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New Integral Inequalities for Co-Ordinated Convex Functions

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Abstract: In this paper, we prove some new integral inequalities for co-ordinated convex functions by using a new lemma and fairly elementary analysis.

Keywords: Co-ordinated convex functions, Hadamard-type inequalities, Hölder inequality, power mean inequality.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is well known in the literature as Hadamard's inequality. Several papers have been written on convexity and inequalities (see [16–27] and references there in).

In [13], Dragomir defined convex functions on the co-ordinates as following:

Definition 1.1 *Let us consider the bidimensional interval* $\Delta = [a,b] \times [c,d]$ *in* $\mathbb{R}^2$ *with* $a < b$, $c < d$. *A function* $f : \Delta \to \mathbb{R}$ *will be called convex on the co-ordinates if the partial mappings* $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ *and* $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ *are convex where defined for all* $y \in [c,d]$ *and* $x \in [a,b]$. *Recall that the mapping* $f : \Delta \to \mathbb{R}$ *is convex on* Δ *if the following inequality holds,*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w)$$

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for all $(x,y),(z,w)\in\Delta$ *and* $\lambda\in[0,1]$.

In [13], Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1.2 *Suppose that* $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ *is convex on the co-ordinates on* Δ. *Then one has the inequalities;*

$$
\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \qquad (1)\\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\\
\leq\ & \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx+\frac{1}{(b-a)}\int_a^b f(x,d)dx\right.\\
& \left.+\frac{1}{(d-c)}\int_c^d f(a,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)dy\right]\\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
$$

The above inequalities are sharp.

In [1], Bakula and Pečarić established several Jensen type inequalities for co-ordinated convex functions and in [14], Hwang *et al.* gave a mapping F, discussed some properties of this mapping and proved some Hadamard-type inequalities for Lipschizian mapping in two variables. In [2], Özdemir *et al.* established new Hadamard-type inequalities for co-ordinated $m-$convex and $(\alpha,m)-$convex functions. On all of these, in [11], Sarıkaya *et al.* proved some Hadamard-type inequalities for co-ordinated convex functions as followings:

Theorem 1.3 *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta:=[a,b]\times[c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$ *and* $c<d$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is a convex function on the co-ordinates on* Δ, *then one has the*

inequalities:

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \tag{2}$$

$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right|$$

$$\leq \frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{4}\right)$$

where

$$A=\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b\left[f(x,c)+f(x,d)\right]dx+\frac{1}{(d-c)}\int_c^d\left[f(a,y)dy+f(b,y)\right]dy\right].$$

Theorem 1.4 *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \tag{3}$$

$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right|$$

$$\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}$$

where

$$A=\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b\left[f(x,c)+f(x,d)\right]dx+\frac{1}{(d-c)}\int_c^d\left[f(a,y)dy+f(b,y)\right]dy\right]$$

and $\frac{1}{p}+\frac{1}{q}=1.$

Theorem 1.5 *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\geq 1$, is a convex function on the co-ordinates on Δ, then one*

has the inequalities:

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right| \\
\leq\ &\frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}
\end{aligned} \tag{4}$$

where

$$A=\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b\left[f(x,c)+f(x,d)\right]dx+\frac{1}{(d-c)}\int_c^d\left[f(a,y)dy+f(b,y)\right]dy\right].$$

In [15], Özdemir *et al.* proved following inequalities for co-ordinated convex functions.

Theorem 1.6 *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is a convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \\
&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right| \\
\leq\ &\frac{(b-a)(d-c)}{64}\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right].
\end{aligned} \tag{5}$$

Remark 1.7 *Suppose that all the assumptions of Theorem 5 are satisfied. If we choose $\frac{\partial^2 f}{\partial t\partial s}$ is bounded, i.e.,*

$$\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty=\sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|<\infty,$$

we get

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \qquad (6)\\
&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right|\\
\leq\ & \frac{(b-a)(d-c)}{16}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty .
\end{aligned}$$

Theorem 1.8 *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right. \qquad (7)\\
&\left.-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\right|\\
\leq\ & \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

Remark 1.9 *Suppose that all the assumptions of Theorem 6 are satisfied. If we choose $\frac{\partial^2 f}{\partial t\partial s}$ is bounded, i.e.,*

$$\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty=\sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|<\infty,$$

we get

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \qquad (8)\\
&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right|\\
\leq\ & \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty .
\end{aligned}$$

Theorem 1.10 *Let* $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta=[a,b]\times[c,d]$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\geq 1$, *is a convex function on the co-ordinates on* Δ, *then the following inequality holds;*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right. \qquad (9)\\&\left.-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\right|\\ \leq\ &\frac{(b-a)(d-c)}{16}\\&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}.\end{aligned}$$

Recent results, generalizations, new methods and further properties of different kinds of co-ordinated convex functions can be found in [3–12].

The main purpose of this paper is to prove some new inequalities of Hadamard-type for co-ordinated convex functions on the co-ordinates by using a new Lemma. These estimations give new upper bounds.

2. Main Results

To prove our main results, we need following Lemma.

Lemma 2.1 *Let* $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta=[a,b]\times[c,d]$. *If* $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, *then the following equality holds:*

$$\begin{aligned}\Lambda&=f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\end{aligned}$$

$$= \frac{(b-a)(d-c)}{16}\left[\int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)dsdt\right.$$

$$+\int_0^1\int_0^1 (t-1)(s-1)\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)dsdt$$

$$+\int_0^1\int_0^1 t(s-1)\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, sd+(1-s)\frac{c+d}{2}\right)dsdt$$

$$\left.+\int_0^1\int_0^1 s(t-1)\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)dsdt\right]$$

for all $t, s \in [0,1]$.

Proof Let we start with the following statement

$$I_1 = \int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)dsdt.$$

Integration by parts, we can write

$$\int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)dsdt$$

$$= \int_0^1 t\left[\frac{2s}{d-c}\frac{\partial f}{\partial t}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)\Big|_0^1\right.$$

$$\left.-\frac{2}{d-c}\int_0^1 \frac{\partial f}{\partial t}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)ds\right]dt$$

$$= \frac{2}{d-c}\int_0^1 t\frac{\partial f}{\partial t}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)dt$$

$$-\frac{2}{d-c}\int_0^1\int_0^1 t\frac{\partial f}{\partial t}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)dsdt.$$

By integrating again, we get

$$I_1$$

$$= \int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)dsdt$$

$$= \frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_0^1 f\left(t\frac{a+b}{2}+(1-t)a,\frac{c+d}{2}\right)dt\right.$$

$$\left.-\int_0^1 f\left(\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)ds+\int_0^1\int_0^1 f\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)dtds\right].$$

Analogously, we can easily obtain the followings

$$
\begin{aligned}
& I_2 \\
= & \int_0^1 \int_0^1 (t-1)(s-1) \frac{\partial^2 f}{\partial t \partial s}\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) dsdt \\
= & \frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_0^1 f\left(tb+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)dt\right. \\
& \left.-\int_0^1 f\left(\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)ds+\int_0^1\int_0^1 f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)dtds\right],
\end{aligned}
$$

$$
\begin{aligned}
& I_3 \\
= & \int_0^1 \int_0^1 t(s-1) \frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2}+(1-t)a, sd+(1-s)\frac{c+d}{2}\right) dsdt \\
= & \frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_0^1 f\left(t\frac{a+b}{2}+(1-t)a,\frac{c+d}{2}\right)dt\right. \\
& \left.-\int_0^1 f\left(\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)ds+\int_0^1\int_0^1 f\left(t\frac{a+b}{2}+(1-t)a, sd+(1-s)\frac{c+d}{2}\right)dtds\right]
\end{aligned}
$$

and

$$
\begin{aligned}
& I_4 \\
= & \int_0^1 \int_0^1 (t-1)s \frac{\partial^2 f}{\partial t \partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right) dsdt \\
= & \frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_0^1 f\left(tb+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)dt\right. \\
& \left.-\int_0^1 f\left(\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)ds+\int_0^1\int_0^1 f\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)dtds\right].
\end{aligned}
$$

By adding $I_1,\ I_2,\ I_3,\ I_4$ and changing of the variables, we get the desired result. □

Theorem 2.2 *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is co-ordinated convex function on the co-ordinates on Δ, then the following inequality holds;*

$$
\begin{aligned}
& |\Lambda| \qquad (10) \\
\leq & \frac{(b-a)(d-c)}{144}\left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{4}\right. \\
& +4\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right| \\
& \left.+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|\right].
\end{aligned}
$$

Proof From Lemma 1 and by using the absolute value for the both sides of the equality, we can write

$$
\begin{aligned}
&|\Lambda| \\
=\ & \frac{(b-a)(d-c)}{16}\left[\int_0^1\int_0^1 ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)\right|dsdt\right. \\
&+\int_0^1\int_0^1 |(t-1)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|dsdt \\
&+\int_0^1\int_0^1 |t(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, sd+(1-s)\frac{c+d}{2}\right)\right|dsdt \\
&\left.+\int_0^1\int_0^1 |s(t-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|dsdt\right].
\end{aligned}
$$

If we use the co-ordinated convexity of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$, we have

$$
\begin{aligned}
&|\Lambda| \\
=\ & \frac{(b-a)(d-c)}{16}\left[\int_0^1\int_0^1 ts\left\{t\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|\right.\right. \\
&\left.+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a, s\frac{c+d}{2}+(1-s)c\right)\right|\right\}dsdt \\
&+\int_0^1\int_0^1 |(t-1)(s-1)|\left\{t\left|\frac{\partial^2 f}{\partial t\partial s}\left(b, sd+(1-s)\frac{c+d}{2}\right)\right|\right. \\
&\left.+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|dsdt\right\} \\
&+\int_0^1\int_0^1 |t(s-1)|\left\{t\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|dsdt\right. \\
&\left.+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a, sd+(1-s)\frac{c+d}{2}\right)\right|dsdt\right\} \\
&+\int_0^1\int_0^1 |s(t-1)|\left\{t\left|\frac{\partial^2 f}{\partial t\partial s}\left(b, s\frac{c+d}{2}+(1-s)c\right)\right|\right. \\
&\left.\left.+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|dsdt\right\}\right].
\end{aligned}
$$

By computing these integrals, we obtain

$$
\begin{aligned}
&|\Lambda| \\
= &\frac{(b-a)(d-c)}{16}\left[\frac{1}{3}\int_0^1 s\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|ds\right. \\
&+\frac{1}{6}\int_0^1 s\left|\frac{\partial^2 f}{\partial t\partial s}\left(a, s\frac{c+d}{2}+(1-s)c\right)\right|ds \\
&+\frac{1}{6}\int_0^1 |s-1|\left|\frac{\partial^2 f}{\partial t\partial s}\left(b, sd+(1-s)\frac{c+d}{2}\right)\right|ds \\
&+\frac{1}{3}\int_0^1 |s-1|\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|ds \\
&+\frac{1}{3}\int_0^1 |s-1|\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|ds \\
&+\frac{1}{6}\int_0^1 |s-1|\left|\frac{\partial^2 f}{\partial t\partial s}\left(a, sd+(1-s)\frac{c+d}{2}\right)\right|ds \\
&+\frac{1}{6}\int_0^1 s\left|\frac{\partial^2 f}{\partial t\partial s}\left(b, s\frac{c+d}{2}+(1-s)c\right)\right|ds \\
&\left.+\frac{1}{3}\int_0^1 s\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|ds\right].
\end{aligned}
$$

By using co-ordinated convexity of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ again and computing the integrals, we get desired result.

□

Remark 2.3 *Suppose that all the assumptions of Theorem 8 are satisfied. If we choose* $\frac{\partial^2 f}{\partial t\partial s}$ *is bounded, i.e.,*

$$
\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty = \sup_{(t,s)\in(0,1)\times(0,1)}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right| < \infty,
$$

we get

$$
|\Lambda| \leq \frac{(b-a)(d-c)}{16}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty
$$

which is the inequality (6).

Theorem 2.4 *Let* $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta=[a,b]\times[c,d]$.

If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ *is a convex function on the co-ordinates on* Δ*, then the following inequality holds;*

$$\begin{aligned}
&|\Lambda| &(11)\\
\leq\ & \frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|}{2}\right.\\
&\times\left(\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}
\end{aligned}$$

where $q>1$.

Proof From Lemma 1, we have

$$\begin{aligned}
&|\Lambda|\\
=\ & \frac{(b-a)(d-c)}{16}\left[\int_0^1\int_0^1 ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)\right|dsdt\right.\\
&+\int_0^1\int_0^1|(t-1)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right|dsdt\\
&+\int_0^1\int_0^1|t(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,sd+(1-s)\frac{c+d}{2}\right)\right|dsdt\\
&\left.+\int_0^1\int_0^1|s(t-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)c\right)\right|dsdt\right].
\end{aligned}$$

By applying the well-known Hölder inequality for double integrals, then one has

$$\begin{aligned}
&|\Lambda| &(12)\\
=\ & \frac{(b-a)(d-c)}{16}\left[\left(\int_0^1\int_0^1(ts)^p\,dsdt\right)^{\frac{1}{p}}\right.\\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)\right|^q dsdt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1|(t-1)(s-1)|^p\,dsdt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right|^q dsdt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1|t(s-1)|^p\,dsdt\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a,sd+(1-s)\,\frac{c+d}{2}\right)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1|s\,(t-1)|^p\,dsdt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\,\frac{a+b}{2},s\frac{c+d}{2}+(1-s)\,c\right)\right|^q dsdt\right)^{\frac{1}{q}}\Bigg].$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is a co-ordinated convex function on Δ, we can write for all $(t,s)\in[0,1]\times[0,1]$

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a,s\frac{c+d}{2}+(1-s)\,c\right)\right|^q \qquad (13)\\
\leq\;& ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\\
&+t\,(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\\
&+(1-t)\,s\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q\\
&+(1-t)\,(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\,(a,c)\right|^q.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\,\frac{a+b}{2},sd+(1-s)\,\frac{c+d}{2}\right)\right|^q \qquad (14)\\
\leq\;& ts\left|\frac{\partial^2 f}{\partial t\partial s}\,(b,d)\right|^q\\
&+t\,(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q\\
&+(1-t)\,s\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q\\
&+(1-t)\,(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q,
\end{aligned}$$

$$\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, sd+(1-s)\frac{c+d}{2}\right)\right|^q \tag{15}$$

$$\begin{aligned}\leq\ & ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q \\ & +t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \\ & +(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \\ & +(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q,\end{aligned}$$

$$\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|^q \tag{16}$$

$$\begin{aligned}\leq\ & ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q \\ & +t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q \\ & +(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \\ & +(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q.\end{aligned}$$

Using inequalities of (13)-(16) in (12), we get

$$\begin{aligned}&|\Lambda| \\ \leq\ & \frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|}{2}\right. \\ & \times\left(\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}\end{aligned}$$

This completes the proof. □

Remark 2.5 *Suppose that all the assumptions of Theorem 9 are satisfied. If we choose* $\frac{\partial^2 f}{\partial t\partial s}$ *is bounded, i.e.,*

$$\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty=\sup_{(t,s)\in(0,1)\times(0,1)}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|<\infty,$$

we get

$$|\Lambda| \leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty$$

which is the inequality (8).

Theorem 2.6 *Let $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$. If $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q$ is a convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned} &|\Lambda| \\ \leq\ & \frac{(b-a)(d-c)}{16} \left[\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{36} \right. \\ & \left. + \frac{4\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, d\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right|^q}{9} \right]^{\frac{1}{q}}. \end{aligned}$$

Proof From Lemma 1, we have

$$\begin{aligned} &|\Lambda| \\ =\ & \frac{(b-a)(d-c)}{16} \left[\int_0^1 \int_0^1 ts \left| \frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2} + (1-t)a, s\frac{c+d}{2} + (1-s)c \right) \right| dsdt \right. \\ & + \int_0^1 \int_0^1 |(t-1)(s-1)| \left| \frac{\partial^2 f}{\partial t \partial s}\left(tb + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2} \right) \right| dsdt \\ & + \int_0^1 \int_0^1 |t(s-1)| \left| \frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2} + (1-t)a, sd + (1-s)\frac{c+d}{2} \right) \right| dsdt \\ & \left. + \int_0^1 \int_0^1 |s(t-1)| \left| \frac{\partial^2 f}{\partial t \partial s}\left(tb + (1-t)\frac{a+b}{2}, s\frac{c+d}{2} + (1-s)c \right) \right| dsdt \right]. \end{aligned}$$

By applying the well-known Power mean inequality for double integrals, then one has

$$
\begin{aligned}
&|\Lambda| \qquad (17)\\
=\ & \frac{(b-a)(d-c)}{16}\left[\left(\int_0^1\int_0^1 (ts)\,dsdt\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^1\int_0^1 (ts)\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)\right|^q dsdt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1 |(t-1)(s-1)|\,dsdt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\int_0^1 |(t-1)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|^q dsdt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1 |t(s-1)|\,dsdt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\int_0^1 |t(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, sd+(1-s)\frac{c+d}{2}\right)\right|^q dsdt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1 |s(t-1)|\,dsdt\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\int_0^1\int_0^1 |s(t-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|^q dsdt\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is a co-ordinated convex function on Δ, we can write for all $(t,s)\in[0,1]\times[0,1]$

$$
\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)\right|^q \qquad (18)\\
\leq\ & ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\\
&+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\\
&+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q\\
&+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q.
\end{aligned}
$$

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|^q \qquad (19)\\
\leq\ & ts\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\\
&+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q\\
&+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q\\
&+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q,
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, sd+(1-s)\frac{c+d}{2}\right)\right|^q \qquad (20)\\
\leq\ & ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q\\
&+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\\
&+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\\
&+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q,
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|^q \qquad (21)\\
\leq\ & ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q\\
&+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\\
&+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\\
&+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q.
\end{aligned}$$

If we use (18)-(21) in (17), we get

$$
\begin{aligned}
&|\Lambda| \\
\leq\ & \frac{(b-a)(d-c)}{16}\left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{36}\right. \\
&\left.+\frac{4\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q}{9}\right]^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof. □

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