SKEW SEMI-INVARIANT SUBMANIFOLDS OF GENERALIZED KENMOTSU MANIFOLDS

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Abstract

In this paper, we study a special submanifold of a generalized Kenmotsu manifold. Firstly, we define a skew semi-invariant submanifold of a generalized Kenmotsu manifold and give an example. Later, we obtain some basic results for such submanifolds. Finally, we investigate the geometry of distributions.

Keywords: Semi-invariant submanifold, skew semi-invariant submanifold, generalized Kenmotsu manifold

1. Introduction

In 1972 K.Kenmotsu [1] defined a new type of contact manifolds. This type of manifolds known as Kenmotsu manifolds and they are an important class of contact manifolds. There are many geometric properties different from Sasakian manifolds. For example, although a Kenmotsu manifold is normal, it is not Sasakian. Also, a Kenmotsu manifold is not compact and it has negative sectional curvature. For a cemperensive introduction to Kemotsu manifolds, we refer to reader [2]. M. Falcitelli and A. M. Pastore [3] introduced Kenmotsu $f.pk$ -manifold which has an f -structure. In [4], Turgut Vanlı and Sarı defined and studied generalized Kenmotsu manifold (GK-manifold). GK-manifolds have been studied in [4-8].

Submanifold theory is a major notion in the contact geometry. We classify submanifolds of contact manifolds such as submanifolds of complex manifolds. One kind of submanifolds is CR-submanifold which is defined and studied by A. Bejancu [9] for a Kaehlerian manifold. By the following works, this subject has been carried to contact manifolds. Some different classes of submanifolds which have been defined for almost contact structures, are semi-invariant, invariant, anti-invariant etc. Almost semi-invariant submanifolds of a Sasakian manifold was studied by Bejancu and Papaghuic [10]. They also gave a classification for submanifold classes of a Sasakian manifold. Except for these submanifold classes also we have slant submanifolds

of complex and contact manifolds. Ronsse [11] defined the notion of skew CR-submanifolds of Kaehler manifolds. Also, we have the notion skew semi-invariant submanifolds for contact manifolds. Many researchers studied on this subject in [12-15].

In this study, we work on skew semi-invariant submanifolds of GK-manifolds. After present fundamental facts on GK-manifolds and submanifold theory, we define skew semi-invariant submanifolds of GK-manifolds. Then, we obtain some basic properties of such manifolds. Finally, we examine the geometry of distributions.

2. Preliminaries

In this section we present some fundamental facts on GK-manifolds. For detail we refer to reader [4]. Also, we give basic tools of the submanifold theory.

Definition 2.1 A $(2n+s)-$dimensional differentiable manifold $\bar{M}$ is called metric $f-$manifold if there exist an $(1,1)$ type tensor field φ, s vector fields $\xi_1,...,\xi_s$, s 1-forms $\eta^1,...,\eta^s$ and a Riemannian metric g on $\bar{M}$ such that

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i \tag{2.1}$$

$$\eta^i(\xi_j) = \delta_{ij}, \varphi\xi_i = 0, \eta^i \circ \varphi = 0 \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y) \tag{2.3}$$

$$\eta^i(X) = g(X,\xi_i),\ g(X,\varphi Y) = -g(\varphi X,Y), \tag{2.4}$$

for any $X,Y \in \Gamma(T\bar{M})$, $i,j \in \{1,...,s\}$ [16].

Second fundamental 2-form Φ is defined by $\Phi(X,Y) = g(X,\varphi Y)$, for any $X,Y \in \Gamma(T\bar{M})$. If we denote $\mathcal{L} = \{X : \eta^\alpha(X) = 0, 1 \le \alpha \le s\}$ and $\mathcal{M} = sp\{\xi_1,\xi_2,...,\xi_s\}$ then we have following direct sum : $T\bar{M} = \mathcal{L} \oplus \mathcal{M}$.

A metric f-manifold is *normal* if $[\varphi,\varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0$, where $[\varphi,\varphi]$ is the Nijenhuis tensor of φ. For a 2-form Φ on $\bar{M}$ such that $\eta^1 \wedge ... \wedge \eta^s \wedge \Phi^n \ne 0$, M is called an *almost s-contact metric manifold*. If M is also normal, then it is called an *s-contact metric manifold*.

Definition 2.2 Let $\bar{M}$ be an almost $s-$contact metric manifold of dimension $(2n+s)$, $s \ge 1$, with $(\varphi, \xi_i, \eta^i, g)$. $\bar{M}$ is said to be a generalized almost Kenmotsu manifold if for all $1 \le i \le s$, 1-forms η^i are closed and $d\Phi = 2\sum_{i=1}^{s} \eta^i \wedge \Phi$. A normal generalized almost Kenmotsu manifold $\bar{M}$ is called a generalized Kenmotsu manifold (GK-manifold) [4].

By the following theorem we characterize the s-contact metric manifolds as GK-manifold.

Theorem 2.3 An almost s-contact metric manifold $\left(\bar{M}, \varphi, \xi_i, \eta^i, g\right)$ is a GK- manifold if and only if

$$\left(\bar{\nabla}_X \varphi\right) Y = \sum_{i=1}^{s} \left\{ g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X \right\} \tag{2.5}$$

for all $X, Y \in \Gamma(T\bar{M})$, $i \in \{1, 2, ..., s\}$, where $\bar{\nabla}$ is Riemannian connection on M [4].

Let $\bar{M}$ be $(2n+s)$-dimensional a GK- manifold with structure $\left(\varphi, \xi_i, \eta^i, g\right)$. Then we have $\bar{\nabla}_X \xi_i = -\varphi^2 X$ for all $X \in \Gamma(T\bar{M})$ [4].

Let M be an m-dimensional submanifold isometrically immersed in a GK-manifold $\bar{M}$. Denote by TM and $TM^\perp$ the tangent bundle and the normal bundle of M, respectively. Suppose that the structure vectors field ξ_i be tangent to the submanifold M, and let denote $\{\bar{\xi}\}$ as s-dimensional distribution spanned by $\{\xi_1, ..., \xi_s\}$ on M and $\{\bar{\xi}\}^\perp$ be orthogonal complementary of $\{\bar{\xi}\}$ in TM.

For any $X \in \Gamma(TM)$ we have $g(\varphi X, \xi_i) = 0$. Then, we put

$$\varphi X = bX + cX \tag{2.6}$$

for $bX \in \Gamma(TM)$ and $cX \in \Gamma(TM^\perp)$, where b is an endomorphism of the tangent bundle TM and c is a normal bundle valued 1-form on M. On the other hand, for each $x \in M$ we define the following subspaces of TM.

$$D_x = \{X_x \in \{\bar{\xi}\}_x^\perp : c(X_x) = 0\}$$

and

$$D_x^\perp = \{X_x \in \{\bar{\xi}\}_x^\perp : b(X_x) = 0\}.$$

We note that D_x and $D_x^\perp$ are two orthogonal subspaces of the tangent space $T_x M$ [10]. In fact, using (2.1), (2.4) and (2.6), we have $g(X, Y) = g(\varphi X, \varphi Y) = g(bX, cY) = 0$ for any $X_x \in D_x$ and $Y_x \in D_x^\perp$.

Definition 2.4 A submanifold M of the GK-manifold $\bar{M}$ is said to be a skew semi-invariant submanifold if its tangent bundle TM has the decomposition $TM = D \oplus D^\perp \oplus \tilde{D} \oplus \{\bar{\xi}\}$ such as

   i. $\{\bar{\xi}\}$ is distribution spanned on M by the vector field $\{\xi_1, ..., \xi_s\}$
   ii. D is an invariant distribution on M, that is $\varphi(D_x) = D_x$ for each $x \in M$
   iii. $D^\perp$ is an anti invariant distribution on M, that is, $\varphi(D_x^\perp) \subset T_x M^\perp$, for each $x \in M$
   iv. D is neither an invariant nor an anti invariant distribution on M that is $bX_x \neq 0$ and $cX_x \neq 0$ for any $x \in M$ and $X_x \in D_x$.

From the definition, we have following classification:

A skew semi-invariant submanifold M in a GK-manifold $\bar{M}$ is said to be,

1. a semi-invariant submanifold if we have $D = \{0\}$

2. a semi-slant submanifold if we have $D^\perp = \{0\}$

3. a hemi-slant submanifold if we have $D = \{0\}$

4. an invariant submanifold if we have $D^\perp = \{0\}$ and $D = \{0\}$

5. an anti invariant submanifold if we have $D = \{0\}$ and $D = \{0\}$

6. a slant-submanifold if we have $D = \{0\}$ and $D^\perp = \{0\}$.

In the following example we give a skew semi-invariant submanifold of a canonical GK-manifold's example $\mathbb{R}^{2n+s}$.

Example 2.5 Let take a usual GK-structure $(\varphi, \eta^j, \xi_j, g)$ on $\mathbb{R}^{2n+s}$ such as

$$\eta^j = dz_j, \ \xi_i = \frac{\partial}{\partial z_i},$$

$$\varphi(\sum_{i=1}^{n}(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + \sum_{j=1}^{s} Z_j \frac{\partial}{\partial z_j}) = \sum_{i=1}^{n}(Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i}) + \sum_{i=1}^{n}\sum_{j=1}^{s} Y_i \frac{\partial}{\partial z_j}$$

$$g = e^{-2z}(\sum_{i=1}^{n} dx_i \otimes dx_i + dy_i \otimes dy_i) - \sum_{j=1}^{s}\eta^j \otimes \eta^j$$

$x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_s$ denoting the cartesian coordinates on $\mathbb{R}^{2n+1}$.

Let consider a submanifold of $\mathbb{R}^{10}$ defined by

$$M = X(u, v, k, l, s, w, t_1, t_2) = (u, 0, k, s, v, l, \cos w, \sin w, t_1, t_2).$$

Then the local frame of TM is given by

$$e_1 = \frac{\partial}{\partial x_1}, e_2 = \frac{\partial}{\partial y_1}, e_3 = \frac{\partial}{\partial x_3}, e_4 = \frac{\partial}{\partial y_2}, e_5 = \frac{\partial}{\partial x_4},$$

$$e_6 = -\sin w \frac{\partial}{\partial y_3} + \cos w \frac{\partial}{\partial y_4}, e_7 = \frac{\partial}{\partial z_1} = \xi_1, e_8 = \frac{\partial}{\partial z_2} = \xi_2$$

and $e_1^* = \frac{\partial}{\partial x_2}, e_2^* = \frac{\partial}{\partial y_3}$.

Thus, we determine $D = sp\{e_1, e_2\}$, $D^\perp = sp\{e_3, e_4\}$, and $D = sp\{e_5, e_6\}$. Then D, $D^\perp$ and D become invariant, anti-invariant and neither an invariant nor an anti-invariant distribution, respectively. Thus $TM = D \oplus D^\perp \oplus D \oplus sp\{\xi_1, \xi_2\}$ is a skew semi-invariant submanifold of $\mathbb{R}^{10}$

3. Basic properties of skew semi-invariant submanifolds of GK-manifold

Let M be a skew semi-invariant submanifold in a GK-manifold $\bar{M}$. The projection morphisms of M to distributions $D, D^\perp$ and D are denoted by P, Q and L, respectively. Then for each $X \in \Gamma(TM)$ we can write

$$X = PX + QX + LX + \eta^i(X)\xi_i \tag{3.1}$$

For non-zero vector field $X \in \Gamma(D)$ we note that $bX \neq 0$ and $cX \neq 0$. Thus c defines a vector subbundle $cD : x \to cD_x$ of $TM^\perp$. Also, we have $g(\varphi D^\perp, cD) = 0$. For any $N \in \Gamma(TM^\perp)$ we put

$$\varphi N = tN + fN \tag{3.2}$$

where tN and fN are the tangential and normal components of φN, respectively. Next, we denote by μ the orthogonal complementary vector bundle to $\varphi D^\perp \oplus cD$ in $TM^\perp$. Thus we have that the normal bundle to M has the decomposition $TM^\perp = \varphi D^\perp \oplus cD \oplus \mu$. The Gauss equation is given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{3.3}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM)^\perp$, where $\bar{\nabla}$ and ∇ Levi-Civita connection on M and induced connection on M, respectively.
The Weingarten equation is given by

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N \tag{3.4}$$

for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, where $\nabla^\perp$ is the connection in the normal bundle, h is the second fundamental from of M. The second fundamental form h and the shape operator A have a relation as $g(h(X,Y),V) = g(A_V X, Y)$. By the following lemmas we obtain some basic relations on components of vector fields on M. The proofs could be obtained from Gauss and Weingarten formulas with using (3.1).

Lemma 3.1 Let M be a skew semi-invariant submanifold of a GK-manifold $\bar{M}$. Then;

1. The decompositions of the vector fields on M is given by

    $$P\nabla_X bY = \varphi P(\nabla_X Y) + PA_{cY} X - \eta(Y)PX,$$
    $$Q\nabla_X bY = QA_{cY} X + Qth(X,Y) - \eta(Y)QX,$$
    $$L\nabla_X bY = bL(\nabla_X Y) + LA_{cY} X + Lth(X,Y) - \eta(Y)LX,$$
    $$\eta(\nabla_X bY) = \eta(A_{cY} X) + g(\varphi X, \varphi Y),$$
    $$h(X,bY) + \nabla_X^\perp cY = c\nabla_X Y + fh(X,Y),$$
    for all $X,Y \in \Gamma(TM)$.

2. The covariant derivation of vector fields in distributions are given by

    $$\nabla_X \xi_i = PX \text{ and } h(X,\xi_i) = 0, \text{ for any } X \in \Gamma(D), \tag{3.5}$$
    $$\nabla_Y \xi_i = 0 \text{ and } h(Y,\xi_i) = -\varphi^2 Y, \text{ for any } Y \in \Gamma(D^\perp), \tag{3.6}$$
    $$\nabla_Z \xi_i = 0 \text{ and } h(Z,\xi_i) = -\varphi cLZ, \text{ for any } Z \in \Gamma(D) \tag{3.7}$$

$$\nabla_{\xi_i}\xi_j = 0 \text{ and } h(\xi_i,\xi_j) = 0. \tag{3.8}$$

3. The covariant derivation of vector fields on the direction of ξ_i are given by

$$\nabla_{\xi_i} U \in \Gamma(D) \text{ for any } U \in \Gamma(D),$$
$$\nabla_{\xi_i} V \in \Gamma(D^\perp) \text{ for any } V \in \Gamma(D^\perp),$$
$$\nabla_{\xi_i} W \in \Gamma(D) \text{ for any } W \in \Gamma(D).$$

4. The Lie derivations of any vector fields are given by

$$[X,\xi_i] \in \Gamma(D) \text{ for any } X \in \Gamma(D),$$
$$[Y,\xi_i] \in \Gamma(D) \text{ for any } Y \in \Gamma(D^\perp),$$
$$[Z,\xi_i] \in \Gamma(D) \text{ for any } Z \in \Gamma(D).$$

4. The geometric properties of distributions

In this section, we examine that the distributions $D, D^\perp, D, D \oplus \{\bar{\xi}\}$ are involutive or not.

Theorem 4.1 Let M be a skew semi-invariant submanifold of a GK-manifold $\bar{M}$. Then the distribution D is not involutive.

Proof. For all $X,Y \in \Gamma(D)$, we have $g(X,\xi_i) = 0$. Then, we get $g(\nabla_Y X, \xi_i) = g(\nabla_Y \xi_i, X)$. Thus, we obtain $g([X,Y],\xi_i) = g(\nabla_X \xi_i, Y) - g(\nabla_Y \xi_i, X)$. By using (3.5), we get $g([X,Y],\xi_i) = 2g(X,PY)$ which completes the proof.

Theorem 4.2 Let M be a skew semi-invariant submanifold of a GK-manifold $\bar{M}$. Then the distribution $D^\perp$ is always involutive.

Proof. For all $U,V \in \Gamma(D^\perp)$ and by using (3.6) we obtain $g([U,V],\xi_i) = g(\nabla_U \xi_i, V) - g(\nabla_V \xi_i, U) = 0$. This shows us $D^\perp$ is involutive.

Theorem 4.3 Let M be a skew semi-invariant submanifold of a GK-manifold $\bar{M}$. Then, the distribution D is always involutive.

Proof. For all $K,W \in \Gamma(D)$, by using (3.7) we have $g([K,W],\xi_i) = g(\varphi LK, bLW) - g(\varphi LW, bLK)$. It is seen from here that D is involutive.

Theorem 4.4 Let M be a skew semi-invariant submanifold of a GK-manifold $\bar{M}$. The distribution $D \oplus \bar{\xi}$ is involutive if and only if $h(U,\varphi V) = h(U,\varphi V)$ for all $U,V \in \Gamma(D)$.

Proof. For all $U,V \in \Gamma(D \oplus \bar{\xi})$, we get $\varphi([U,V]) = \bar{\nabla}_U \varphi V - (\bar{\nabla}_U \varphi)V - \bar{\nabla}_V \varphi U + (\bar{\nabla}_V \varphi)U$. Then, using (2.5) and (3.3) we have

$$\varphi([U,V]) = \nabla_U \varphi V + h(U, \varphi V) - \sum_{i=1}^{s} \left\{ g(\varphi U, V)\xi_i - \eta^i(U)\varphi V \right\}$$

$$- \nabla_V \varphi U - h(V, \varphi U) + \sum_{i=1}^{s} \left\{ g(\varphi V, U)\xi_i - \eta^i(V)\varphi U \right\}.$$

Then we obtain that $[U,V] \in \Gamma(D \oplus \overline{\xi})$ if and only if $h(U, \varphi V) = h(V, \varphi U)$ where $\varphi([U,V])$ is the component of $\nabla_U V$ from orthogonal complementary distribution of $D \oplus \overline{\xi}$.

Corollary 4.5 Let M be a skew semi-invariant submanifold of a GK-manifold $\overline{M}$. The distribution $D^\perp \oplus \xi$ is involutive if and only $A_{\varphi U}V = A_{\varphi V}U$ for $U, V \in \Gamma(D^\perp)$.

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