] have all studied the class $\Sigma^*(\varrho)$ and numerous other subclasses of Σ extensively.

For functions $\aleph \in \Sigma$ given by (1) and $g \in \Sigma$ given by

$$g(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} b_\ell \hbar^\ell,$$

we define the Hadamard product of $\aleph$ and g by

$$(\aleph * g)(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} a_\ell b_\ell \hbar^\ell.$$

Jung et al. defined the integral operator on normalised analytic functions in [6] and Lashin [7] updated their operator for meromorphic functions in the following manner:

Lemma 1. *For $\aleph \in \Sigma$ given by (1), if the operator $S_\mu^\theta : \Sigma \to \Sigma$ is defined by*

$$S_\mu^\theta \aleph(\hbar) = \frac{1}{(1-\mu)^\theta \Gamma(\theta+1)} \int_0^\infty t^{\theta+1} e^{\frac{-t}{1-\mu}} \aleph(t\hbar) dt, \tag{2}$$

$(0 \leq \mu < 1, 0 \leq \theta \leq 1$ and $\hbar \in \Upsilon)$ then

$$S_\mu^\theta \aleph(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} \phi_\ell(\theta, \mu) a_\ell \hbar^\ell \tag{3}$$

where $\phi_\ell(\theta, \mu) = (1-\mu)^{\ell+1} \frac{\Gamma(\ell+\theta+2)}{\Gamma(\theta+1)}$ and Γ is the familiar Gamma function.

Using the equation (3), it is easily seen that

$$\hbar(S_\mu^\theta \aleph(\hbar))' = \mu S_\mu^{\theta-1} \aleph(\hbar) - (\mu+1) S_\mu^\theta \aleph(\hbar), (0 \leq \mu \leq 1, 0 \leq \theta \leq 1). \tag{4}$$

We define a new subclass $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ of Σ based on Sivaprasad Kumar et al. [13] and Venkateswarlu et al. [14] $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ of Σ.

Definition 2. *For $0 \leq \vartheta < 1$, $\varrho \geq 0$, $0 \leq \wp < \frac{1}{2}$, we let $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ be the subclass of Σ consisting of functions of the form (1) and satisfying the analytic condition*

$$-\Re\left(\frac{\hbar(S_\mu^\theta \aleph(\hbar))'^2 (S_\mu^\theta \aleph(\hbar))''}{(1-\wp) S_\mu^\theta \aleph(\hbar) + \wp \hbar (S_\mu^\theta \aleph(\hbar))'} + \vartheta\right) > \varrho \left|\frac{\hbar(S_\mu^\theta \aleph(\hbar))'^2 (S_\mu^\theta \aleph(\hbar))''}{(1-\wp) S_\mu^\theta \aleph(\hbar) + \wp \hbar (S_\mu^\theta \aleph(\hbar))'} + 1\right|. \tag{5}$$

The following lemmas are needed to prove our findings [3].

Lemma 3. *If η is a real number and ω is a complex number then*

$$\Re(\omega) \geq \eta \Leftrightarrow |\omega + (1-\eta)| - |\omega - (1+\eta)| \geq 0.$$

Lemma 4. *If ω is a complex number and η, ℓ are real numbers then*

$$-\Re(\omega) \geq \ell|\omega+1| + \eta \Leftrightarrow -\Re\left(\omega(1+\ell e^{i\theta}) + \ell e^{i\theta}\right) \geq \eta, \ (-\pi \leq \ \theta \leq \pi).$$

The key purpose of this paper is to look at some traditional geometric function theory properties for the class of geometric functions, such as coefficient bounds, distortion properties, closure theorems, Hadamard product, and integral transforms.

2. Coefficient estimates

We obtain required and adequate conditions for a function $\aleph$ to be in the class in this section.

Theorem 5. *Let $\aleph \in \Sigma$ be given by* (1). *Then $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ iff*

$$\sum_{\ell=1}^{\infty}[(1+(\ell-1)\wp)][\ell(\varrho+1)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)a_\ell \leq (1-\vartheta)(1-2\wp). \tag{6}$$

Proof. Let $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. Then by Definition 2 and using Lemma 4, It suffices to demonstrate that

$$-\Re\left\{\frac{\hbar(S_\mu^\theta \aleph(\hbar))'^2 (S_\mu^\theta \aleph(\hbar))''}{(1-\wp)S_\mu^\theta \aleph(\hbar) + \wp\hbar(S_\mu^\theta \aleph(\hbar))'}(1+\varrho e^{i\theta}) + \varrho e^{i\theta}\right\} \geq \vartheta, \ (-\pi \leq \ \theta \leq \pi). \tag{7}$$

For convenience

$$\begin{aligned} C(\hbar) =& -\left[\hbar(S_\mu^\theta \aleph(\hbar))'^2 (S_\mu^\theta \aleph(\hbar))''\right](1+\varrho e^{i\theta}) \\ & -\varrho e^{i\theta}\left[(1-\wp)S_\mu^\theta \aleph(\hbar) + \wp\hbar(S_\mu^\theta \aleph(\hbar))'\right] \\ D(\hbar) =& (1-\wp)S_\mu^\theta \aleph(\hbar) + \wp\hbar(S_\mu^\theta \aleph(\hbar))'. \end{aligned}$$

That is, the equation (7) is equivalent to

$$-\Re\left(\frac{C(\hbar)}{D(\hbar)}\right) \geq \vartheta.$$

We only need to prove that in light of Lemma 3

$$|C(\hbar) + (1-\vartheta)D(\hbar)| - |C(\hbar) - (1+\vartheta)D(\hbar)| \geq 0.$$

Therefore

$$\begin{aligned} & |C(\hbar) + (1-\vartheta)D(\hbar)| \\ \geq & (2-\vartheta)(1-2\wp)\frac{1}{|\hbar|} - \sum_{\ell=1}^{\infty}[\ell-(1-\vartheta)][1+\wp(\ell-1)]\phi_\ell(\theta,\mu)a_\ell|\hbar|^\ell \\ & -\varrho\sum_{\ell=1}^{\infty}(\ell+1)[1+\wp(\ell-1)]\phi_\ell(\theta,\mu)a_\ell|\hbar|^\ell \end{aligned}$$

and $|C(\hbar) - (1+\vartheta)D(\hbar)|$

$$\leq \vartheta(1-2\wp)\frac{1}{|\hbar|}+\sum_{\ell=1}^{\infty}[\ell+(1+\vartheta)][1+\wp(\ell-1)]\phi_{\ell}(\theta,\mu)a_{\ell}|\hbar|^{\ell}$$
$$+\varrho\sum_{\ell=1}^{\infty}(\ell+1)[1+\wp(\ell-1)]\phi_{\ell}(\theta,\mu)a_{\ell}|\hbar|^{\ell}.$$

It is to show that

$$|C(\hbar)+(1-\vartheta)D(\hbar)|-|C(\hbar)-(1+\vartheta)D(\hbar)|$$
$$\geq 2(1-\vartheta)(1-2\wp)\frac{1}{|\hbar|}-2\sum_{\ell=1}^{\infty}[(\ell+\vartheta)(1+(\ell-1)\wp)]\phi_{\ell}(\theta,\mu)a_{\ell}|\hbar|^{\ell}$$
$$-2\varrho\sum_{\ell=1}^{\infty}(\ell+1)(1+(\ell-1)\wp)\phi_{\ell}(\theta,\mu)a_{\ell}|\hbar|^{\ell}$$
$$\geq 0, \text{ by the given condition (6).}$$

Conversely suppose $\aleph \in \Sigma^{*}(\vartheta,\varrho,\wp,\theta,\mu)$. Then by Lemma 3, we have (7).

The inequality (7) is reduced to when the values of $\hbar$ are chosen on the positive real axis

$$\Re\left\{\frac{[(1-2\wp)(1-\vartheta)(1+\varrho e^{i\theta})]\frac{1}{\hbar^{2}}+\sum_{\ell=1}^{\infty}\{\ell+\varrho e^{i\theta}(\ell+1)+\vartheta\}[1+\wp(\ell-1)]\phi_{\ell}(\theta,\mu)\hbar^{\ell-1}}{(1-2\wp)\frac{1}{\hbar^{2}}+\sum_{\ell=1}^{\infty}[1+\wp(\ell-1)]\phi_{\ell}(\theta,\mu)a_{\ell}\hbar^{\ell-1}}\right\}\geq 0.$$

Since $\Re(-e^{i\theta})\geq -|e^{i\theta}|=-1$, the above inequality is reduced to

$$\Re\left\{\frac{[(1-2\wp)(1-\vartheta)(1+\varrho e^{i\theta})]\frac{1}{r^{2}}+\sum_{\ell=1}^{\infty}\{\ell+\varrho(\ell+1)+\vartheta\}[1+\wp(\ell-1)]\phi_{\ell}(\theta,\mu)a_{\ell}r^{\ell-1}}{(1-2\wp)\frac{1}{r^{2}}+\sum_{\ell=1}^{\infty}[1+\wp(\ell-1)]\phi_{\ell}(\theta,\mu)r^{\ell-1}}\right\}\geq 0.$$

We obtained the inequality (6) by letting $r\to 1^{-}$ and using the mean value theorem. □

Corollary 6. *If* $\aleph \in \Sigma^{*}(\vartheta,\varrho,\wp,\theta,\mu)$ *then*

$$a_{\ell}\leq \frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\vartheta+\varrho)]\phi_{\ell}(\theta,\mu)}. \tag{8}$$

The estimate is sharp for the function

$$\aleph(\hbar)=\frac{1}{\hbar}+\frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\vartheta+\varrho)]\phi_{\ell}(\theta,\mu)}\hbar^{\ell}. \tag{9}$$

We get the following corollary by taking $\wp=0$ in Theorem 5.

Corollary 7. *If* $\aleph \in \Sigma^*(\vartheta, \varrho, \theta, \mu)$ *then*

$$a_\ell \leq \frac{1-\vartheta}{[\ell(1+\varrho)+(\vartheta+\varrho)]\phi_\ell(\theta,\mu)}. \tag{10}$$

3. Distortion theorem

Theorem 8. *If* $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ *then for* $0 < |\hbar| = r < 1$,

$$\frac{1}{r} - \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}\ r \leq |\aleph(\hbar)| \leq \frac{1}{r} + \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)} r. \tag{11}$$

This estimate is sharp for the function

$$\aleph(\hbar) = \frac{1}{\hbar} + \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}\ \hbar. \tag{12}$$

Proof. Since $\aleph(\hbar) = \frac{1}{\hbar} + \sum\limits_{\ell=1}^{\infty} a_\ell \hbar^\ell$, we have

$$|\aleph(\hbar)| = \frac{1}{r} + \sum_{\ell=1}^{\infty} a_\ell r^\ell \leq \frac{1}{r} + r\sum_{\ell=1}^{\infty} a_\ell. \tag{13}$$

Since $\ell \geq 1,\ (2\varrho+\vartheta+1)\phi_1(\theta,\mu) \leq [1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)$, using Theorem 5, we have

$$\begin{aligned}(2\varrho+\vartheta+1)\phi_1(\theta,\mu)\sum_{\ell=1}^{\infty} a_\ell &\leq \sum_{\ell=1}^{\infty}[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)\\ &\leq (1-\vartheta)(1-2\wp)\\ \Rightarrow \sum_{\ell=1}^{\infty} a_\ell &\leq \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}.\end{aligned}$$

Using the above inequality in (13), we have

$$\begin{aligned}|\aleph(\hbar)| &\leq \frac{1}{r} + \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)} r\\ \text{and } |\aleph(\hbar)| &\geq \frac{1}{r} - \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)} r.\end{aligned}$$

The estimate is sharp for the function $\aleph(\hbar) = \frac{1}{\hbar} + \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}\hbar$. □

We omit the proof of the following corollary since it is similar to that of Theorem 8.

Corollary 9. *If* $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ *then*

$$\frac{1}{r^2} - \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}\ \leq |\aleph'(\hbar)| \leq \frac{1}{r^2} + \frac{((1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}.$$

The estimate is sharp for the function given by (12).

4. Closure theorems

Let the function $\aleph_j$ be defined, for $j = 1, 2, \cdots, m$, by

$$\aleph_j(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} a_{\ell,j}\hbar^{\ell}, a_{\ell,j} \geq 0. \tag{14}$$

Theorem 10. *Let the functions $\aleph_j, j = 1, 2, \cdots, m$ defined by (14) be in the class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. Then the function h defined by*

$$h(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} a_{\ell,j} \right) \hbar^{\ell} \tag{15}$$

also belongs to the class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$.

Proof. Since $\aleph_j, j = 1, 2, \cdots, m$ are in the class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$, it follows from Theorem 5, that

$$\sum_{\ell=1}^{\infty} [1 + \wp(\ell - 1)][\ell(1 + \varrho) + (\varrho + \vartheta)]\phi_{\ell}(\theta, \mu)a_{\ell,j} \leq (1 - \vartheta)(1 - 2\wp),$$

for every $j = 1, 2, \cdots, m$. Hence

$$\begin{aligned} &\sum_{\ell=1}^{\infty} [1 + \wp(\ell - 1)][\ell(1 + \varrho) + (\varrho + \vartheta)]\phi_{\ell}(\theta, \mu) \left(\frac{1}{m} \sum_{j=1}^{m} a_{\ell,j} \right) \\ &= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{\ell=1}^{\infty} [1 + \wp(\ell - 1)][\ell(1 + \varrho) + (\varrho + \vartheta)]\phi_{\ell}(\theta, \mu)a_{\ell,j} \right) \\ &\leq (1 - \vartheta)(1 - 2\wp). \end{aligned}$$

From Theorem (6), it follows that $h \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$.

Hence the proof. □

Theorem 11. *The class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ is closed under convex linear combinations.*

Proof. Let the functions $\aleph_j, j = 1, 2, \cdots, m$ defined by (14) be in the class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. Then one need only show that function

$$h(\hbar) = \varsigma\aleph_1(\hbar) + (1 - \varsigma)\aleph_2(\hbar), \ 0 \leq \varsigma \leq 1 \tag{16}$$

is in the class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. Since for $0 \leq \varsigma \leq 1$,

$$h(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} [\varsigma a_{\ell,1} + (1 - \varsigma)a_{\ell,1}]\hbar^{\ell}, \tag{17}$$

with the assistance of the Theorem5, we have

$$\begin{aligned}&\sum_{\ell=1}^{\infty}[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)[\varsigma a_{\ell,1}+(1-\varsigma)a_{\ell,1}]\\ \leq&\varsigma(1-\vartheta)(1-2\wp)+(1-\varsigma)(1-\vartheta)(1-2\wp)\\ =&(1-\vartheta)(1-2\wp),\end{aligned}$$

which implies that $h \in \Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$. □

Theorem 12. *Let $\xi \geq 0$. Then $\Sigma^{*\xi}(\vartheta,\varrho,\wp,\theta,\mu) \subseteq N(\varrho,\xi)$, where*

$$\xi = 1 - \frac{2(1-\vartheta)(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)+(1-\vartheta)(1-2\wp)}. \tag{18}$$

Proof. If $\aleph \in \Sigma^{*\xi}(\vartheta,\varrho,\wp,\theta,\mu)$ then

$$\sum_{\ell=1}^{\infty}\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}a_\ell \leq 1.$$

We need to find the value of ξ such that

$$\sum_{\ell=1}^{\infty}\frac{[\ell(1+\varrho)+(\varrho+\xi)]\phi_\ell(\theta,\mu)}{1-\xi}a_\ell \leq 1.$$

Thus it is sufficient to show that

$$\frac{[\ell(1+\varrho)+(\varrho+\xi)]\phi_\ell(\theta,\mu)}{1-\xi} \leq \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}.$$

Then

$$\xi \leq 1 - \frac{(\ell+1)(1-\vartheta)(1-2\wp)(1+\varrho)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]+(1-\vartheta)(1-2\wp)}.$$

Since

$$G(\ell) = 1 - \frac{(\ell+1)(1-\vartheta)(1-2\wp)(1+\varrho)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]+(1-\vartheta)(1-2\wp)}$$

is an increasing function of $\ell,\ \ell \geq 1$, we obtain

$$\xi \leq G(1) = 1 - \frac{2(1-\vartheta)(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)+(1-\vartheta)(1-2\wp)}.$$

□

Theorem 13. *Let $\aleph_0(\hbar) = \frac{1}{\hbar}$ and*

$$\aleph_\ell(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty}\frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}\hbar^\ell,\ \ell \geq 1. \tag{19}$$

Then $\aleph$ is in the class $\Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$ iff can be expressed in the form

$$\aleph(\hbar) = \sum_{\ell=0}^{\infty}\omega_\ell\aleph_\ell(\hbar), \tag{20}$$

where $\omega_\ell \geq 0$ *and* $\sum_{\ell=0}^{\infty} \omega_\ell = 1$.

Proof. Assume that

$$\begin{aligned}\aleph(\hbar) &= \sum_{\ell=0}^{\infty} \omega_\ell \aleph_\ell(\hbar) \\ &= \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} \frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)} \hbar^\ell.\end{aligned}$$

Then it follows that

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} \frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)} \hbar^\ell$$
$$= \sum_{\ell=1}^{\infty} \omega_\ell = 1 - \omega_0 \leq 1$$

which implies that $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$.

On the other side, assume that the function $\aleph$ defined by (1) be in the class $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. Then

$$a_\ell \leq \frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}.$$

Setting

$$\omega_\ell = \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} a_\ell,$$

where

$$\omega_0 = 1 - \sum_{\ell=0}^{\infty} \omega_\ell,$$

$\aleph$ can be expressed in the form (20), as can be shown. □

Corollary 14. *The extreme points of the class* $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ *are the functions* $\aleph_0(\hbar) = \frac{1}{\hbar}$ *and*

$$\aleph_\ell(\hbar) = \frac{1}{\hbar} + \frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)} \hbar^\ell. \tag{21}$$

5. Modified Hadamard products

Let the functions $\aleph_j (j = 1, 2)$ defined by (14). The modified Hadamard product of $\aleph_1$ and $\aleph_2$ is defined by

$$(\aleph_1 * \aleph_2)(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} a_{\ell,1} a_{\ell,2} \hbar^\ell = (\aleph_2 * \aleph_1)(\hbar). \tag{22}$$

Theorem 15. *Let the function* $\aleph_j (j = 1, 2)$ *defined by* (14) *be in the class* $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. *Then* $\aleph_1 * \aleph_2 \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$, *where*

$$\varphi = 1 - \frac{2(1-\vartheta)^2(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)^2\phi_1(\theta,\mu) + (1-\vartheta)^2(1-2\wp)}. \tag{23}$$

The estimate is sharp for the functions $\aleph_j (j = 1, 2)$ *given by*

$$\aleph_j(\hbar) = \frac{1}{\hbar} + \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}\hbar, \ (j = 1, 2). \tag{24}$$

Proof. Using the same method that Schild and Silverman [12] used earlier, we need to find the largest real parameter φ such that

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\varphi)]\phi_\ell(\theta,\mu)}{(1-\varphi)(1-2\wp)} a_{\ell,1}a_{\ell,2} \le 1. \tag{25}$$

Since $\aleph_j \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu), \ j = 1, 2$, we readily see that

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} a_{\ell,1} \le 1$$

and

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} a_{\ell,2} \le 1.$$

By Cauchy- Schwarz inequality, we have

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} \sqrt{a_{\ell,1}a_{\ell,2}} \le 1. \tag{26}$$

Then merely demonstrating that is necessary

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\varphi)]\phi_\ell(\theta,\mu)}{(1-\varphi)(1-2\wp)} a_{\ell,1}a_{\ell,2}$$
$$\le \sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} \sqrt{a_{\ell,1}a_{\ell,2}}$$

or equivalently that

$$\sqrt{a_{\ell,1}a_{\ell,2}} \le \frac{[\ell(1+\varrho)+(\varrho+\vartheta](1-\varphi)}{[\ell(1+\varrho)+(\varrho+\varphi](1-\vartheta)}.$$

Hence, it light of the inequality (26), then merely demonstrating that is necessary

$$\frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]\phi_\ell(\theta,\mu)} \le \frac{[\ell(1+\varrho)+(\varrho+\vartheta](1-\varphi)}{[\ell(1+\varrho)+(\varrho+\varphi](1-\vartheta)}. \tag{27}$$

It follows from (27) that

$$\varphi \le 1 - \frac{(1-\vartheta)^2(1-2\wp)(1+\varrho)(\ell+1)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]^2\phi_\ell(\theta,\mu) + (1-\vartheta)^2(1-2\wp)}.$$

Now defining the function $E(\ell)$,

$$E(\ell) = 1 - \frac{(1-\vartheta)^2(1-2\wp)(1+\varrho)(\ell+1)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]^2\phi_\ell(\theta,\mu)+(1-\vartheta)^2(1-2\wp)}.$$

We see that $E(\ell)$ is an increasing of $\ell,\ \ell \geq 1$. Therefore, we conclude that

$$\varphi \leq E(\ell) = 1 - \frac{2(1-\vartheta)^2(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)^2\phi_1(\theta,\mu)+(1-\vartheta)^2(1-2\wp)},$$

Hence the proof. □

The following theorem is obtained using arguments close to those used in the proof of 15,

Theorem 16. *Let the function $\aleph_1$ defined by (14) be in the class $\Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$. Suppose also that the function $\aleph_2$ defined by (14) be in the class $\Sigma^*(\rho,\vartheta,\varrho,\wp,\theta,\mu)$. Then $\aleph_1 * \aleph_2 \in \Sigma^*(\zeta,\vartheta,\varrho,\wp,\theta,\mu)$, where*

$$\zeta = 1 - \frac{2(1-\vartheta)(1-\rho)(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)(2\varrho+\rho+1)\phi_1(\theta,\mu)+(1-\vartheta)(1-\rho)(1-2\wp)}. \tag{28}$$

The estimate is sharp for the functions $\aleph_j (j=1,2)$ given by

$$\aleph_1(\hbar) = \frac{1}{\hbar} + \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}\hbar$$

and

$$\aleph_2(\hbar) = \frac{1}{\hbar} + \frac{(1-\rho)(1-2\wp)}{(2\varrho+\rho+1)\phi_1(\theta,\mu)}\hbar.$$

Theorem 17. *Let the function $\aleph_j (j=1,2)$ defined by (14) be in the class $\Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$. Then the function*

$$h(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty}(a_{\ell,1}^2 + a_{\ell,2}^2)\hbar^\ell \tag{29}$$

belongs to the class $\Sigma^*(\varepsilon,\vartheta,\varrho,\wp,\theta,\mu)$, where

$$\varepsilon = 1 - \frac{4(1-\vartheta)^2(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)^2\phi_1(\theta,\mu)+2(1-\vartheta)^2(1-2\wp)}. \tag{30}$$

The estimate is sharp for the functions $\aleph_j (j=1,2)$ given by (24).

Proof. By using Theorem 5, we obtain

$$\begin{aligned}&\sum_{\ell=1}^{\infty}\left\{\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}\right\}^2 a_{\ell,1}^2\\ \leq&\sum_{\ell=1}^{\infty}\left\{\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}a_{\ell,1}\right\}^2 \leq 1\end{aligned} \tag{31}$$

and

$$\sum_{\ell=1}^{\infty}\left\{\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}\right\}^2 a_{\ell,2}^2$$
$$\leq\sum_{\ell=1}^{\infty}\left\{\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}a_{\ell,2}\right\}^2\leq 1. \tag{32}$$

It follows from (31) and (32) that

$$\sum_{\ell=1}^{\infty}\frac{1}{2}\left\{\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}\right\}^2(a_{\ell,1}^2+a_{\ell,2}^2)\leq 1.$$

Therefore, we need to find the largest ε such that

$$\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\varepsilon)]\phi_\ell(\theta,\mu)}{(1-\varepsilon)(1-2\wp)}$$
$$\leq\frac{1}{2}\left\{\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}\right\}^2,$$

that is

$$\varepsilon\ \leq 1-\frac{2(1-\vartheta)^2(1-2\wp)(1+\varrho)(\ell+1)}{1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]^2\phi_\ell(\theta,\mu)+2(1-\vartheta)^2(1-2\wp)}.$$

Since

$$G(\ell)=\ 1-\frac{2(1-\vartheta)^2(1-2\wp)(1+\varrho)(\ell+1)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]^2\phi_\ell(\theta,\mu)+2(1-\vartheta)^2(1-2\wp)}$$

is an increasing function of $\ell,\ \ell\geq 1$, we obtain

$$\varepsilon\leq G(1)=\frac{4(1-\vartheta)^2(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)^2\phi_1(\theta,\mu)+2(1-\vartheta)^2(1-2\wp)}$$

and hence the proof. □

6. Integral operators

Theorem 18. *Let the functions* $\aleph$ *given by* (1) *be in the class* $\Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$. *Then the integral operator*

$$F(\hbar)=c\int_0^1 u^c\aleph(u\hbar)du,\ 0<u\leq 1,\ c>0 \tag{33}$$

is in the class $\Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$, *where*

$$\xi=1-\frac{2c(1-\vartheta)(1+\varrho)}{(c+2)(2\varrho+\vartheta+1)+c(1-\vartheta)}. \tag{34}$$

The estimate is sharp for the function $\aleph$ *given by* (12).

Proof. Let $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. Then

$$\begin{aligned} F(\hbar) =& c\int_0^1 u^c \aleph(u\hbar)du \\ =& \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} \frac{c}{\ell + c + 1} a_\ell \hbar^\ell. \end{aligned}$$

Thus it is enough to show that

$$\sum_{\ell=1}^{\infty} \frac{c[1 + \wp(\ell - 1)][\ell(1 + \varrho) + (\varrho + \xi)]\phi_\ell(\theta, \mu)}{(\ell + c + 1)(1 - \xi)(1 - 2\wp)} a_\ell \leq 1. \tag{35}$$

Since $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$, then

$$\sum_{\ell=1}^{\infty} \frac{[1 + \wp(\ell - 1)][\ell(1 + \varrho) + (\varrho + \vartheta)]\phi_\ell(\theta, \mu)}{(1 - \vartheta)(1 - 2\wp)} a_\ell \leq 1. \tag{36}$$

From (35) and (36), we have

$$\frac{[\ell(1 + \varrho) + (\varrho + \xi)]}{(\ell + c + 1)(1 - \xi)} \leq \frac{[\ell(1 + \varrho) + (\varrho + \vartheta)]}{(1 - \vartheta)}.$$

Then

$$\xi \leq 1 - \frac{c(1 - \vartheta)(\ell + 1)(1 + \varrho)}{(\ell + c + 1)[\ell(1 + \varrho) + (\varrho + \vartheta)] + c(1 - \vartheta)}.$$

Since

$$Y(\ell) = 1 - \frac{c(1 - \vartheta)(\ell + 1)(1 + \varrho)}{(\ell + c + 1)[\ell(1 + \varrho) + (\varrho + \vartheta)] + c(1 - \vartheta)}$$

is an increasing function of $\ell,\ \ell \geq 1$, we obtain

$$\xi \leq Y(1) = 1 - \frac{2c(1 - \vartheta)(1 + \varrho)}{(c + 2)(2\varrho + \vartheta + 1) + c(1 - \vartheta)}$$

and hence the proof. □

7. Conclusion

This research has introduced a new subclass of meromorphic functions defined by Rapid operator and studied some basic properties of geometric function theory. Accordingly, some results to coefficient estimates, distortion properties, closure theorems, hadamard product and integral transforms have been considered, inviting further research for this field of study.

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