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New Δ_q^v-difference operator and topological features

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Abstract

We extented Δ^v by using difference operator Δ_q^v. We generated the difference sequence space $l_p(\Delta_q^v)$ and investigated some of their properties. We showed that, if $l_p(\Delta_q^v)$ is supplied with an proper norm $\|.\|_{p,\Delta_q^v}$ then it will be a Banach space. We further more showed that, the sequence spaces $\left(l_p(\Delta_q^v),\|.\|_{p,\Delta_q^v}\right)$ and $\left(l_p,\|.\|_p\right)$ are linearly isometric. At the end of this studies, it was shown that $l_p(\Delta_q^v)\subset l_p(\mathcal{M},\Delta_q^v)$. The family of the Orlicz functions $\mathcal{M}$ is coincides the Δ_2-condition.

Keywords: *Difference sequence spaces, isometric sequence spaces, sequence spaces.*

Yeni Δ_q^v-fark operatörü ve topolojik özellikleri

Öz

Δ_q^v fark operatörünü kullanarak Δ^v'yi genişlettik. $l_p(\Delta_q^v)$ fark dizi uzayını oluşturduk ve bazı topolojik özelliklerini inceledik. Eğer $l_p(\Delta_q^v)$ uygun bir $\|.\|_{p,\Delta_q^v}$ normu verilirse bunun bir Banach uzayı olacağını gösterdik. Ayrıca $\left(l_p(\Delta_q^v),\|.\|_{p,\Delta_q^v}\right)$ ve $\left(l_p,\|.\|_p\right)$ dizi uzaylarının lineer izometrik olduklarını gösterdik. Çalışmanın sonunda ise $l_p(\Delta_q^v)\subset l_p(\mathcal{M},\Delta_q^v)$ olduğu gösterildi. Orlicz fonksiyonlarının ailesi $\mathcal{M}$, Δ_2-şartı ile örtüşmektedir.

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Anahtar kelimeler: *Fark dizi uzayları, izometrik dizi uzayları, dizi uzayları.*

1. Introduction

Let c, l_∞ and c_0 be the Banach spaces of convergent, bounded and null sequences $u=(u_k)_1^\infty$ respectively with complex terms, normed by

$$\|u\|_\infty = \sup_k |u_k|,$$

where $k \in \mathbb{N}$.

Kızmaz [1] presented the difference sequence spaces,

$$U(\Delta)=\{u=(u_k):\Delta u\in U\}$$

for $U=c,$ and l_∞, c_0 where

$$\Delta u=(\Delta u_k)=(u_k-u_{k+1}).$$

We have the norm for these Banach spaces as:

$$\|u\|_\Delta = |u_1| + \|\Delta u\|_\infty.$$

Çolak and Et [2] have extended the spaces $U(\Delta)$ to the $U(\Delta^v)$ for $U=c, l_\infty$ and c_0. Let U be any sequence spaces and defined

$$U(\Delta^v)=\{u=(u_k):\Delta^v u\in U\}$$

where $v\in\mathbb{N}$ and $\Delta^v u=\left((\Delta\circ\Delta^{v-1})u_k\right)$ for all $k\in\mathbb{N}$ and prove that $c(\Delta^v), l_\infty(\Delta^v)$ and $c_0(\Delta^v)$ are Banach spaces with the norm

$$\Delta^v u_k=\sum_{t=0}^{v}(-1)^t\binom{v}{t}u_{k+t},\ \|u\|_{\Delta^v}=\sum_{i=1}^{v}|u_i|+\|\Delta^v u\|_\infty .$$

Karakaş et al. [3] have defined the sequence spaces $c(\Delta_q), l_\infty(\Delta_q)$ and $c_0(\Delta_q)$. He also presented

$$\Delta_q u=(\Delta_q u_k)=(qu_k-u_{k+1})$$

for $q\in\mathbb{N}$. Karakaş et al. [4] have presented

$$U(\Delta_q^v)=\{u=(u_k):\Delta_q^v u\in U\}$$

for $U = c, l_\infty$ and c_0, where $q, v \in \mathbb{N}$. They showed that the spaces $U(\Delta_q^v)$ are Banach spaces by:

$$\|u\|_{\Delta_q^v} = \sum_{i=1}^{v} |u_i| + \|\Delta_q^v u\|_\infty ,$$

where

$$\Delta_q^v u = (\Delta_q^v u_k) = (q\Delta_q^{v-1} u_k - \Delta_q^{v-1} u_{k+1})$$

and

$$\Delta_q^v u = (\Delta_q^v u_k) = \sum_{t=0}^{v} (-1)^t \binom{v}{t} q^{v-t} u_{k+t} .$$

Recently, Peralta [5] has studied $l_p(\Delta^v)$ and investigated the topological features of this space. In this work, we choose $p \in [1, \infty)$. By ω, we denote the space of all sequences $u = (u_k)$, for $u_k \in \mathbb{C}$ and all $k \in N$. Taken $u \in \omega$, describe $\|u\|_p = \left(\sum_{k=1}^{\infty} |u_k|^p \right)^{1/p}$ and let

$$l_p = \left\{ u = (u_k) : \|u\|_p < \infty \right\}.$$

The linear operator $\Delta_q^v : \omega \to \omega$ is presented recursively as the composition $\Delta_q^v = \Delta_q \circ \Delta_q^{v-1}$ for $v \geq 2$ and $q \in \mathbb{N}$. It is obvious that for $u \in \omega$ and $v \geq 1$ we have the following Binomial representation

$$\Delta_q^v u_k = \sum_{t=0}^{v} (-1)^t \binom{v}{t} q^{v-t} u_{k+t}$$

for all $k \in \mathbb{N}$.

Let $v \in \mathbb{N}$ and define the sequence space $l_p(\Delta_q^v)$ by

$$l_p(\Delta_q^v) = \left\{ u = (u_k) : \Delta_q^v u \in l_p \right\}.$$

The sequence spaces are Banach spaces normed by

$$\|u\|_{p,\Delta_q^v} = \left(\sum_{i=1}^{v} |u_i|^p + \|\Delta_q^v\|_p^p \right)^{1/p} \tag{1.1}$$

For Euler difference sequence spaces and sequence spaces generated by a sequence of Orlicz functions, the reader can consult Altay and Polat [6], Altay and Başar [7] and Qamaruddin and Saifi [8], respectively.

2. Main results

Theorem 2.1. The sequence space $l_p(\Delta_q^v)$ is a Banach space with the norm $\|\cdot\|_{p,\Delta_q^v}$.

Proof: Let $(u^{(n)}) = \left((u_k^{(n)})\right)$ is a Cauchy sequence in $l_p(\Delta_q^v)$. Thus, for $\varepsilon > 0$ we may find a positive integer N such that

$$\left\|u^{(n)} - u^{(r)}\right\|_{p,\Delta_q^v} < \varepsilon$$

whenever $n, r \geq N$. In other words, we have

$$\left(\sum_{i=1}^{v}\left|u_i^{(n)} - u_i^{(r)}\right|^p + \left\|\Delta_q^v u^{(n)} - \Delta_q^v u^{(r)}\right\|_p^p\right)^{\frac{1}{p}} < \varepsilon,$$

for $n, r \geq N$.

Since

$$\left|u_i^{(n)} - u_i^{(r)}\right| \leq \left\|u^{(n)} - u^{(r)}\right\|_{p,\Delta_q^v}$$

for $i = 1, 2, 3, \ldots, v$ and

$$\left\|\Delta_q^v u^{(n)} - \Delta_q^v u^{(r)}\right\|_p \leq \left\|u^{(n)} - u^{(r)}\right\|_{p,\Delta_q^v}.$$

Therefore, $(u_i^{(n)})$ and $(\Delta_q^v u^{(n)})$ are Cauchy sequences in $\mathbb{C}$ and l_p, respectively. The completeness of the spaces $\mathbb{C}$ and l_p show the existence of elements $y_i \in \mathbb{C}$, $i = 1, 2, 3, \ldots, v$, and $z = (z_k) \in l_p$ such that

$$\lim_n \left|u_i^{(n)} - y_i\right| = 0 \tag{2.1}$$

for $i = 1, 2, 3, \ldots, v$ and

$$\lim_n \left\|\Delta_q^v u^{(n)} - z\right\|_p = 0. \tag{2.2}$$

Since

$$\left|\Delta_q^v u_k^{(n)} - z_k\right| \le \left\|\Delta_q^v u^{(n)} - z\right\|_p$$

we get

$$\left|\Delta_q^v u_k^{(n)} - z_k\right| \to 0$$

as $n \to \infty$ for all $k \in \mathbb{N}$ by equation (2.2).

We obtain a recursive formula for $\lim_n u_{v+i}^{(n)}, i \ge 1,$ as $n \to \infty$. We have

$$(-1)^v u_{v+1}^{(n)} = \Delta_q^v u_1^{(n)} - \sum_{t=0}^{v-1} (-1)^t \binom{v}{t} q^{v-t} u_{t+1}^{(n)}$$

and so

$$w_{v+1} := \lim_n u_{v+i}^{(n)} = (-1)^v \left(z_1 - \sum_{t=0}^{v-1} (-1)^t \binom{v}{t} q^{v-t} y_{v+1} \right)$$

Assume that $w_{v+1}, ..., w_{v+k-1}, 1 < k \le v,$ have been established. Where

$$w_{v+i} : \lim_n u_{v+i}^{(n)}, i = 1, 2, ..., k-1.$$

Using these, we acquire, for $1 < k \le v$

$$w_{v+k} := \lim_n u_{v+k}^{(n)} = (-1)^v \begin{pmatrix} z_k - \sum_{t=0}^{v-k} (-1)^t \binom{v}{t} q^{v-t} y_{t+k} \\ - \sum_{t=1}^{k-1} (-1)^{v-k+t} \binom{v}{v-k+t} q^{k-t} w_{v+t} \end{pmatrix}$$

On the other side, for $k > v$ we get

$$(-1)^v u_{v+k}^{(n)} = \Delta_q^v u_k^{(n)} - \sum_{t=0}^{v-1} (-1)^t \binom{v}{t} q^{v-t} u_{t+k}^{(n)}.$$

So that

$$w_{v+k} := \lim_n u_{v+k}^{(n)} = (-1)^v \left(z_k - \sum_{t=0}^{v-1} (-1)^t \binom{v}{t} q^{v-t} w_{k+t} \right).$$

Let $w = (y_1, ..., y_v, w_{v+1}, w_{v+2}, ...)$. We assert that $w \in l_p(\Delta_q^v)$, that is, $\Delta_q^v w \in l_p$. First, show that

$$(\Delta_q^v w)_1 = \sum_{t=0}^{v-1}(-1)^t \binom{v}{t} q^{v-t} y_{t+1} + (-1)^v w_{v+1}$$
$$= \sum_{t=0}^{v-1}(-1)^t \binom{v}{t} q^{v-t} y_{t+1} + \left[z_1 - \sum_{t=0}^{v-1}(-1)^t \binom{v}{t} q^{v-t} y_{t+1} \right]$$
$$= z_1$$

Also, for $k = 2,3,...,v$. We get

$$(\Delta_q^v w)_k = \sum_{t=0}^{v-k}(-1)^t \binom{v}{t} q^{v-t} y_{t+k} + \sum_{t=v-k+1}^{v-1}(-1)^t \binom{v}{t} q^{v-t} w_{t+k} + (-1)^v w_{v+k}$$
$$= z_k$$

Similarly, for $k > v$ we acquire

$$(\Delta_q^v w)_k = \sum_{t=0}^{v-1}(-1)^t \binom{v}{t} q^{v-t} w_{t+k} + (-1)^v w_{v+k}$$
$$= z_k.$$

Thus we have presented that $\Delta_q^v w = z \in l_p$. It remains to prove that

$$\left\| u^{(n)} - w \right\|_{p,\Delta_q^v} \to 0 \text{ as } n \to \infty$$

Then, we obtain

$$\lim_n \left\| u^{(n)} - w \right\|_{p,\Delta_q^v}^p = \lim_n \left(\sum_{k=1}^{v} \left| u_k^{(n)} - y_k \right|^p + \left\| \Delta_q^v u^{(n)} - \Delta_q^v w \right\| \right)$$
$$= \sum_{k=1}^{v} \lim_n \left| u_k^{(n)} - y_k \right|^p + \lim_n \left\| \Delta_q^v u^{(n)} - z \right\|_p^p$$
$$= 0.$$

This is proof of the theorem.

Theorem 2.2. The sequence spaces $\left(l_p(\Delta_q^v), \|.\|_{p,\Delta_q^v} \right)$ and $\left(l_p, \|.\|_p \right)$ are linearly isometric.

Proof: Take in to consideration the map $T : l_p(\Delta_q^v) \to l_p$ given by $Ty = u$, where $y = (y_k) \in l_p(\Delta_q^v)$ and $u = (u_k)$ with

$$u_k = \begin{cases} y_k, & \text{if } 1 \le k \le v; \\ \Delta_q^v y_{k-v}, & \text{if } k > v. \end{cases}$$

The linearity of the difference operator Δ refers the linearity of T. If $y \in l_p(\Delta_q^v)$ and $Ty = u$, then

$$\|Ty\|_p^p = \|u\|_p^p = \sum_{k=1}^{v} |y_k|^p + \sum_{k=v+1}^{\infty} \left|\Delta_q^v y_{k-v}\right|^p$$
$$= \sum_{k=1}^{v} |y_k|^p + \sum_{k=1}^{\infty} \left|\Delta_q^v y_k\right|^p$$
$$= \|y\|_{p,\Delta_q^v}^p < \infty.$$

This demonstrates that T is well-defined and it is also norm preserving. We presented that T is one-to-one and onto. Assume that $Ty = 0$.
Then, we obtain

$$\Delta_q^v y_k = 0 \text{ for all } k \geq 1, \tag{2.3}$$
$$y_1 = y_2 = ... = y_v = 0. \tag{2.4}$$

We show that the difference equation (2.3) with initial conditions (2.4) refers that $y_k = 0$ for all $k \geq 1$, , that is, $y = (0,0,...)$. Therefore, T is one-to-one.

Assume that $u = (u_k) \in l_p$. Describe the sequence $y = (y_k)$ as follows. Let $y_k = u_k$ for $u_{k+v} = \Delta_q^v u_k, k = 1,2,...,v.$

The succeeding terms of the sequence y is then showed recursively by

$$y_{v+1} = (-1)^v \left[u_{v+1} - \sum_{t=0}^{v-1} (-1)^t \binom{v}{t} q^{v-t} u_{t+1} \right]$$
$$y_{v+k} = (-1)^v \left[\begin{array}{l} u_{v+k} - \sum_{t=0}^{v-k} (-1)^t \binom{v}{t} q^{v-t} u_{t+k} \\ -\sum_{t=1}^{k-1} (-1)^{t}{}^{v-k+t} \binom{v}{v-k+t} q^{k-t} y_{v+t} \end{array} \right], \ 1 < k \leq v$$

and

$$y_{v+k} = (-1)^v \left[u_{v+k} - \sum_{t=0}^{v-1} (-1)^t \binom{v}{t} q^{v-t} y_{t+k} \right], \ k > v.$$

Utilizing a similar argument as in the proof of the previous theorem, we prove that

$$\Delta_q^v y_k = u_{k+v}$$

for $k \in \mathbb{N}$. Therefore it follows that $Ty = u$.

Thus, we obtain

$$\begin{aligned}\left\|\Delta_q^v y\right\|_p^p &= \sum_{k=1}^{\infty}\left|\Delta_q^v y_k\right|^p \\ &= \sum_{k=1}^{\infty}\left|u_{k+v}\right|^p \\ &= \left\|u\right\|_p^p < \infty.\end{aligned}$$

So that $y \in l_p(\Delta_q^v)$. Since T is onto, $l_p(\Delta_q^v)$ and l_p are linearly isometric.

Definition 2.3. An Orlicz function is a continuous, convex function and nondecreasing $M:[0,\infty)\to[0,\infty)$ such that $M(z)=0$, if and only if $z=0$, $M(u)>0$, and $M(u)\to\infty$ as $u\to\infty$. M is said to fulfil Δ_2-condition if there exists a positive constant K such that $M(2z)\le KM(z)$ for all $z\ge 0$. Let $\mathcal{M}=(M_k)$ be a sequence of Orlicz functions meeting the Δ_2-condition [9]. An Orlicz function M has been defined in [10] also see [11] for a more general representation in thise direction in the following from:

$$M(u)=\int_0^u p(t)dt$$

where p, know as the kernel of M, is right-differentable for $t\ge 0, p(t)>0, p(0)=0$ for $t>0,$ p is nondecreasing, and $t\to\infty, p(t)\to\infty.$

Lindenstrauss and Tzafriri [12] have utilized the view of Orlicz function to find the sequence space,

$$l_p(\mathcal{M})=\left\{u=(u_k):\sum_{k=1}^{\infty}\left|M_k\left(\left|u_k\right|/\rho\right)\right|^p<\infty, \text{for some } \rho>0\right\},$$

which is a Banach Spaces with respect to the norm

$$\left\|(u_k)\right\|=\inf\left\{\rho>0:\sum_{k=1}^{\infty}\left|M_k\left(\left|u_k\right|/\rho\right)\right|\le 1\right\}.$$

The space $l(\mathcal{M})$ is closely related to space l_p, which is an Orlicz sequence space with $M(u)=\left|u\right|^p$, for $1\le p<\infty$.

Describe the sequence spaces as:

$$l_p(\mathcal{M})=\left\{u=(u_k):\sum_{k=1}^{\infty}\left|M_k\left(\left|u_k\right|/\rho\right)\right|^p<\infty, \text{for some } \rho>0\right\},$$

and

$l_p(\mathcal{M}, \Delta_q^v) = \{u = (u_k) : \Delta_q^v u \in l_p(\mathcal{M})\}$.

Theorem 2.4. Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions fulfil the $\Delta_2 -$condition. If

$$\sum_{k=1}^{\infty} \left| M_k\left(|u_k|/\rho\right) \right|^p < \infty \qquad (2.5)$$

for all $t, \rho > 0$ then $l_p(\Delta_q^v) \subset l_p(\mathcal{M}, \Delta_q^v)$.

Proof: Assume that condition (2.5) exists and let $u = (u_k) \in l_p(\Delta_q^v)$. Then, we get

$$\sum_{k=1}^{\infty} \left| \Delta_q^v u_k \right|^p < \infty. \qquad (2.6)$$

The convergence of

$$\sum_{k=1}^{\infty} \left| \Delta_q^v u_k \right|^p < \infty$$

implies that

$$\lim_k \left| \Delta_q^v u_k \right| = 0.$$

Thus, we can find $n \in \mathbb{N}$ such that $\left| \Delta_q^v u_k \right| \leq 1$ for all $k \geq N$.

Let

$$K = \max\left\{ \left| \Delta_q^v u_1 \right|, \ldots \left| \Delta_q^v u_{N-1} \right|, 1 \right\}.$$

Then $\left| \Delta_q^v u_k \right| \leq K$ for all $k \in \mathbb{N}$. For $\rho > 0$, utilizing the monotonicity of M_k , we get $M_k\left(\left| \Delta_q^v u_k \right| / \rho \right) \leq M_k\left(K/\rho \right)$ for all $k \in \mathbb{N}$.

This inequality shows that

$$\sum_{k=1}^{\infty} \left| M_k\left(\left| \Delta_q^v u_k \right| / \rho \right) \right|^p \leq \sum_{k=1}^{\infty} \left| M_k\left(K/\rho \right) \right|^p .$$

This estimate proves that $\Delta_q^v u \in l_p(\mathcal{M})$ that is, $u \in l_p(\mathcal{M}, \Delta_q^v)$. By equation (2.5) Therefore, the inclusion $l_p(\Delta_q^v) \subset l_p(\mathcal{M}, \Delta_q^v)$ holds.

3. Results and discussion

Peralta [5] studied $l_p(\Delta_q^v)$ and checked the topological properties of this space. Later Karakaş et al. [4] defined difference operator Δ_q^v. We used Peralta' s [5] studies and extented it by used the generalized difference operator Δ_q^v. We generated the difference sequence space $l_p(\Delta_q^v)$ and $\|.\|_{p,\Delta_q^v}$, and investigated some of their properties. We showed that, if $l_p(\Delta_q^v)$ is equipped with an appropriate norm $\|.\|_{p,\Delta_q^v}$ is a Banach space. We further more showed that, the sequence spaces $\left(l_p(\Delta_q^v), \|.\|_{p,\Delta_q^v}\right)$ and $\left(l_p, \|.\|_p\right)$ are linearly isometric. It is shown that $l_p(\Delta_q^v) \subset l_p(\mathcal{M}, \Delta_q^v)$. Where $\mathcal{M}$ a family of Orlicz functions, is coincides the Δ_2-condition.

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