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The Padovan- Padovan p-Sequences in Groups

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Abstract. Erdag and Deveci [13] defined the Padovan-Padovan p-sequence and they studied properties of this sequence. Then, Akuzum and Deveci [1] studied the Padovan-Padovan p-sequence modulo m . Also, they discussed the connections between the order the cyclic groups obtained and the periods of the Padovan-Padovan p-sequence according to modulo m . In this paper, we redefine the Padovan-Padovan p-sequence by means of the elements of the groups and then, we examine this sequence in the finite groups in detail. Also, we obtain the lengths of the periods of the Padovan-Padovan 4-sequence in the semidihedral group SD_{2^m} as applications of the results obtained.

1. Introduction

Erdag and Deveci [13] defined the Padovan-Padovan p-sequence as shown:

$$Pa^{P,p}_{n+p+5} = 2Pa^{P,p}_{n+p+3} + Pa^{P,p}_{n+p+2} - Pa^{P,p}_{n+p+1} - Pa^{P,p}_{n+p} + Pa^{P,p}_{n+3} - Pa^{P,p}_{n+1} - Pa^{P,p}_{n}$$

for $p\,(4,5,6,\ldots)$ and $n \geq 0$ with initial constants $Pa^{P,p}_{0} = \cdots = Pa^{P,p}_{p+3} = 0$, $Pa^{P,p}_{p+4} = 1$.

Also, they gave the Padovan-Padovan p-matrix as shown:

$$C_p = \begin{bmatrix} 0 & 2 & 1 & -1 & -1 & 0 & \cdots & 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{(p+5)\times(p+5)}.$$

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Then by an inductive argument, they derived that

$$\left(C_p\right)^n = \left[\begin{array}{cccccccc} Pa^{P,p}_{n+p+4} & Pa^{P,p}_{n+p+5} & -Pa^{P,p}_{n+p+4}+Pa^{P,p}_{n+p+3}+Pap\left(n+p+1\right)+Pap\left(n+1\right) & -Pa^{P,p}_{n+p+5}+Pap\left(n+p+4\right) & -Pa^{P,p}_{n+p+3}+Pap\left(n+3\right) & Pap\left(n+4\right) & \cdots & \\ Pa^{P,p}_{n+p+3} & Pa^{P,p}_{n+p+4} & -Pa^{P,p}_{n+p+3}+Pa^{P,p}_{n+p+2}+Pap\left(n+p\right)+Pap\left(n\right) & -Pa^{P,p}_{n+p+4}+Pap\left(n+p+3\right) & -Pa^{P,p}_{n+p+2}+Pap\left(n+2\right) & Pap\left(n+3\right) & \cdots & \\ Pa^{P,p}_{n+p+2} & Pa^{P,p}_{n+p+3} & -Pa^{P,p}_{n+p+2}+Pa^{P,p}_{n+p+1}+Pap\left(n+p-1\right)+Pap\left(n-1\right) & -Pa^{P,p}_{n+p+3}+Pap\left(n+p+2\right) & -Pa^{P,p}_{n+p+1}+Pap\left(n+1\right) & Pap\left(n+2\right) & \cdots & C_p^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ Pa^{P,p}_{n+1} & Pa^{P,p}_{n+2} & -Pa^{P,p}_{n+1}+Pa^{P,p}_{n}+Pap\left(n-2\right)+Pap\left(n-p-2\right) & -Pa^{P,p}_{n+2}+Pap\left(n+1\right) & -Pa^{P,p}_{n}+Pap\left(n-p\right) & Pap\left(n-p+1\right) & \cdots & \\ Pa^{P,p}_{n} & Pa^{P,p}_{n+1} & -Pa^{P,p}_{n}+Pa^{P,p}_{n-1}+Pap\left(n-3\right)+Pap\left(n-p-3\right) & -Pa^{P,p}_{n+1}+Pap\left(n\right) & -Pa^{P,p}_{n-1}+Pap\left(n-p-1\right) & Pap\left(n-p\right) & \cdots & \end{array}\right],$$

where C_p^* is a matrix as follows:

$$C_p^* = \left[\begin{array}{cccc} Pap\left(n+p\right) & -Pa^{P,p}_{n+p+4}+Pap\left(n+p+1\right) & -Pa^{P,p}_{n+p+5}+Pap\left(n+p+2\right) & -Pa^{P,p}_{n+p+3} \\ Pap\left(n+p-1\right) & -Pa^{P,p}_{n+p+3}+Pap\left(n+p\right) & -Pa^{P,p}_{n+p+4}+Pap\left(n+p+1\right) & -Pa^{P,p}_{n+p+2} \\ Pap\left(n+p-2\right) & -Pa^{P,p}_{n+p+2}+Pap\left(n+p-1\right) & -Pa^{P,p}_{n+p+3}+Pap\left(n+p\right) & -Pa^{P,p}_{n+p+1} \\ \vdots & \vdots & \vdots & \vdots \\ Pap\left(n-3\right) & -Pa^{P,p}_{n+1}+Pap\left(n-2\right) & -Pa^{P,p}_{n+2}+Pap\left(n-1\right) & -Pa^{P,p}_{n} \\ Pap\left(n-4\right) & -Pa^{P,p}_{n}+Pap\left(n-3\right) & -Pa^{P,p}_{n+1}+Pap\left(n-2\right) & -Pa^{P,p}_{n-1} \end{array}\right].$$

Akuzum and Deveci [1] obtained the following repeating sequence, reducing the Padovan-Padovan p-sequences $\left\{Pa^{P,p}_n\right\}$ by a modulus m:

$$\left\{Pa^{P,p,m}_n\right\} = \left\{Pa^{P,p,m}_o, Pa^{P,p,m}_1, Pa^{P,p,m}_2, \ldots, Pa^{P,p,m}_i, \ldots\right\}$$

where $Pa^{P,p,m}_i = Pa^{P,p}_i \ (modm)$.

It is well-known that a sequence is periodic if, after certain points, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence.

Theorem 1. (Akuzum and Deveci [1]). The sequence $\left\{Pa^{P,p,m}_n\right\}$ is simply periodic for every positive integer m.

The linear recurrence sequences in groups were firstly studied by Wall [15] who calculated the periods of the Fibonacci sequences in cyclic groups. In the mid-eighties, Wilcox [16] extended the problem to abelian groups and Campbell et al. [5] expanded the theory to some finite simple groups. Further, the concept extended to some special linear recurrence sequences by several authors; see for example, [2–4, 6–14]. In this paper, we redefine the Padovan-Padovan p-sequence by means of the elements of the groups and then, we examine this sequence in the finite groups in detail. Also, we obtain the lengths of the periods of the Padovan-Padovan 4-sequence in the semidihedral group SD_{2^m} as applications of the results obtained.

2. The Padovan- Padovan p-Sequences in Groups

Let G be a finite j-generator group and let X be the subset of $\underbrace{G \times G \times G \cdots \times G}_{j}$ such that $\left(x_0, x_2, \ldots, x_{j-1}\right) \in X$ if and only if G is generated by $x_0,\ x_1,\ \ldots,\ x_{j-1}$. We call $\left(x_0, x_2, \ldots, x_{j-1}\right)$ a generating j-tuple for G.

Definition 2.1. *For a j-tuple* $\left(x_0, x_1, \ldots, x_{j-1}\right) \in X$, *we define the Padovan-Padovan p-orbit* $PA^p\left(G : x_0, x_1, \ldots, x_{j-1}\right) = \left\{a_p\left(n\right)\right\}$ *as shown:*

$$a_p\left(n+p+5\right) = \left(a_p\left(n\right)\right)^{-1}\left(a_p\left(n+3\right)\right)\left(a_p\left(n+p\right)\right)^{-1}\left(a_p\left(n+p+1\right)\right)^{-1}\left(a_p\left(n+p+2\right)\right)\left(a_p\left(n+p+3\right)\right)^{2}$$

where $n \geq 0$ and

$$\begin{cases} a_p(0) = x_0, a_p(1) = x_1, \ldots, a_p(j-1) = x_j, a_p(j) = e, \ldots, a_p(p+4) = e & \text{if} \quad j < p+4, \\ a_p(0) = x_0, a_p(1) = x_1, a_p(2) = x_2, \ldots, a_p(p+4) = x_{p+4} & \text{if} \quad j = p+4. \end{cases}$$

Theorem 2.2. *If G is a finite group, then a Padovan-Padovan p-orbit of the group G is simply periodic.*

Proof. Suppose that t is the order of the group G. Since there are t^{p+5} distinct $p+5$-tuples of elements of G, at least one of the $p+5$-tuples appears twice in a Padovan-Padovan p-orbit of the group G. Because of the repeating, the Padovan-Padovan p-orbit of the group G is periodic. Since the orbit $PA^p\left(G : x_0, x_1, \ldots, x_{j-1}\right)$ is periodic, there exist natural numbers i and j, with $i \equiv j\,(mod p+5)$, such that

$$a_p\left(i\right) = a_p\left(j\right), a_p\left(i+1\right) = a_p\left(j+1\right), \ldots, a_p\left(i+p+5\right) = a_p\left(j+p+5\right).$$

By the definition of the Padovan-Padovan p-orbit, it is clear that

$$a_p\left(n\right) = \left(a_p\left(n+3\right)\right)\left(a_p\left(n+p\right)\right)^{-1}\left(a_p\left(n+p+1\right)\right)^{-1}\left(a_p\left(n+p+2\right)\right)\left(a_p\left(n+p+3\right)\right)^{2}\left(a_p\left(n+p+5\right)\right)^{-1}.$$

Therefore, we obtain $a_p\left(i\right) = a_p\left(j\right)$, and hence

$$a_p\left(i-j\right) = a_p\left(0\right), a_p\left(i-j+1\right) = a_p\left(1\right), \ldots, a_p\left(i-j+p+5\right) = a_p\left(p+5\right),$$

which implies that the Padovan-Padovan p-orbit is simply periodic. □

We denote the length of the period of Padovan-Padovan p-orbit $PA^p\left(G : x_0, x_1, \ldots, x_{j-1}\right)$ by $hPA^p\left(G : x_0, x_1, \ldots, x_{j-1}\right)$.

In **[1]**, Akuzum and Deveci denoted the period of the sequence $\left\{Pa_n^{P,p,m}\right\}$ by $h_p\left(m\right)$.

Now we give the lengths of the periods of the Padovan-Padovan 4-orbit of the semidihedral group SD_{2^m} as applications of the results obtained.

The semidihedral group SD_{2^m}, $(m \geq 4)$ is defined by the presentation

$$SD_{2^m} = \left\langle x, y : x^{2^{m-1}} = y^2 = e,\ \ yxy = x^{2^{m-2}-1}\right\rangle.$$

Note that $|SD_{2^m}| = 2^m$, $|x| = 2^{m-1}$ and $\left|y\right| = 2$.

Theorem 2.3. *The length of the period of the Padovan-Padovan 4-orbit of the semidihedral group SD_{2^m} is* $2^{m-2} \cdot h_4(2)$.

Proof. We consider the length of the period of the the Padovan-Padovan 4-orbit in the semidihedral group by the aid of the period $h_4(2) = 14$. The orbit $PA^4(SD_{2^m} : x, y)$ is

$$a_4(0) = x,\ a_4(1) = y,\ a_4(2) = e, \ldots, a_4(8) = e.$$

Thus, we also have

$$a_4(28i) = x^{4ir_1+1},\ a_4(28i+1) = x^{8ir_2}y,\ a_4(28i+2) = e,\ a_4(28i+3) = x^{8ir_3},\ a_4(28i+4) = x^{8ir_4},$$
$$a_4(28i+5) = x^{-4ir_5},\ a_4(28i+6) = x^{8ir_6},\ a_4(28i+7) = x^{4ir_7},\ a_4(28i+8) = x^{4ir_8},$$

where $r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8$ are positive integers such that $\gcd(r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8) = 1$ So we need the smallest $i \in \mathbb{N}$ such that $4i = 2^{m-1}.k$ $(k \in \mathbb{N})$. If we choose $i = 2^{m-3}$,we obtain

$$a_4\left(2^{m-2}14\right) = x,\ a_4\left(2^{m-2}14+1\right) = y,\ a_4\left(2^{m-2}14+2\right) = e,\ a_4\left(2^{m-2}14+3\right) = e,\ a_4\left(2^{m-2}14+4\right) = e,$$
$$a_4\left(2^{m-2}14+5\right) = e,\ a_4\left(2^{m-2}14+6\right) = e,\ a_4\left(2^{m-2}14+7\right) = e,\ a_4\left(2^{m-2}4+8\right) = e.$$

Since the elements succeeding $a_4\left(2^{m-2}14\right)$, $a_4\left(2^{m-2}14+1\right)$,$a_4\left(2^{m-2}14+2\right)$,. . .,$a_4\left(2^{m-2}14+8\right)$ depend on x, y, e for their values and $h_4(2) = 14$, the cycle begins again with the $2^{m-2}.h_4(2)nd$ element. Thus it is verifed that the length of the period of the Padovan-Padovan 4-orbit of the semidihedral group SD_{2^m} is $2^{m-2} \cdot h_4(2)$. □

Example 2.4. *For $m = 4$, we consider the length of the period of the Padovan-Padovan 4-orbit in the semidihedral group SD_{2^4}. Since $h_4(2) = 14$, we have the sequence*

$$a_4(0) = x,\ a_4(1) = y,\ a_4(2) = e,\ a_4(3) = e,\ a_4(4) = e,\ a_4(5) = e,\ a_4(6) = e,\ a_4(7) = e,\ a_4(8) = e, \ldots,$$
$$a_4(28) = x^5,\ a_4(29) = y,\ a_4(30) = e,\ a_4(31) = e,\ a_4(32) = e, a_4(33) = x^4,\ a_4(34) = e,\ a_4(35) = x^4,\ a_4(36) = x^4, \ldots,$$
$$a_4(56) = x^5,\ a_4(57) = y,\ a_4(58) = e,\ a_4(59) = e,\ a_4(60) = e, a_4(61) = x,\ a_4(62) = e,\ a_4(63) = e,\ a_4(64) = e, \ldots.$$

Since $a_4(0) = a_4(56)$,$a_4(1) = a_4(57)$,$a_4(2) = a_4(58)$,$a_4(3) = a_4(59)$,$a_4(4) = a_4(60)$, $a_4(5) = a_4(61)$,$a_4(6) = a_4(61)$, $a_4(7) = a_4(62)$, $a_4(8) = a_4(63)$ the length of the period of the the Padovan-Padovan 4-orbit $PA^4(SD_{2^4} : x, y)$ is 56.

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