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The generating matrices of the bivariate Balancing and Lucas-Balancing polynomials

İki değişkenli Balans ve Lucas-Balans polinomlarının üreteç matrisleri

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Abstract

The objective of this paper is to express the bivariate Balancing and Lucas-Balancing polynomials in terms of determinants of tridiagonal matrices. In addition, we obtained the inverses of the tridiagonal matrices. We finalized the general results to construct families of the tridiagonal matrices whose determinants generate arbitrary linear subsequence with positive and negative indices of the bivariate Balancing and Lucas-Balancing polynomials.

Keywords: Balancing polynomials, Determinant, Inverse of matrix, Lucas-Balancing polynomials, Tridiagonal matrix.

Öz

Makalenin amacı iki değişkenli Balans ve Lucas-Balans polinomlarını üçgensel matrislerin determinantları ile ifade etmektir. Ek olarak bu üçgensel matrislerin terslerini elde ettik. Determinantları iki değişkenli Balans ve Lucas-Balans polinomlarının herhangi pozitif ve negatif indisli lineer alt dizilerini üreten üçgensel matrislerin ailesini veren genel sonuçlar ile sonlandırdık.

***Anahtar kelimeler**: Balans polinomları, Determinant, Matris tersi, Lucas-Balans polinomları, Üçgensel matris.*

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1. Introduction

The study of number sequences has been a source of attraction to the mathematicians since ancient times. Since then many of them are focusing their interest on the study of the fascinating triangular numbers. Behera and Panda in 1999, introduced the notion of Balancing numbers $(B_n)_{n\in\mathbb{N}}$ as solutions to a certain Diophantine equation:

$$1+2+\ldots+(n-1)=(n+1)+(n+2)+\ldots+(n+r),$$

for some positive integer r which is called *balancer* or *cobalancing number* (Behera and Panda, 1999). Then, the recurence relation of this number sequence is $B_{n+1}=6B_n-B_{n-1}$ for $n\geqslant 1$, where $B_0=0, B_1=1$. A study on the Lucas-Balancing numbers $C_n=\sqrt{8B_n^2+1}$ was published in 2009 by Panda (Panda, 2009). The recurrence relation of this number sequence is $C_{n+1}=6C_n-C_{n-1}$ for $n\geqslant 1$, where $C_0=1, C_1=3$.

In the recent years many number theorists from all over the world are taking interest in this beatiful number system and studying the generalizations of this numbers. Interested reader may follow (Frontczak, 2019; Ozkoc, 2015; Ozkoc and Tekcan, 2017; Patel et. al., 2018; Ray, 2017; Ray, 2018; Yilmaz, 2020). One of these generalizations is the Balancing polynomials. A natural extension is to consider for $x\in\mathbb{C}$ sequence of bivariate Balancing polynomials $(B_n(x,y))_{n\in\mathbb{N}}$ (Yakar, 2020).

The bivariate Balancing polynomial is denoted by the recurrence relation

$$B_{n+1}(x,y)=6xB_n(x,y)-yB_{n-1}(x,y), \tag{1}$$

where $B_0(x,y)=0, B_1(x,y)=1$ in (Aşcı and Yakar, 2020).

The explicit form of the bivariate Balancing polynomials is given by the equation

$$B_n(x,y)=\frac{\alpha^n-\beta^n}{\alpha-\beta}, \tag{2}$$

where $9x^2-y>0$, $\alpha=3x+\sqrt{9x^2-y}$ and $\beta=3x-\sqrt{9x^2-y}$, in (Aşcı and Yakar, 2020).

In similar manner, the bivariate Lucas-Balancing polynomial is defined by the recurrence relation

$$C_{n+1}(x,y)=6xC_n(x,y)-yC_{n-1}(x,y), \tag{3}$$

where $C_0(x,y)=1, C_1(x,y)=3x$ and it's Binet formula is

$$C_n(x,y)=\frac{\alpha^n+\beta^n}{2}, \tag{4}$$

where $9x^2-y>0$, $\alpha=3x+\sqrt{9x^2-y}$ and $\beta=3x-\sqrt{9x^2-y}$.

Let $A(k)$ be a family of tridiagonal matrices which is form as following

$$A(k)=\begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{k-1} \\ & & & c_{k-1} & a_k \end{bmatrix}.$$

Theorem 1.1 (Cahill and Narayan, 2004) *The determinant of* $A(k)$ *is*

$$|A(1)|=a_1,$$
$$|A(2)|=a_1a_2-b_1c_1,$$
$$|A(k)|=a_k|A(k-1)|-b_{k-1}c_{k-1}|A(k-2)|,\quad k\geq 3.$$

Theorem 1.2 (Usmani, 1994a; 1994b) *The inverse of a non-singular tridiagonal matrix* $A(k)$ *is*

$$(A(k))^{-1}_{i,j}=\begin{cases} (-1)^{i+j}\dfrac{1}{\theta_k}b_i\cdots b_{j-1}\theta_{i-1}\phi_{j+1}, & i\leq j \\ \\ (-1)^{i+j}\dfrac{1}{\theta_k}c_j\cdots c_{i-1}\theta_{j-1}\phi_{i+1}, & i>j \end{cases},$$

where θ_i and ϕ_i satisfy the following recurrence relations:

$$\theta_i=a_i\theta_{i-1}-b_{i-1}c_{i-1}\theta_{i-2} \text{ for } i=2,\ldots,k$$

with the initial conditions $\theta_0 = 1$ and $\theta_1 = a_1$. Theorem 1.1 is one special case of this one. Observe that $\theta_k = \det A(k)$,

$$\phi_i = a_i\phi_{i+1} - b_i c_i \phi_{i+2} \text{ for } i = k-1,\ldots,1$$

with the initial conditions $\phi_{k+1} = 1$ and $\phi_k = a_k$.

There are many relations between Fibonacci, Lucas, Balancing, Lucas-Balancing numbers and tridiagonal matrices. (Chen, 2020; Falcon, 2013; Feng, 2011; Goy, 2018; Nalli and Civciv, 2009; Ozkoc, 2015; Ray, 2012; Ray and Panda, 2015; Taskara et. al., 2011; Trojovsky, 2016; Yilmaz and Kirklar, 2015). As a brief antecedents, the Fibonacci numbers as tridiagonal matrix determinants were construct by Strang (Strang 1997; 1998), and further authors generalized results to establish families of tridiagonal matrices whose determinants create any linear subsequence $F_{\alpha k+\beta}$ or $L_{\alpha k+\beta}$ of the Fibonacci and Lucas numbers in (Nalli and Civciv, 2009). Additionally, in (Ray and Panda, 2015), the authors give the linear subsequence of the Balancing and Lucas-Balancing numbers by using the determinant of tridiagonal matrices.

Under these conditions, the main aim of this work is to construct the bivariate Balancing and Lucas-Balancing polynomials with positive and negative indices in the meaning of determinants of tridiagonal matrices. Then, we formulate the inverses of some of these matrices.

2. Main results

In this section, by a different approximation, we present the bivariate Balancing and Lucas-Balancing polynomials with the determinant of the tridiagonal matrices. To do that, firstly, we give the proposition in the following.

In the following proposition, we reveal the relationship between the bivariate Balancing and Lucas-Balancing numbers.

Proposition 2.1 *The bivariate Balancing and Lucas-Balancing polynomials satisfy the following equalities*

$$(i) B_{n+k}(x,y) = 2C_k(x,y)B_n(x,y) - y^k B_{n-k}(x,y), \tag{5}$$

$$(ii) C_{n+k}(x,y) = 2C_k(x,y)C_n(x,y) - y^k C_{n-k}(x,y), \tag{6}$$

where $0 < k \leqslant n$ and $k, n \in \mathbb{Z}^+$.

Proof. (i) From the Equations (2) and (4), we have

$$2C_k(x,y)B_n(x,y) - y^k B_{n-k}(x,y) = \left(\alpha^k + \beta^k\right)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) - y^k \frac{\alpha^{n-k} - \beta^{n-k}}{\alpha - \beta}.$$

Then, by using the equality $\alpha\beta = y$, we can write

$$\begin{aligned} 2C_k(x,y)B_n(x,y) - y^k B_{n-k}(x,y) &= \frac{\alpha^{n+k} - \beta^{n+k}}{\alpha - \beta} \\ &= B_{n+k}(x,y), \end{aligned}$$

which completes the proof.

The proof of the part (ii) is omitted as it is analogous to the part (i).

In the following theorem, we extend the results to construct families of tridiagonal matrices whose determinants generate any arbitrary linear subsequence of the bivariate Balancing and Lucas-Balancing polynomials.

Theorem 2.1 *For $n, r \in \mathbb{Z}^+$ and $s \in \mathbb{N}$,*

(i) Let $M_{r,s}(n)$ be the symmetric tridiagonal family of matrices. That is,

$$\begin{aligned} &m_{1,1} = B_{r+s}(x,y),\ m_{2,2} = \left\lceil \frac{B_{2r+s}(x,y)}{B_{r+s}(x,y)} \right\rceil, \\ &m_{j,j} = 2C_r(x,y),\ 3 \le j \le n, \\ &m_{1,2} = m_{2,1} = \sqrt{m_{2,2}B_{r+s}(x,y) - B_{2r+s}(x,y)}, \\ &m_{j,j+1} = y^r,\ 2 \le j < n, \end{aligned}$$

$m_{j+1,j} = 1, 2 \le j < n.$

Then, we have

$$\det M_{r,s}(n) = B_{rn+s}(x,y). \qquad (7)$$

(ii) Let $K_{r,s}(n)$ be the symmetric tridiagonal family of matrices. That is,

$$k_{1,1} = C_{r+s}(x,y),\ k_{2,2} = \left\lceil \frac{C_{2r+s}(x,y)}{C_{r+s}(x,y)} \right\rceil,$$
$$k_{j,j} = 2C_r(x,y),\ 3 \le j \le n,$$
$$k_{1,2} = k_{2,1} = \sqrt{k_{2,2}C_{r+s}(x,y) - C_{2r+s}(x,y)},$$
$$k_{j,j+1} = y^r,\ 2 \le j < n,$$
$$k_{j+1,j} = 1,\ 2 \le j < n.$$

Then, we have

$$\det K_{r,s}(n) = C_{rn+s}(x,y). \qquad (8)$$

Proof. We prove (i), since the proof of (ii) can be showed similar to it. We use the principle of finite induction on n to prove the Equation (7). For $n=1$,

$$\det M_{r,s}(1) = B_{r+s}(x,y),$$

it is easy to see that for $n=2$,

$$\det M_{r,s}(2) = \det \begin{bmatrix} B_{r+s}(x,y) & \sqrt{m_{2,2}B_{r+s}(x,y) - B_{2r+s}(x,y)} \\ \sqrt{m_{2,2}B_{r+s}(x,y) - B_{2r+s}(x,y)} & \left\lceil \frac{B_{2r+s}(x,y)}{B_{r+s}(x,y)} \right\rceil \end{bmatrix}$$
$$= B_{2r+s}(x,y).$$

Now assume that it is true for n. That is,

$$\det M_{r,s}(n) = B_{rn+s}(x,y).$$

Then, by considering Theorem 1.1 and the our assumption, we have

$$\det M_{r,s}(n+1) = 2C_r(x,y)\det M_{r,s}(n) - y^r \det M_{r,s}(n-1)$$
$$= 2C_r(x,y)B_{rn+s}(x,y) - y^r B_{rn-r+s}(x,y).$$

From the part (i) of Proposition 2.1, we get

$$\det M_{r,s}(n+1) = B_{r(n+1)+s}(x,y)$$

which ends up the proof.

Using Theorem 2.1, one can construct a family of tridiagonal matrices whose successive determinants form any linear subsequences of the bivariate Balancing and Lucas-Balancing polynomials. For example, the determinants of the following $n \times n$ tridiagonal matrices are

$$M_{1,1}(n) = T(n) = \begin{bmatrix} 6x & \sqrt{y} & & & \\ \sqrt{y} & 6x & y & & \\ & 1 & 6x & \ddots & \\ & & \ddots & \ddots & y \\ & & & 1 & 6x \end{bmatrix} \qquad (9)$$

and

$$K_{1,0}(n) = E(n) = \begin{bmatrix} 3x & \sqrt{y} & & & \\ \sqrt{y} & 6x & y & & \\ & 1 & 6x & \ddots & \\ & & \ddots & \ddots & y \\ & & & 1 & 6x \end{bmatrix} \qquad (10)$$

are given by $B_{n+1}(x,y)$ and $C_n(x,y)$, respectively.

Theorem 2.2 *The inverses of matrices* $T(n)$ *and* $E(n)$ *are*

$$t'_{i,j}=\begin{cases}\dfrac{B_n(x,y)}{B_{n+1}(x,y)}, & i=j=1\\ (-1)^{j+1}\dfrac{B_{n-j+1}(x,y)}{B_{n+1}(x,y)}\sqrt{y}y^{j-2}, & i<j, i=1\\ (-1)^{i+j}\dfrac{B_{n-j+1}(x,y)}{B_{n+1}(x,y)}B_i(x,y)y^{j-i}, & i\le j, i\ne 1\\ (-1)^{i+1}\dfrac{B_{n-i+1}(x,y)}{B_{n+1}(x,y)}\sqrt{y}, & i>j, j=1\\ (-1)^{i+j}\dfrac{B_{n-i+1}(x,y)}{B_{n+1}(x,y)}B_j(x,y), & i>j, j\ne 1\end{cases}$$

and

$$e'_{i,j}=\begin{cases}\dfrac{B_n(x,y)}{C_n(x,y)}, & i=j=1\\ (-1)^{j+1}\dfrac{B_{n-j+1}(x,y)}{C_n(x,y)}\sqrt{y}y^{j-2}, & i<j, i=1\\ (-1)^{i+j}\dfrac{B_{n-j+1}(x,y)}{C_n(x,y)}C_{i-1}(x,y)y^{j-i}, & i\le j, i\ne 1\\ (-1)^{i+1}\dfrac{B_{n-i+1}(x,y)}{C_n(x,y)}\sqrt{y}, & i>j, j=1\\ (-1)^{i+j}\dfrac{B_{n-i+1}(x,y)}{C_n(x,y)}C_{j-1}(x,y), & i>j, j\ne 1\end{cases}$$

where $x,y\ne 0$ and $n\in\mathbb{Z}^+$.

Proof. It is easily seen that the matrices $T(n)$ and $E(n)$ are invertible matrices, since every $B_n(x,y)$ and $C_n(x,y)$ are nonzero for $x,y\ne 0$ and $n\in\mathbb{Z}^+$. For the inverse matrix of $T(n)$, if we choose

$$\begin{aligned}&a_j=6x,\ j=1,2,\dots,n\\ &b_1=c_1=\sqrt{y},\ b_i=y,\ c_i=1,\ i=2,\dots,n\\ &\theta_i=B_{i+1}(x,y),\ \ i=2,\dots,n\\ &\phi_j=B_{n-j+2}(x,y),\ j=n-1,n-2,\dots,1\end{aligned}$$

in Theorem 1.2, then we obtain

$$t'_{i,j}=\begin{cases}\dfrac{B_n(x,y)}{B_{n+1}(x,y)}, & i=j=1\\ (-1)^{j+1}\dfrac{B_{n-j+1}(x,y)}{B_{n+1}(x,y)}\sqrt{y}y^{j-2}, & i<j, i=1\\ (-1)^{i+j}\dfrac{B_{n-j+1}(x,y)}{B_{n+1}(x,y)}B_i(x,y)y^{j-i}, & i\le j, i\ne 1\\ (-1)^{i+1}\dfrac{B_{n-i+1}(x,y)}{B_{n+1}(x,y)}\sqrt{y}, & i>j, j=1\\ (-1)^{i+j}\dfrac{B_{n-i+1}(x,y)}{B_{n+1}(x,y)}B_j(x,y), & i>j, j\ne 1\end{cases}.$$

For the inverse matrix of $E(n)$, if we choose

$$\begin{aligned}&a_1=3x,\ a_j=6x,\ j=1,2,\dots,n\\ &b_1=c_1=\sqrt{y},\ b_i=y,\ c_i=1,\ i=2,\dots,n\\ &\theta_i=C_i(x,y),\ \ i=2,\dots,n\\ &\phi_1=3x,\ \phi_j=B_{n-j+2}(x,y),\ j=n-1,n-2,\dots,2\end{aligned}$$

in Theorem 1.2, then it is as required.

In the following theorem, it give us the bivariate Balancing and Lucas-Balancing polynomials with negative indices in terms of the determinants of the special tridiagonal matrices.

Theorem 2.3 *For* $n,r\in\mathbb{Z}^+$ *and* $s\in\mathbb{N}$,

(i) Let $M_{-r,-s}(n)$ be the symmetric tridiagonal family of matrices. That is,

$$\begin{aligned}&m_{1,1}=B_{-r-s}(x,y),\ m_{2,2}=\left\lceil\frac{B_{-2r-s}(x,y)}{B_{-r-s}(x,y)}\right\rceil,\\ &m_{j,j}=2C_{-r}(x,y),\ 3\le j\le n,\\ &m_{1,2}=m_{2,1}=\sqrt{m_{2,2}B_{-r-s}(x,y)-B_{-2r-s}(x,y)},\\ &m_{j,j+1}=y^{-r},\ 2\le j<n,\\ &m_{j+1,j}=1,\ 2\le j<n.\end{aligned}$$

Then, we have

$$\det M_{-r,-s}(n)=B_{-rn-s}(x,y). \tag{11}$$

(ii) Let $K_{-r,-s}(n)$ be the symmetric tridiagonal family of matrices. That is,

$$k_{1,1} = C_{-r-s}(x,y),\ k_{2,2} = \left\lceil \frac{C_{-2r-s}(x,y)}{C_{-r-s}(x,y)} \right\rceil,$$
$$k_{j,j} = 2C_{-r}(x,y),\ 3 \le j \le n,$$
$$k_{1,2} = k_{2,1} = \sqrt{k_{2,2}C_{-r-s}(x,y) - C_{-2r-s}(x,y)},$$
$$k_{j,j+1} = y^{-r},\ 2 \le j < n,$$
$$k_{j+1,j} = 1,\ 2 \le j < n.$$

Then, we have

$$\det K_{-r,-s}(n) = C_{-rn-s}(x,y). \qquad (12)$$

Proof. The proof can be seen similar to Theorem 2.1 .

3. Conclusion

In this paper, to obtain the bivariate Balancing and Lucas-Balancing polynomials, we define the tridiagonal matrices and present the determinants and inverses of these matrices. By the results in Sections 2 of this paper, we have a major chance to crosscheck and acquire some new properties over these sequences. This is the key goal of this paper. Thus, we enlarge some recent result in the literature. That is, the Balancing and Lucas-Balancing numbers are a special case of $B_n(x,y)$ and $C_n(x,y)$ with $x = y = 1$. k-Balancing and k-Lucas Balancing numbers are $B_n(x,y)$ and $C_n(x,y)$ with $x = k, y = 1$.

In the future studies on the tridiagonal matrices for number sequences, we hope that the following topic will bring a new comprehension. Also, it would be interesting to research the tridiagonal matrices for generalized Balancing, bivariate cobalancing and Lucas-Balancing polynomials.

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