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Finite-Time Stability of Time-Delay Dynamical System for the Outbreak of COVID-19

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ABSTRACT: In this study, the finite-time stability of the time-delay system representing the COVID-19 outbreak is analyzed. The infection dynamics is stated with the new kernel function to express the distribution of exposed people in the model. A history-wise Lyapunov functional is used to show the finite-time stability of the proposed system. A condition in terms of linear matrix inequalities is given to ensure finite-time stability. With this condition, it is guaranteed that the norm of the variables which are infected, confirmed, isolated and cured/recovered people do not exceed a certain bound in a fixed finite time interval. The solution of the generalized minimum/maximum parameters is explained and a numerical example is demonstrated to show the validity of the proposed method.

Keywords *– Finite-time stability, time-delay systems, COVID-19, infection dynamics.*

1. Introduction

The novel coronavirus (COVID-19) was first identified in Wuhan, China on December 2019 (World Health Organization, 2020b) and has resulted in an ongoing pandemic. Due to its highly contagious behavior, World Health Organization (WHO) classified the current outbreak as a pandemic on 12 March 2020 (World Health Organization, 2020a, Situation Report - 52). Shortly after this declaration, the virus has spread rapidly and WHO announced Europe as the new epicenter of the pandemic on March 14, 2020 (World Health Organization, 2020a, Situation Report - 54). At the end of March, outbreak has reached to USA which has the highest number of confirmed cases in the world (World Health Organization, 2020a, Situation Report - 68). As of 26 May 2020, Brazil has the second-highest number of confirmed cases in the world behind the USA (World Health Organization, 2020a, Situation Report - 127).

Currently, the impact area of the coronavirus has expanded worldwide and according to recent statistics, nearly 66 million people have infected while over one and a half million people passed away due to this disease (World Health Organization, 2020a, Weekly Operational Update - 07 December 2020). Several precautions taken both by national and international organizations, including travel restrictions, quarantine etc., have resulted in a control of the disease during summer, however, as of the September 2020, the number of cases has started to increase again. It is crucial to analyze infection control, impact of prevention and duration of isolation parameters to slow down and finally prevent such rise.

WHO also has announced that the mathematical models of this outbreak play a decisive role for policy makers. The scholars proposed new models to simulate the sanitary phenomena: (Kermack and McKendrick, 1932), (Kermack and McKendrick, 1933), (Kermack and

McKendrick, 1937), (Kermack and McKendrick, 1939), (Kermack and McKendrick, 1991) and (Hethcote, 2000). In the recent outbreak of COVID-19, the latency period become prominent and the common models such as SIR, SEIR, SEIS or SEIJR fail to describe the current outbreak. Therefore, new models (Chen et al., 2020a) and (Chen et al., 2020b) are interpreted to describe the latent period as well as the isolation measures which plays a key role describing the infection dynamics of COVID-19.

Stability is one of the fundamental research topic on dynamical systems. Most of the studies related to the stability focus on asymptotic or exponential stability, which is defined over an infinite time interval. However, in many practical applications, finite-time (FT) stability of a system is the main concern, which means keeping the system behavior/state within the specified bounds in a fixed FT interval. By definition, FT stability and asymptotic stability demonstrate two different concepts. Asymptotically stable systems may not be FT stable or FT stable systems may not be asymptotically stable.

This paper gives insights to the behavior of COVID-19 in terms of FT stability. In contrast to the other stability definitions which consider the asymptotic behavior or saturating behavior of the infection dynamics, we analyze to determine whether the norm of the infected, confirmed, isolated and cured/recovered people exceeds a certain bound. Here are the contributional highlights of this study:

- In this paper, we propose a novel kernel function to describe the distribution of exposed people in the model.
- We also state a new condition in linear matrix inequalities to guarantee FT stability of the corresponding infection dynamics.
- Moreover, this condition is tested in a numerical example to show the validity.

The notation in this study is fairly standard. Throughout the paper, given $x \in \mathbb{R}^n$, $|x|$, denotes its Euclidean norm. Given $\phi \in X^n \coloneqq C([-\delta, 0], \mathbb{R}^n)$, $||\phi||_{[-\delta,0]} := \sup_{\tau \in [-\delta,0]} |\phi(\tau)|$ and ``*" denotes the term that is induced by symmetry.

The organization of the paper is as follows. Definitions and technical lemmas are given in Section 2. The system under study is given in Section 3. Section 4 presents the main result where in Theorem 1, the sufficient conditions to guarantee FT stability is expressed. Finally, a numerical example is presented to illustrate the effectiveness of the proposed method in Section 5.

2. Definitions and Technical Lemmas

Consider the nonlinear TDS

$$\dot{x}(t) = f(x_t), \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the is the current value of the solution and $x_t \in X^n$ is the state history defined with the maximum delay $\delta \geq 0$ as

$$x_t(s) := x(t+s), \qquad \forall\, s \in [-\delta, 0]. \tag{2.2}$$

Our result requires the definition of FTS as well as the use of partial upper/lower right Dini derivatives and total upper/lower right Dini derivatives for the functions of variables.

Definition 1. (Finite-Time Stability) Given three positive numbers ϵ, δ, T_f with $\delta \leq \epsilon$, the system (2.1) is said to be FTS with respect to (ϵ, δ, T_f), if for every trajectory $x(t)$, $||x_0||_{[-\delta,0]} < \delta$ implies $|x(t)| < \epsilon$, $\forall\, t \in [0, T_f]$.

Definition 2. (Partial Dini Derivatives) Let $I, J \subset \mathbb{R}$ denote intervals (possibly unbounded), and consider a continuous function $F : I \times J \to \mathbb{R}$. Then, the partial upper right Dini derivatives of F with respect to its first and second arguments are respectively defined as:

$$\begin{aligned} \mathcal{D}^{+}_{,t}[F(t,y)] &:= \limsup_{h\to 0^{+}} \frac{F(t+h,y)-F(t,y)}{h}, \\ \mathcal{D}^{+}_{,y}[F(t,y)] &:= \limsup_{h_y\to 0^{+}} \frac{F(t,y+h_y)-F(t,y)}{h_y}. \end{aligned} \tag{2.3}$$

Similarly, its partial lower right Dini derivatives of F with respect to its first and second arguments are respectively defined as:

$$\begin{aligned} \mathcal{D}_{+,t}[F(t,y)] &:= \liminf_{h\to 0^{+}} \frac{F(t+h,y)-F(t,y)}{h}, \\ \mathcal{D}_{+,y}[F(t,y)] &:= \liminf_{h_y\to 0^{+}} \frac{F(t,y+h_y)-F(t,y)}{h_y}, \end{aligned} \tag{2.4}$$

Definition 3. (Total Dini Derivatives) Let $I, J \subset \mathbb{R}$ denote intervals (possibly unbounded), and consider continuous functions $F : I \times J \to \mathbb{R}$ and $b : I \to J$. Then the total upper right Dini derivative of F along b is defined as:

$$D^{+}_{,t}[F(t,b(t))] := \limsup_{h\to 0^{+}} \frac{F(t+h,b(t+h))-F(t,b(t))}{h}, \tag{2.5}$$

Similarly, the total lower right Dini derivative of F along b is defined as:

$$D_{+,t}[F(t,b(t))] := \liminf_{h\to 0^{+}} \frac{F(t+h,b(t+h))-F(t,b(t))}{h}, \tag{2.6}$$

The proof of our main result relies on the following technical results. The first one is an extension of the chain rule for total upper right Dini derivative. Its proof is provided in Appendix.

Lemma 1. (Chain Rule for Upper Right Dini Derivative) Let $I, J \subset \mathbb{R}$ be intervals (possibly unbounded), $F : I \times J \to \mathbb{R}$ be a function having finite partial upper and lower right Dini derivatives with respect to its first and second arguments on I and J respectively, and $b : I \to J$ be a function having continuous non-negative derivative on I. Then, it holds that

$$D^{+}_{,t}\big[F\big(t,b(t)\big)\big] \leq \big[\mathcal{D}^{+}_{,t}\, F(t,y)\big]_{(t,b(t))} + \big[\mathcal{D}^{+}_{,y}\, F(t,y)\big]_{(t,b(t))} \cdot\, b'(t), \forall\, t \in I, \tag{2.7}$$

where $\left[\mathcal{D}^{+}_{,t} F(t,y)\right]_{(t,b(t))}$ denotes the partial upper right Dini derivative with respect to t evaluated at $(t, b(t))$, and $\left[\mathcal{D}^{+}_{,y} F(t,y)\right]_{(t,b(t))}$ is the partial upper right Dini derivative with respect to y evaluated at $(t, b(t))$.

A similar chain rule holds for partial lower right Dini derivative, as stated next.

Corollary 1. Under the assumptions of Lemma 1, it holds that

$$D^{+}_{,t}\left[F\big(t,b(t)\big)\right] \geq \left[\mathcal{D}_{+,t}\, F(t,y)\right]_{(t,b(t))} + \left[\mathcal{D}_{+,y}\, F(t,y)\right]_{(t,b(t))} \cdot\, b'(t), \tag{2.8}$$

for all $t \in I$. The proof of this result is omitted as it follows along the sames lines as that of Lemma 1 by noticing that (See for instance (Royden, 1988)).

$$D^{+}_{,t}\left[F\big(t,b(t)\big)\right] \geq D_{,t}[F(t,b(t)], \qquad \forall\, t \in I.$$

Based on these chain rules, we can extend the Leibniz integral rule to total upper right Dini derivative, as follows. Its proof is provided in Appendix.

Lemma 2. (Leibniz Rule for Upper Right Dini Derivative) Let $I, J \subset \mathbb{R}$ be intervals (possibly unbounded), $f : I \times J \to \mathbb{R}$ be a continuous function having finite partial upper right Dini derivative with respect to its first argument on I and $a, b : I \to J$ be functions having continuous non-negative derivatives on I. Then, it holds that

$$\begin{aligned} D^{+}_{,t}\left[\int_{a(t)}^{b(t)} f(t,\tau)d\tau\right] \leq\ & f\big(t,b(t)\big)b'(t) - f\big(t,a(t)\big)a'(t) \\ & + \int_{a(t)}^{b(t)} \left[\mathcal{D}^{+}_{,t}\, f(t,y)\right]_{(t,\tau)} d\tau, \end{aligned} \tag{2.9}$$

for all $t \in I$.

3. System Under Study

In this study, we consider the time-delay system for the COVID-19 outbreak (Chen et al., 2020a). Recalling that study, we have the four category of variables for the infection dynamics

- $I(t)$: cumulative infected people at time t,
- $J(t)$: cumulative confirmed people at time t,
- $G(t)$: currently isolated people who are infected but still in latent period at time t,
- $R(t)$: cumulative cured/recovered people at time t.

For an area with no inflow, the authors proposed the following infection dynamics

$$\begin{aligned} \frac{dI}{dt} &= \mathfrak{T}(t), \\ \frac{dJ}{dt} &= \gamma \int_0^t h_1(t-\delta_1, t')\, \mathfrak{T}(t')dt', \end{aligned} \tag{3.1}$$

$$\frac{dG}{dt} = \mathcal{G}(t) - \int_0^t h_2(t-\delta_1', t')\, G(t')dt',$$
$$\frac{dR}{dt} = \kappa \int_0^t h_3(t-\delta_1-\delta_2, t')\, \mathfrak{T}(t')dt',$$

where the increment of the number of infected people in this region is selected as

$$\mathfrak{T}(t) \coloneqq \beta(I(t) - J(t) - G(t)), \tag{3.2}$$

and the newly isolated and infected people in the region is determined as

$$\mathcal{G}(t) \coloneqq \ell(I(t) - J(t) - G(t)), \tag{3.3}$$

Let us recall that, the term $I(t) - J(t) - G(t)$ represents the number of exposed people at time t assuming that the individual would no longer transmit the coronavirus to others when he/she is isolated or in the treatment. For the dynamics above, the following constants are as follows:

- γ: morbidity rate,
- κ and $(1-\kappa)$: are the cured and death rates,
- β: spread rate,
- ℓ: isolation rate of the currently exposed people,
- δ_1: latent period,
- ${\delta_1}'$: confirm period ($\delta_1' < \delta_1$),
- δ_2: days in average for the confirmed people become cured with rate κ or dead with rate $(1-\kappa)$,
- $h_i(t_1, t_2)$, $i = 1, 2, 3$ are the kernel functions representing the normalized probability distributions between times t_1 and t_2.

Here, we consider the choice of kernel functions $h_i(t_1, t_2)$, $i = 1, 2, 3$. By practical reasons to modify the lower bounds of the integrals of (3.1), we take triangular type of kernel functions rather than a Gaussian one of the form

$$\begin{aligned} h_1(t_1, t_2) &:= \max\left(\frac{h_{1,max}}{\delta_1}|t_1 - t_2| - h_{1,max}, 0\right), \\ h_2(t_1, t_2) &\coloneqq \max\left(\frac{h_{2,max}}{{\delta_1}'}|t_1 - t_2| - h_{2,max}, 0\right), \\ h_3(t_1, t_2) &:= \max\left(\frac{h_{3,max}}{\delta_1 + \delta_2}|t_1 - t_2| - h_{3,max}, 0\right). \end{aligned} \tag{3.4}$$

Please note that, regardless of choosing these kernel functions either of triangular or Gaussian form, we have $h_i(t_1, t_2) \le h_{i,max}$ for some positive $h_{i,max}$, $i = 1, 2, 3$ which is attained in $[t_1, t_2]$. To normalize the probability distribution, we will use $h_{1,max} = \frac{2}{\delta_1}$, $h_{1,max} = \frac{2}{\delta_1'}$ and $h_{3,max} = \frac{2}{\delta_1+\delta_2}$ (for the normalization procedure, see (Chen et al., 2020a)). In this regard, the altered dynamics is presented as follows:

$$\frac{dI}{dt} = \mathfrak{T}(t), \tag{3.5}$$

$$\frac{dJ}{dt} = \gamma \int_{t-\delta_1}^{t} h_1(t-\delta_1, t')\,\mathfrak{T}(t')dt',$$
$$\frac{dG}{dt} = \mathcal{G}(t) - \int_{t-\delta_1'}^{t} h_2(t-\delta_1', t')\,G(t')dt',$$
$$\frac{dR}{dt} = \kappa \int_{t-\delta_1-\delta_2}^{t} h_3(t-\delta_1-\delta_2, t')\,\mathfrak{T}(t')dt',$$

where $\mathfrak{T}(t)$ and $\mathcal{G}(t)$ are defined as in (3.2) and (3.3), respectively.

4. Main Results

Now, we are ready to propose our main result.

Theorem 1. Consider the TDS to describe the outbreak of COVID-19 (3.5). Assume that there exists $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \alpha_1, \alpha_2, \alpha_3 > 0$ satisfying

$$\Xi \le 0, \tag{4.1}$$

$$\frac{\lambda_2 \gamma h_{1,max}^2}{\delta_1} \le \lambda_5 e^{\alpha_1 \delta_1}, \tag{4.2}$$

$$\frac{\lambda_3 \ell h_{2,max}^2}{\delta_1'} \le \lambda_6 e^{\alpha_2 \delta_1'}, \tag{4.3}$$

$$\frac{\lambda_4 \kappa h_{3,max}^2}{\delta_1 + \delta_2} \le \lambda_7 e^{\alpha_3(\delta_1+\delta_2)}, \tag{4.4}$$

$$\Lambda(\lambda_{max}, \lambda_5, \lambda_6, \lambda_7) \le \frac{\epsilon^2}{e^{\alpha T_f} \delta^2} \lambda_{min}, \tag{4.5}$$

where

$$\Xi := \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & 0 \\ * & \Xi_{22} & \Xi_{23} & 0 \\ * & * & \Xi_{33} & 0 \\ * & * & * & \Xi_{44} \end{bmatrix},$$
$\Xi_{11} = -\alpha\lambda_1 + 2\lambda_1\beta + \lambda_5\delta_1\beta^2 + \delta_7(\delta_1+\delta_2)\beta^2,$
$\Xi_{12} = -\lambda_1\beta - \lambda_5\delta_1\beta^2 - \delta_7(\delta_1+\delta_2)\beta^2,$
$\Xi_{13} = -\lambda_1\beta - \lambda_3\ell - \lambda_5\delta_1\beta^2 - \delta_7(\delta_1+\delta_2)\beta^2,$
$\Xi_{22} = -\alpha\lambda_2 + \lambda_2\gamma + \lambda_5\delta_1\beta^2 + \delta_7(\delta_1+\delta_2)\beta^2,$
$\Xi_{23} = -\lambda_3\ell + \lambda_5\delta_1\beta^2 + \lambda_7(\delta_1+\delta_2)\beta^2,$
$\Xi_{33} = -\alpha\lambda_3 + 3\lambda_3\ell + \lambda_5\delta_1\beta^2 + \lambda_6\delta_1' + \lambda_7(\delta_1+\delta_2)\beta^2,$
$\Xi_{44} = -\alpha\lambda_4 + \lambda_4\kappa,$
$\lambda_{min} = \inf_{i=\{1,2,3,4\}}\{\lambda_i\}$, $\lambda_{max} = \sup_{i=\{1,2,3,4\}}\{\lambda_i\},$
$\alpha = \max\{\alpha_1, \alpha_2, \alpha_3\},$
$$\Lambda(\lambda_{max}, \lambda_5, \lambda_6, \lambda_7) = \lambda_{max} + 3\lambda_5\delta_1^2 e^{\alpha_1\delta_1}\beta^2 + 3\lambda_6{\delta_1'}^2 e^{\alpha_2\delta_1'}\ell^2 + 3\lambda_7(\delta_1+\delta_2)^2 e^{\alpha_3(\delta_1+\delta_2)}\beta^2$$

Then, the TDS (3.5) is FTS with respect to (ϵ, δ, T_f).

Proof. Consider the LKF candidate for the TDS (3.5) defined as

$$V(\phi_I,\phi_J,\phi_G,\phi_R)=\sum_{i=1}^{4} V_i(\phi_I,\phi_J,\phi_G,\phi_R), \tag{4.6}$$

where

$$V_1(\phi_I,\phi_J,\phi_G,\phi_R)=\lambda_1\phi_I(0)^2+\lambda_2\,\phi_J(0)^2+\lambda_3\phi_G(0)^2+\lambda_4\phi_R(0)^2,$$

$$V_2(\phi_I,\phi_J,\phi_G,\phi_R)=\lambda_5\int_{-\delta_1}^{0}\int_{s}^{0} e^{-\alpha_1 t'}\,\phi_{\mathfrak{I}}(t')^2dt'ds\,,$$

$$V_3(\phi_I,\phi_J,\phi_G,\phi_R)=\lambda_6\int_{-\delta_1'}^{0}\int_{s}^{0} e^{-\alpha_2 t'}\,\phi_G(t')^2dt'ds\,, \tag{4.7}$$

$$V_4(\phi_I,\phi_J,\phi_G,\phi_R)=\lambda_7\int_{-\delta_1-\delta_2}^{0}\int_{s}^{0} e^{-\alpha_3 t'}\,\phi_{\mathfrak{I}}(t')^2dt'ds\,,$$

for all $\phi_I,\phi_J,\phi_G,\phi_R\in C([-\delta_1-\delta_2,0],\mathbb{R})$ and where $\phi_{\mathfrak{I}}\equiv\beta(\phi_I-\phi_J-\phi_G)$. By applying Lemma 2, we have

$$D_{,t}^{+}V_2(I_t,J_t,G_t,R_t)\le\alpha_1V_2(I_t,J_t,G_t,R_t)+\lambda_5\delta_1\mathfrak{I}(t)^2-\lambda_5e^{\alpha_1\delta_1}\int_{t-\delta_1}^{t}\mathfrak{I}(t')^2dt'\,,$$

$$D_{,t}^{+}V_3(I_t,J_t,G_t,R_t)\le\alpha_2V_3(I_t,J_t,G_t,R_t)+\lambda_6\delta_1'G(t)^2-\lambda_6e^{\alpha_2\delta_1'}\int_{t-\delta_1'}^{t}G(t')^2dt'\,, \tag{4.8}$$

$$D_{,t}^{+}V_4(I_t,J_t,G_t,R_t)\le\alpha_3V_4(I_t,J_t,G_t,R_t)+\lambda_7(\delta_1+\delta_2)\mathfrak{I}(t)^2$$
$$-\lambda_7e^{\alpha_3(\delta_1+\delta_2)}\int_{t-\delta_1-\delta_2}^{t}\mathfrak{I}(t')^2dt'.$$

The derivative of V along the solutions of (3.5) reads

$$\begin{aligned}D_{,t}^{+}V(I_t,J_t,G_t,R_t)\le{}&2\lambda_1I(t)\mathfrak{I}(t)+2\lambda_2\gamma J(t)\int_{t-\delta_1}^{t}h_1(t-\delta_1,t')\mathfrak{I}(t')^2dt'\\&+2\lambda_3G(t)\left[G(t)-\int_{t-\delta_1'}^{t}h_2(t-\delta_1',t')G(t')^2dt'\right]\\&+2\lambda_4\kappa R(t)\int_{t-\delta_1-\delta_2}^{t}h_3(t-\delta_1-\delta_2,t')\mathfrak{I}(t')^2dt'\\&+\alpha V_2(I_t,J_t,G_t,R_t)+\lambda_5\delta_1\mathfrak{I}(t)^2\end{aligned} \tag{4.8}$$

$$
\begin{aligned}
&-\lambda_5 e^{\alpha\delta_1} \int_{t-\delta_1}^{t} \mathfrak{T}(t')^2 dt' \\
&+\alpha V_3(I_t, J_t, G_t, R_t) + \lambda_6 \delta_1' G(t)^2 \\
&-\lambda_6 e^{\alpha\delta_1'} \int_{t-\delta_1'}^{t} G(t')^2 dt' \\
&+\alpha V_4(I_t, J_t, G_t, R_t) + \lambda_7(\delta_1 + \delta_2)\mathfrak{T}(t)^2 \\
&-\lambda_7 e^{\alpha(\delta_1+\delta_2)} \int_{t-\delta_1-\delta_2}^{t} \mathfrak{T}(t')^2 dt'
\end{aligned}
$$

By using $2f(t)g(t) \leq f(t)^2 + g(t)^2$ and

$$
2f(t)\int_{t-\delta}^{t} g(t')dt' \leq \left[f(t)^2 + \left(\int_{t-\delta}^{t} g(t')dt'\right)^2\right] \leq \left[f(t)^2 + \frac{1}{\delta}\int_{t-\delta}^{t} g(t')^2 dt'\right],
$$

and after arranging the terms, we have

$$
\begin{aligned}
D_{,t}^{+}V(X_t) \leq \alpha V(X_t) &+ X^T(t)\Xi X(t) \\
&+ \left(\frac{\lambda_2 \gamma h_{1,max}^2}{\delta_1} - \lambda_5 e^{\alpha_1\delta_1}\right) \int_{t-\delta_1}^{t} \mathfrak{T}(t')^2 dt' \\
&+ \left(\frac{\lambda_3 \ell h_{2,max}^2}{\delta_1'} - \lambda_6 e^{\alpha_2\delta_1'}\right) \int_{t-\delta_1'}^{t} G(t')^2 dt' \\
&+ \left(\frac{\lambda_4 \kappa h_{3,max}^2}{\delta_1 + \delta_2} - \lambda_7 e^{\alpha_3(\delta_1+\delta_2)}\right) \int_{t-\delta_1-\delta_2}^{t} \mathfrak{T}(t')^2 dt'
\end{aligned}
\tag{4.9}
$$

where $X(t) := [I(t) \quad J(t) \quad G(t) \quad R(t)]^T$. Using (4.1)-(4.4), we have the following

$$
D_{,t}^{+}V(X_t) \leq \alpha V(X_t). \tag{4.10}
$$

Integrating (4.10) by virtue of Theorem 11 in (Hagood and Thomson, 2006), we have

$$
V(X_t) \leq e^{\alpha t}V(X_0) \leq e^{\alpha T_f}V(X_0). \tag{4.11}
$$

By definition of $V(X_t)$ in (4.6) and (4.7), we obtain

$$
\begin{aligned}
V(X_0) \leq (\lambda_{max} &+ 3\lambda_5\delta_1^2 e^{\alpha_1\delta_1}\beta^2 + 3\lambda_6 {\delta_1'}^2 e^{\alpha_2\delta_1'}\ell^2 \\
&+ 3\lambda_7(\delta_1+\delta_2)^2 e^{\alpha_3(\delta_1+\delta_2)}\beta^2)||X_0||_{[-\delta_1-\delta_2,0]},
\end{aligned}
\tag{4.12}
$$

and

$$
\lambda_{min}|X(t)|^2 \leq V(X_t) \tag{4.13}
$$

The inequality $|X(t)|^2 \leq \epsilon^2$ holds, if (4.5) holds, which implies $|X(t)| \leq \epsilon$ for all $t \in [0, T_f]$. This indicates that (3.5) is FTS with respect to (ϵ, δ, T_f). ■

Note that, there is no expression for the existence of λ_{min} and λ_{max} in the condition in Theorem 1. This is due to the fact that λ_{min} and λ_{max} rely on the existence of λ_i, $i = 1, 2, 3, 4$, by definition, and we can easily convert their definition into an inequality constraint as

$$\lambda_{\min} \leq \lambda_1, \lambda_{min} \leq \lambda_2, \lambda_{min} \leq \lambda_3, \lambda_{min} \leq \lambda_4, \quad (4.14)$$

$$\lambda_1 \leq \lambda_{\max}, \lambda_2 \leq \lambda_{\max}, \lambda_3 \leq \lambda_{\max}, \lambda_4 \leq \lambda_{\max}, \quad (4.15)$$

In the following section, we show a numerical example to show the validity of the proposed theorem.

5. Numerical Example

Here, we present a numerical example to verify the validity and efficiency of the proposed methodologies to ensure the FTS of the TDS to describe the outbreak of COVID-19 (3.5)

$$\beta = 0.27, \ell = 0.6, \kappa = 0.97, \gamma = 0.99, \delta_1 = 7, \delta_1' = 4, \delta_2 = 12,$$

To apply Theorem 1, we choose

$$\alpha = 1, \delta = 5, \epsilon = 50000, T_f = 14,$$

and for the simulation, we choose the initial state history as

$$X(t) = \begin{cases} [1 \quad 0 \quad 0 \quad 0]^T, & t \in [-19, -12), \\ [2 \quad 0 \quad 0 \quad 0]^T, & t \in [-12, -6), \\ [4 \quad 0 \quad 0 \quad 0]^T, & t \in [-6, -3), \\ [5 \quad 0 \quad 0 \quad 0]^T, & t \in [-3,0], \end{cases}$$

Note that, this initial state history satisfy $||X_0||_{[-\delta_1-\delta_2,0]} \leq \delta$. By applying Theorem 1, the numbers that safisfy (4.1)-(4.5) are found as

$$\lambda_1 = 438.49, \lambda_2 = 11701, \lambda_3 = 337.67, \lambda_4 = 6347.3, \lambda_5 = 0.48687,$$

$$\lambda_6 = 2.1251, \lambda_7 = 2.3232 \cdot 10^{-7}, \lambda_{min} = 300.18, \lambda_{max} = 12418.$$

Thus, the TDS (3.5) is found FTS with respect to $(\epsilon, \delta, T_f) = (50000,5,14)$.

6. Conclusions

In this study, the finite-time stability analysis of a time-delay system representing the COVID-19 outbreak was performed. The proposed infection dynamics in (Chen et al., 2020a) was restated with a novel kernel function to describe the distribution of exposed people in the existing model. A new condition in terms of linear matrix inequalities was given to guarantee FT stability of the corresponding infection dynamics. As a result of this condition, the norm of the variables which are infected, confirmed, isolated and cured/recovered people did not exceed a certain bound in a fixed finite time interval. Those findings were also successively supported with a numerical example by using the same parameters as in (Chen et al., 2020a) and the validity of the proposed method had been shown.

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Appendix A. Proof of Technical Lemmas

A.1. Proof of Lemma 1

From the total upper right Dini derivative definition (see Definition 3) and the properties of supremum, we have that

$$\begin{aligned} D_{,t}^{+}\big[F\big(t,b(t)\big)\big] &= \limsup_{h\to 0^+}\frac{F\big(t+h,b(t+h)\big)-F\big(t,b(t)\big)}{h} \\ &= \limsup_{h\to 0^+}\frac{F\big(t+h,b(t+h)\big)-F\big(t,b(t+h)\big)+F\big(t,b(t+h)\big)-F\big(t,b(t)\big)}{h} \\ &\le \limsup_{h\to 0^+}\frac{F(t+h,b(t+h))-F\big(t,b(t+h)\big)}{h} \\ &\quad + \limsup_{h\to 0^+}\frac{F\big(t,b(t+h)\big)-F(t,b(t))}{h} \end{aligned}$$

In other words, in view of Definition 2,

$$D_{,t}^{+}\big[F\big(t,b(t)\big)\big] \le \big[\mathcal{D}_{,t}^{+}F(t,y)\big]_{(t,b(t))} + \limsup_{h\to 0^+}\frac{F\big(t,b(t+h)\big)-F(t,b(t))}{h}. \quad \text{(A.1)}$$

Consequently, we are left to show that

$$\limsup_{h\to 0^+}\frac{F\big(t,b(t+h)\big)-F\big(t,b(t)\big)}{h} = \big[\mathcal{D}_{,y}^{+}F(t,y)\big]_{(t,b(t))}\cdot b'(t). \quad \text{(A.2)}$$

Exploiting differentiability of b, it holds that

$$\begin{aligned}\limsup_{h\to 0^+}&\frac{F\big(t,b(t+h)\big)-F\big(t,b(t)\big)}{h}\\&=\limsup_{h\to 0^+}\frac{F\big(t,b(t+h)\big)-F(t,b(t))}{b(t+h)-b(t)}\cdot\frac{b(t+h)-b(t)}{h}\\&=\limsup_{h\to 0^+}\frac{F\big(t,b(t+h)\big)-F(t,b(t))}{b(t+h)-b(t)}\cdot\frac{b(t+h)-b(t)}{h}\cdot b'(t).\end{aligned}\tag{A.3}$$

Furthermore, it holds that

$$b(t+h)=b(t)+\int_t^{t+h} b'(\tau)d\tau.$$

Given any $h>0$, let $h_{\mathcal{I}_b}(h)\coloneqq\int_t^{t+h}b'(\tau)d\tau$. Then $h_{\mathcal{I}_b}(h)$ is finite for all $h>0$ and $h_{\mathcal{I}_b}(h)\to 0^+$ as $h\to 0^+$. It follows that

$$\begin{aligned}\limsup_{h\to 0^+}\frac{F\big(t,b(t+h)\big)-F(t,b(t))}{h_{\mathcal{I}_b}(h)}&=\limsup_{h\to 0^+}\frac{F\big(t,b(t)+h_{\mathcal{I}_b}(h)\big)-F(t,b(t))}{h_{\mathcal{I}_b}(h)}\\&=\big[\mathcal{D}^+_{,y}F(t,y)\big]_{(t,b(t))}.\end{aligned}\tag{A.4}$$

Thus, (2.7) follows from (A.2) and (A.4), which concludes the proof. ■

A.2. Proof of Lemma 2

Let $y^*\in J$ and given any $t\in I$ and any $y^*\in J$, define $F(t,y)\coloneqq\int_{y^*}^{y}f(t,\tau)d\tau$. Then it holds that

$$\int_{a(t)}^{b(t)} f(t,\tau)d\tau=F\big(t,b(t)\big)+F\big(t,a(t)\big),\qquad \forall t\in I.$$

Note that, F is a function having finite partial upper right Dini derivative with respect to its first and second arguments on I and J, respectively and the functions $t\mapsto F(t,b(t))$ and $t\mapsto F(t,a(t))$ have finite upper right Dini derivatives on I. From Lemma 1 and Corollary 1, it follows that

$$\begin{aligned}D^+_{,t}\left[\int_{a(t)}^{b(t)} f(t,\tau)d\tau\right]&=D^+_{,t}\big[F\big(t,b(t)\big)+F\big(t,a(t)\big)\big]\\&\le\big[\mathcal{D}^+_{,t}F(t,y)\big]_{(t,b(t))}+\big[\mathcal{D}^+_{,y}F(t,y)\big]_{(t,b(t))}\cdot b'(t)\\&\quad-\big[\mathcal{D}_{+,t}F(t,y)\big]_{(t,a(t))}-\big[\mathcal{D}_{+,y}F(t,y)\big]_{(t,a(t))}\cdot a'(t)\end{aligned}\tag{A.5}$$

Moreover, for all $t\in I$ and $y\in J$,

$$\mathcal{D}^+_{,t}[F(t,y)]=\limsup_{h\to 0^+}\frac{F(t+h,y)-F(t,y)}{h}\tag{A.6}$$

$$= \limsup_{h\to 0^+} \frac{1}{h}\left(\int_{y^*}^{y} f(t+h,\tau)d\tau - \int_{y^*}^{y} f(t,\tau)d\tau\right)$$

Since, for any fixed τ, f is assumed to have finite partial upper Dini right derivative with respect to its first argument on I, we get from Theorem 11 of (Hagood and Thomson, 2006) that

$$f(t+h,\tau) = f(t,\tau) + \int_{t}^{t+h} \mathcal{D}^+_{,t}[f(t,y)]_{(s,\tau)} ds. \tag{A.7}$$

Consequently,

$$\mathcal{D}^+_{,t}[F(t,y)] = \limsup_{h\to 0^+} \frac{1}{h}\left(\int_{y^*}^{y}\int_{t}^{t+h} \mathcal{D}^+_{,t}[f(t,y)]_{(s,\tau)} ds\, d\tau\right). \tag{A.8}$$

By Fubini's Theorem, it follows that

$$\begin{aligned}\mathcal{D}^+_{,t}[F(t,y)] &= \limsup_{h\to 0^+} \frac{1}{h}\left(\int_{t}^{t+h}\int_{y^*}^{y} \mathcal{D}^+_{,t}[f(t,y)]_{(s,\tau)} d\tau\, ds\right)\\ &= \int_{y^*}^{y} \mathcal{D}^+_{,t}[f(t,y)]_{(t,\tau)} d\tau\end{aligned} \tag{A.9}$$

Similarly, for the lower right Dini derivative:

$$\begin{aligned}\mathcal{D}_{+,t}[F(t,y)] &= \liminf_{h\to 0^+} \frac{F(t+h,y)-F(t,y)}{h}\\ &= \liminf_{h\to 0^+} \frac{1}{h}\left(\int_{y^*}^{y} f(t+h,\tau)d\tau - \int_{y^*}^{y} f(t,\tau)d\tau\right)\end{aligned} \tag{A.10}$$

From (A.7), we have

$$\mathcal{D}_{+,t}[F(t,y)] = \liminf_{h\to 0^+} \frac{1}{h}\left(\int_{y^*}^{y}\int_{t}^{t+h} \mathcal{D}^+_{,t}[f(t,y)]_{(s,\tau)} ds\, d\tau\right). \tag{A.11}$$

Again by Fubini's Theorem, we obtain

$$\begin{aligned}\mathcal{D}_{+,t}[F(t,y)] &= \liminf_{h\to 0^+} \frac{1}{h}\left(\int_{t}^{t+h}\int_{y^*}^{y} \mathcal{D}^+_{,t}[f(t,y)]_{(s,\tau)} d\tau\, ds\right)\\ &= \int_{y^*}^{y} \mathcal{D}^+_{,t}[f(t,y)]_{(t,\tau)} d\tau\end{aligned} \tag{A.12}$$

Going back to (A.5) and using (A.9) and (A.12), we get

$$
\begin{aligned}
D^{+}_{,t}\left[\int_{a(t)}^{b(t)} f(t,\tau)d\tau\right] \le & \int_{y^*}^{b(t)} \mathcal{D}^{+}_{,t}[f(t,y)]_{(t,\tau)}d\tau + \left[\mathcal{D}^{+}_{,y}F(t,y)\right]_{(t,b(t))}\cdot b'(t) \\
& - \int_{y^*}^{a(t)} \mathcal{D}^{+}_{,t}[f(t,y)]_{(t,\tau)}d\tau - \left[\mathcal{D}_{+,y}F(t,y)\right]_{(t,a(t))}\cdot a'(t) \\
= & \int_{a(t)}^{b(t)}\left[\mathcal{D}^{+}_{,t}\, f(t,y)\right]_{(t,\tau)}d\tau + \left[\mathcal{D}^{+}_{,y}F(t,y)\right]_{(t,b(t))}\cdot b'(t) \\
& -\left[\mathcal{D}_{+,y}F(t,y)\right]_{(t,a(t))}\cdot a'(t)
\end{aligned}
$$

which concludes the proof. ■