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Investigation of the Effects of Nonverbal Proof Based Education for Preservice Mathematics Teachers

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ABSTRACT

Nonverbal proofs are diagrams or illustrations that will help us see what a mathematical expression means, why it is true, and how it is proved. The aim of this study is to examine the effects of nonverbal proof-based education on preservice mathematics teachers. The study was conducted using *case study* research methods, one of the qualitative research designs. The participants of the study consisted of 31 preservice mathematics teachers. The data were collected in writing at the beginning and end of the process with questions directed to preservice teachers. These questions in the data collection tool were used to compare the responses of the preservice teachers who had an experience with nonverbal proof in the pre and post assesment and compare the changes. In the analysis of the data, descriptive analysis, which is a qualitative data analysis, was used. Firstly, each preservice teacher's responses in the post assesment are classified according to their similarities and differences and then are categorized. Then, the answers in the pre assesment were examined and the responses were compared whether there was a change or not. The findings of the study showed that the preservice mathematics teachers' experience of nonverbal proof effect on the recognition of the given image and establishing a relationship with different mathematics subjects.

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1. Introduction

One of the most important aims of mathematics education is to gain mathematical thinking skills. Developing mathematical thinking skills is getting through enabling students to give answers to question why, to reason, to produce new ideas. Proving plays an important role both in the development of these skills and in learning mathematics (Knuth, 2002a). The ability to make proof starts to develop with the acquisition of skills such as classification, matching, sorting and comparison in early ages even in pre-school period. It is therefore also a part of the mathematics curriculum at all levels from pre-school to 12th grade (NCTM, 2000). The proof emerges as a skill when it is defined as showing the accuracy of an assumption or expression (Heinze & Reiss, 2004) or convincing someone about its accuracy (Almeida, 1996). On the other hand, the proof is defined as the evidence for the accuracy of an expression and (Rodd, 2000) is a series of expressions based on previously proven axioms and theorems (Morash, 1987). From this point of view, proof is explained as an evidence of the truth of an expression. In the other word, proof is what a claim means (Hanna, 2000). All of these definitions are explained by the functions of proving. Considering the functions of proof, proof is a good tool for communication (Sekiguchi, 2002). It is therefore at the center of both mathematics and mathematics education (Ball,

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Hoyles, Jahnke & Movshovitz-Hadar, 2002; Knuth, 2002a; 2002b). In spite of this, being able to make proof, to explain proof is a skill that is difficult to develop in every period (Jones, 2000). Studies have shown that students and preservice teachers have difficulty in making proof (Harel and Sowder, 1998; 2007; Jones, 2000; Almeida, 2001; Weber, 2001). Although it is a challenging subject or skill, one of the alternative methods to improve the skill of proof is to gain proof of visualization that is, using nonverbal proofs.

Nonverbal proofs

Alsina and Nelsen (2010) described nonverbal proofs as "visual arguments". Seeing is believing (Mudaly, 2013) or a picture is better than a thousand words (Casselman, 2000; Thornton, 2001; Rösken & Rolka, 2006). Indeed, Maanen (2006) stated that a well-defined diagram could explain more than thousands of words and that this could be achieved especially by nonverbal proofs. In fact, despite the fact that visual drawings, pictures and proofs have a long history (Foo, Pagnucco & Nayak, 1999), the discussions about whether nonverbal proofs are a proof method have continued. The visual to proof the expression *"all the triangles are the isosceles"* in Euclid's book of elements (Wallace & West, 2004) was the mainstay of these debates. In his geometry book Davis (1993) states that when Euclid sees this proof, even though his inference was *"when I will believe what I see"*; he stated that the visuals used for proof needed to be explained orally. Norman (2003) suggested that a theorem should be proved independently from each other, based on Euclid's visual evidence. Therefore, the proof of an expression is accepted when it is proved independent of the visuals, but the use of visuals while explaining why this proof is correct is a widespread admission that will facilitate the understanding of the proof. Even if this aspect is not considered as proof, nonverbal proofs are an effective tool that can be used in mathematics teaching or in developing proof skills.

According to Alsina and Nelsen (2010), nonverbal proofs are pictures or diagrams that help to see *why* a particular mathematical expression can be true and how to begin to prove that expression. Similar to this definition, Bell (2011) has described a proof of a mathematical expression as a mathematical drawing, without proving a formal argument by words. In these definitions, it is seen that nonverbal proofs are tools that help to understand the proof. But Bardelle (2009) defined the deductive steps as proofs based on shapes, diagrams and graphs. In its most general form, nonverbal proof is a proof that uses visual representations, meaning that pictures and visuals represent a mathematical equation, theorem or idea (Casselman, 2000; Gierdien, 2007).

Nonverbal proofs play an important role in mathematics teaching at all levels from primary to university level (Alsina & Nelsen, 2010). In many fields such as algebra, analysis, trigonometry, ... (Hammack & Lyons, 2006; Stucky, 2015) there are examples of nonverbal proof. It can also be used in the history of mathematics (Bell, 2011). There are indeed examples of verbal verbal proof with almost any subject. These examples are found in various websites and in papers (Gierdien, 2007; Bell, 2011) and in two books written by Nelsen (1993, 2000). It is stated that nonverbal proofs are useful, pedagogical (Stucky, 2005) and a valuable tool (Miller, 2012), which play an important role in the process of understanding various mathematical expressions and proofs (Štrausová & Hasek; 2013). Miller (2012) states that a student who can demonstrate the accuracy of the formula with the first n integer cannot be convinced why this is true, and that nonverbal proofs can be effective. In addition, it is stated that the students find it more interesting than classical proofs (Štrausová & Hasek, 2013), and that they are effective in understanding the proof process (Bell, 2011).

Although there are many examples of nonverbal proof in almost every subject in the literature, there are not many studies about the effects of nonverbal proofs on teaching, in-class practices and proofing skills. Generally, the studies are related to the theoretical structure of nonverbal proofs, their history and the examples (Maanen, 2006; Alsina & Nelsen, 2010; Miller, 2012), examples of in-class applications (Gierdien, 2007; Bell, 2011), student difficulties in nonverbal proof applications (Bardelle, 2009), the approaches of the talented secondary school students' creating nonverbal proofs (Uğurel, Moralı & Karahan, 2011), and the opinions of the preservice teachers about the proof of concrete models (Doruk, Kıymaz & Horzum, 2012).This course focuses on the opinions of talented secondary school students about the nonverbal proofing approach (Uğurel, Moralı & Karahan, 2011), and the opinions of the preservice teachers about making proof with concrete models (Doruk, Kıymaz & Horzum, 2012). Flores (1992) presented mathematical induction and nonverbal proofs and stated that the students liked the alternative approach with nonverbal proofs in mathematical induction and they made a better sense of the transition from k step to k+1 step. Hammack and Lyons (2006) stated that with the visual proof for the convergence of alternating series, the success has increased and the students are clearer than the formal proofs. Demircioğlu and Polat (2015) stated in their study that preservice teachers stated that they are effective in gaining skills of proof, problem solving, comprehension, mental, reasoning, generalization, processing, analysis and synthesis, seeing and thinking. Moreover, they showed that the most difficult parts of the nonverbal proofs were the lack of understanding, the lack of explanation, the lack of a relationship between nonverbal proof and the algebraic proof, the lack of knowledge of the field, the shortage of resources. Since only one visual is used in literal proofs, it is an important skill to recognize when we see it, to make proof by using the visual and to make comments. According to Bardelle (2009), visual proofs are those which are not presented with any comments in oral language (that is nonverbal), which are based on diagrams, maybe numbers, letters, arrows, dots and symbolic expressions related to each other, whose construction are left to the reader. However, if there are difficulties with proof in the construction of the proof, as Davis (1993) stated, these visuals need to be explained. Indeed, since only one visual is used in nonverbal proofs, it is an important skill to recognize when we see it, to make proof by using the visual and to make comments. On the other hand, considering individual differences in students, we should provide opportunity students to have their own definitions, their own theorems, their own proofs, as Lockhart (2009) stated. In this context, to provide students with a rich learning environment for the purpose of conducting ideas and associating is also among the objectives of the curriculum.

Alternative proof methods such as nonverbal proof will help both to gain different perspectives and to understand and interpret the proof. In this context, it is aimed in this study to introduce preservice mathematics teaches to both nonverbal proofs, to provide a learning environment with more nonverbal proofs, to examine the effects of such an experience, and to offer suggestions in accordance with the findings.

2. Method

In this study, a case study of qualitative research designs was used to examine the effects of a nonverbal proof for mathematics preservice teachers. Qualitative methods are based on text and imaginary data and have specific steps in data analysis. Case study is used to identify and view the details that make up an event, to develop possible explanations for an event, and to evaluate an event. For this reason, case study design, which is one of the qualitative research methods, was used in this study because the opinions of the individuals were consulted.

2.1. Participants

The participants of the study consisted of 57 preservice mathematics teachers who took the research project in "Research Project in Field Education" which was delivered as 4 hours per week in a state university in Central Anatolia. However, some of these 57 preservice teachers did not participate in the pre or post- assesment. Therefore, the data obtained from 31 preservice teachers who participated in both pre, post- assesment were interpreted, and comparisons were made.

2.2. The process

The application process was carried out within the scope of "Research project in Field Education", course which was delivered for 4 hours in the last year of mathematics teaching department in a state university in Central Anatolia. Before the application process, data collection tool consisting of three nonverbal proofs was applied to the participants. The same tool was applied at the end of the process. In this process, the visuals in this tool were not included. During the first three weeks of the application process, theoretical information related to nonverbal proofs was given and different nonverbal proof examples and animations were introduced. At the same time, 5 nonverbal proofs for different subjects were given to each preservice teacher in the first week. These nonverbal proofs were selected from trigonometric, algebraic, .., from Nelsen (1993; 2000). Examples of nonverbal proof in the process are given below.

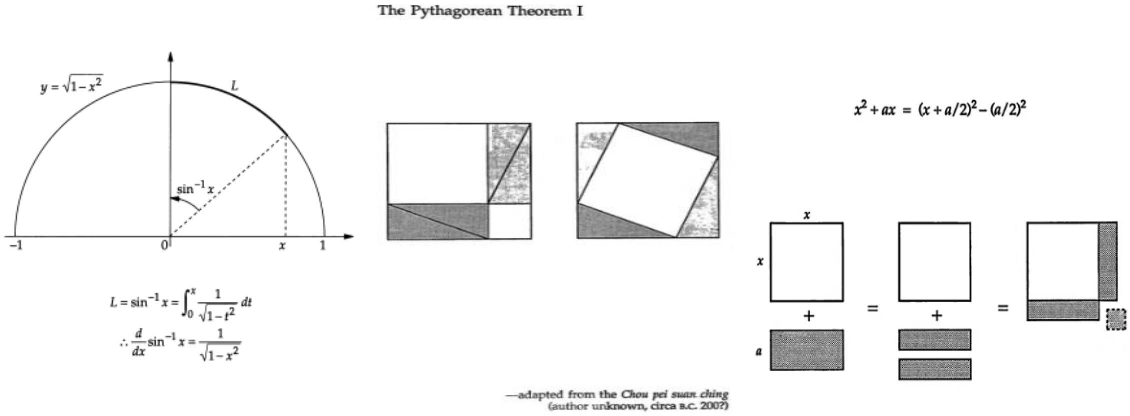


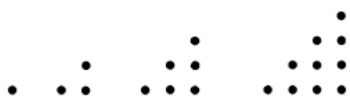
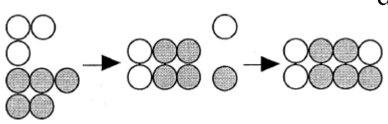
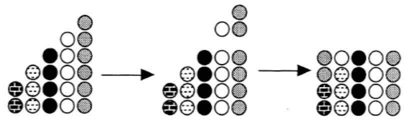
Figure 1. Examples of nonverbal proof in the process

From the 4th week, the preservice teachers shared their proofs with their friends in the classroom. When all preservice teachers finished their first nonverbal proof, the second nonverbal proof was passed. In this way, 5 nonverbal proofs were shared. The nonverbal proof given to each preservice teacher in the process is different from the others. In other words, each preservice teacher was provided with a nonverbal proof like a trigonometric, an algebraic. Therefore, each preservice teacher tried to understand and share with his or her friends nonverbal proofs. At the end of the 13th week, all nonverbal proofs were over. After this stage, the data collection tool applied at the beginning was applied again.

2.3. Data Sources and Collection Methods

The data were gathered in written with three questions shown in Table 1. Prior to the process, written responses were requested. The same questions were given in the same way at the end of the process. Neither a feedback nor a similar activity was made about these three questions in the process. The reason for this was to see how the experience of having experiences with examples in the process would change.

Table 1. Questions in the data collection tool

1.  describe what it means to you.
2.  describe what it means to you.
3.  describe what it means to you.

The first of these questions, which were directed to the pre and post- assesment in comparison, is an example of modeling "the sum of numbers from 1 to n" and the second and third question from Miyazaki (2000). The second question is the "sum of the two odd numbers", and the third question is similar to the first question, but the "sum of the 5 consecutive numbers is 5 times the number in the middle". The first visual was an example that was included in many sources and was associated with the triangular numbers. In addition, it was thought that they could explain better if they have done or encountered many applications related to the pattern. In this way, it was aimed to examine how they think the pre- assesment and see if there was change at the end of the process. The second and third questions were different from the first question. A change with arrows was highlighted. In both questions, a generalization should be reached from the single situation.

2.4. Data Analysis

After the data were collected, each participant was given a number. The data was then transferred to the computer. Three different tables were obtained from their responses to the pre-assesment, their responses to the post- assesment and the pre and post responses of each participant. In the next stage, codes and categories were formed by taking into account the answers given in the post- assesment. Then comparisons were made with the responses of each participant in the pre assesment and it was examined whether there was a change. In this way, the effects of an experience related to nonverbal proofs were examined. Two different experts in the field examined these categories and subcategories and categories were finazlized after the feedback. It was aimed to increase the reliability of the data by making direct quotations from the written statements of the participants.

3. Findings

In this study, it was aimed to raise the awareness of the preservice teachers about the nonverbal proof, to introduce them to many different nonverbal proofs and to examine the changes in such a process. In this section, three questions directed in the data collection tool were handled under separate headings and each preservice teacher's answers were compared with the pre and post-assesment.

3.1. Findings from the first question

The first question directed to preservice teachers is a visual that can be encountered in many sources for the sum of the numbers from 1 to n. The answers of the preservice teachers to this visual were summarized in Table 2.

Table 2. Code and categories derived from the answers of preservice teachers and their pre and post- assesment

Post-assesment	Change Yes/No	Pre-assesment	f		∑f
Triangular numbers	No	Triangular number	1	5	16
		Sum of numbers from 1 to n	4		
	Yes	Pattern	5	11	
		Increase of points	4		
		Isosceles triangle	2		
Sum of numbers from 1 to n	No	Sum of numbers from 1 to n	1	1	12
	Yes	Right triangle, isosceles right triangle	3	11	
		Increase of points	3		
		Pattern	2		
		Binomial expansion	1		
		Bead, increase, right triangle	1		
		Pattern, right triangle	1		
Sum of odd numbers	No	Total of consecutive numbers	1	1	3
	Yes	Isosceles triangle	1	2	
		Pattern, triangle	1		

As seen from Table 2, 28 of 31 preservice teachers participating in the study stated that the visualization given in the post-assesment was either the triangular number or the sum of the numbers from 1 to n. All of these answers were considered as an acceptable answer. As can be seen from the Table 2, 6 preservice teachers expressed both the pre and post-assesment correctly, and only 1 preservice teacher expressed as odd numbers in the post-assesment. These 6 preservice teachers were coded as "no change" at the beginning of the process either because they expressed either the triangular numbers or the sum of the numbers from 1 to n. However, the remaining 22 preservice teachers gave an acceptable answer in the post-assesment while interpreting them as pattern, right triangle, increase of dots in the pre-assesment. The answers of these preservice teachers were coded as there has been change. 2 preservice teachers did not give expected answers in either pre or post-assesment. The answers in this category are explained in detail below. As seen from Table 2, 16 students were told that the given visual was related to the triangular numbers. Only five of these preservice teachers were coded as no change between the pre and post-assesment. It is seen that one of these preservice teacher stated that both of them were modeling of triangular numbers when his/her answers before the study (Figure 2a) and after the study (Figure 2b) were compared.

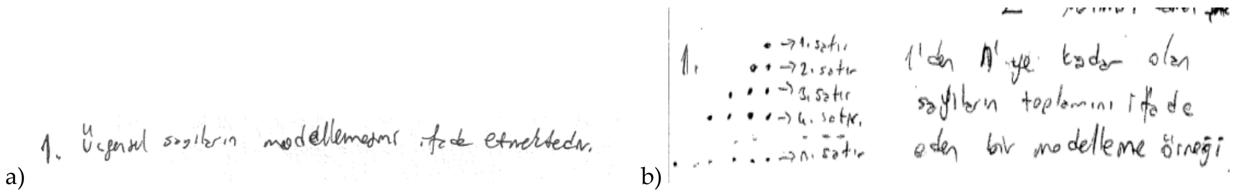


Figure 2. Answer of the preservice teacher who stated that pre-assesment is the modeling of the triangular numbers

This preservice teacher's answer was evaluated as no change. 4 preservice teachers expressed the given visual as the sum of the numbers from 1 to n in the pre-assesment, while in post-assesment they expressed as triangular numbers. One of these preservice teachers was given the answers in the pre-assesment (Figure 3 a) in the post assesment (Figure 3 b).

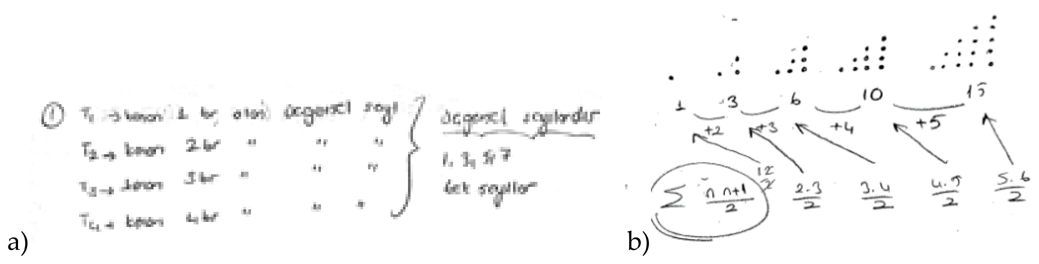


Figure 3. Examples from the answers of preservice teachers who stated that pre-assesment is the sum of numbers from 1 to n.

While these preservice teachers expressed the sum of the numbers from 1 to n, it was observed that they were related to the triangular numbers. Of course, this was also a change, but if these preservice teachers could give the same answers that gave in the pre-assesment take into account, the first answer could be accepted, and if they gave these answers, they could be considered as a change, so it was taken as no change. The answers of 11 preservice teachers were coded as change. While 5 preservice teachers stated that the pre-assesment was a pattern, they stated that it was related to the triangular numbers in the post-assesment. Of course the pattern was acceptable and because there was a regular increase. However, the pattern was considered as a change since it was a more general expression in this sense.

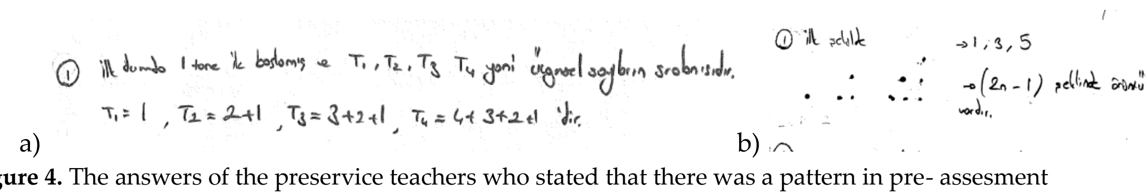


Figure 4. The answers of the preservice teachers who stated that there was a pattern in pre- assesment

As can be seen from Figure 4b, all the preservice teachers in this category stated that they were only patterns and that the points were increased in a certain order. 4 preservice teachers also mentioned the increase of points similar to the pattern. The answers of these preservice teachers were evaluated in different categories which were not included in the pattern category. Because these preservice teachers did not give a mathematical answer, they only emphasized the point increase. Considering that they will become teachers in a near future and the field and field education courses they take, it was interpreted as unexpected answer.

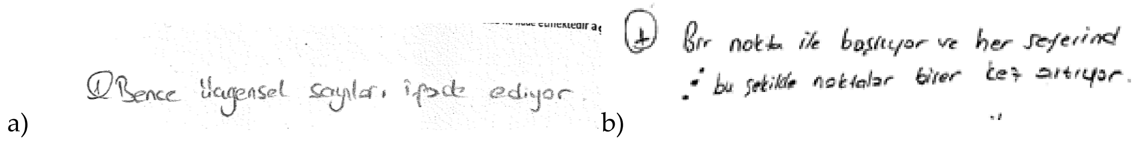


Figure 5. Examples from the answers of preservice teachers who stated that there was an increase in points in pre-assesment

Two preservice teachers differed from the above answers by either emphasizing the increase of number of points nor there was a pattern, they likened the shapes in each step to isosceles right triangle or isosceles triangle.

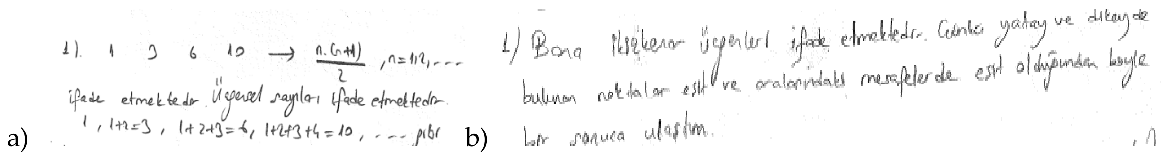


Figure 6. The answers of preservice teachers who stated that there was an isosceles triangle in pre-assesment

When the answers of these preservice teachers were examined, it was seen that in the post-assesment seen in Figure 6a, they mentioned from triangular numbers and even from the sum of of the numbers from 1 to n. The answers of these preservice teachers were evaluated as change. Because they resembled the figure as an isosceles, they at the same time associated with the visual of a mathematical expression. 12 preservice teachers stated that the post-assesment was related to the sum of the numbers from 1 to n. Only one of these preservice teachers gave the same answer in both pre (Fig. 7a) and post (Fig. 7b) assesments.

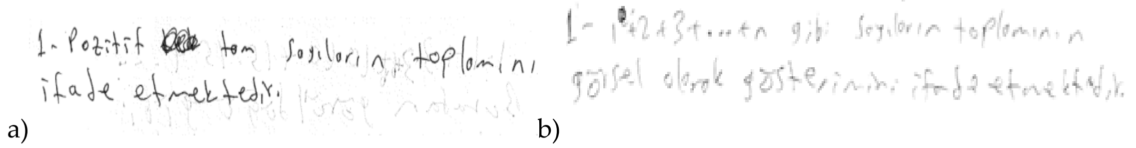


Figure 7. Examples from the answers of preservice teachers who stated that both pre and post-assesment numbers were the sum of numbers from 1 to n.

3 preservice teachers, similar to the preservice teachers above, while stated that the visual given in the pre-assesment as a right triangle or isosceles triangle (Fig.8b), they expressed as the sum of the numbers from 1 to n in the post- assesment (Fig.8a).

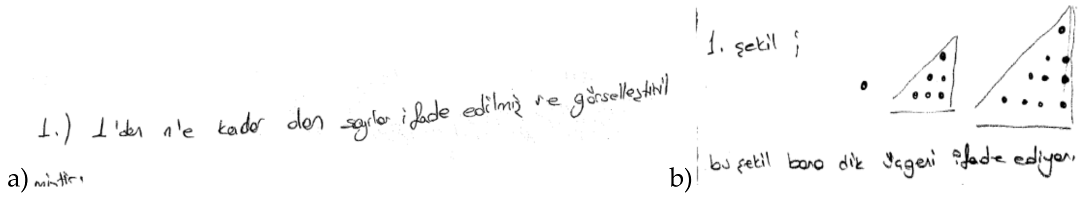


Figure 8. The answers of preservice teachers who stated that there was isosceles triangle in the pre-assesment.

While 3 preservice teachers expressed the increase of points as a pattern in the pre-assesment (Fig.9b), they stated that they were the sum of numbers from 1 to n in the post-assesment (Fig.9a).

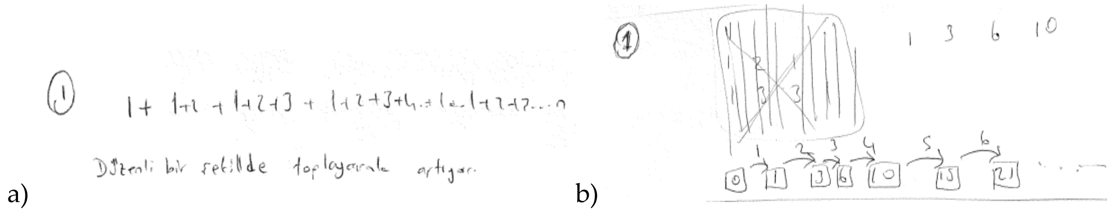


Figure 9.The answers of the preservice teachers who stated that there was an increase in points in the pre-assesment

While two preservice teachers expressed the first assesment as a pattern, in the last assesment they stated that they were the sum of numbers from 1 to n.

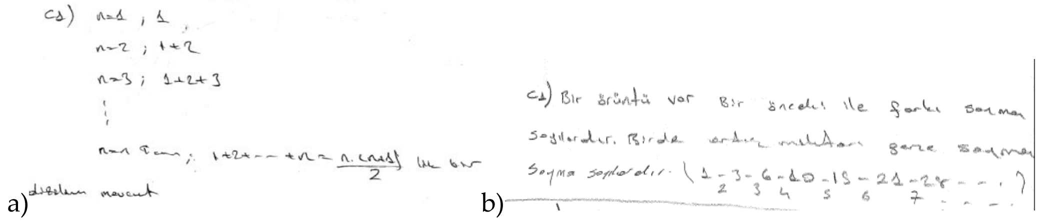


Figure 10. The answers of the preservice teachers who stated that the pre-assesment was a pattern

In the pre-assesment, one preservice teacher was defined as both pattern and right triangle, and one preservice teacher as an increase, right triangle, bead. Both teachers stated that they were the sum of the numbers from 1 to n the end of the process. The answers of these teachers are similar to the answers of the above preservice teachers. In the post -assesment, 3 preservice teachers found the sum of the points in each step, that were triangular numbers as 1,3,9,... and they were probably based upon that they were odd numbers. Only one of these preservice teachers gave the expected answer while he/she stated that they odd numbers in the post- assesment. Since he/she gave acceptable answer in the pre-assesment, it was coded as no change.

3.2 Findings from the second question

The answers of the preservice teachers to the second visual were summarized in Table 3.

Table 3. Code and categories gathered from the **second** question

Category	Post-ssesment	Change Yes/No	Pre-assesment	f		∑f
Operation of addition	The sum of two odd number is even number.	Yes	Generalization	2	3	13
			Customization	1		
			Forming rectangle	5	8	
			Changing color	1		
			Balls	1		
			Class placement	1		
	Formula of the total – Operation of addition		I couldn't see anything	1	2	
			Forming a rectangle	1		
Other	Forming a rectangle	No	I couldn't see anything	1	3	18
			Forming rectangle	2		
	Sum of the area-areass will not change		Forming rectangle	1	4	
			Doesn't mean anything	1		
			Area calculation	1		
			No answer	1		
	Conservation		No answer	1	1	
	Replacement		Replacement	1	1	
	Effectivate the visual		Replacement	1	1	
	Having double group		Replacement	1	1	
	Combination		Replacement	1	1	
	No answer		No answer	2	6	
			Forming rectangle	3		
			Ball pulling	1		

It is seen from Table 3, 13 preservice teachers said the given visual was related to the sum of odd numbers was even or the addition. 3 of these preservice teachers gave the same answers in the pre-assesment and in the post-assesment, they were coded as no change. Only 2 of these preservice teachers expressed the general rule based on the sample situation, that is, when making generalization, a preservice teacher only commented on the special case.

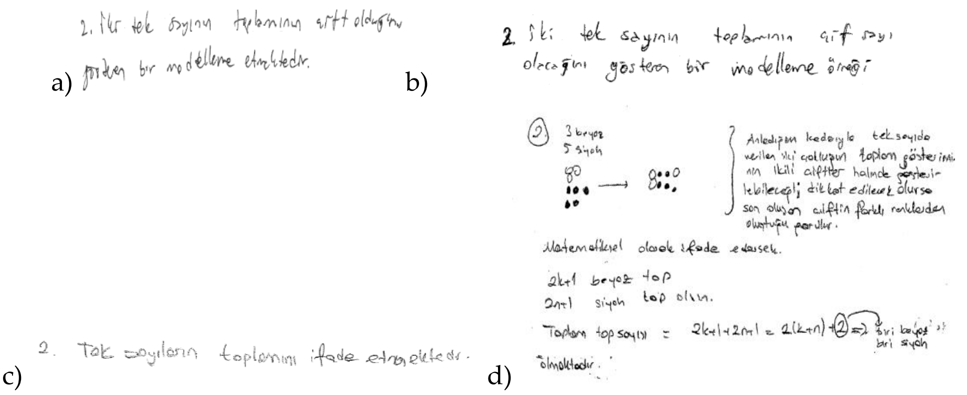


Figure 11. The answers of preservice teachers expressed the general rule in pre- and post-assesment

The first preservice teacher gave almost the same answer to the pre- (Figure 11b) and the post-assesment (Figure 11a). The other preservice teacher took 2k + 1 and 2n + 1 odd numbers as general state based from the special case upon (Figure 11d), that is 3 + 5, and showed the sum was even. If the answer would be examined, it could be seen that he/she took any odd numbers and proved based upon the visual. This preservice teacher expressed only the general rule in the post-assesment (Figure 11c). Therefore, these preservice teachers were coded as no change since they gave same answers at the beginning and end of the process. Generalization, customization and proof are sub-dimensions of mathematical thinking. These three skills were seen in this preservice teacher's answer. Based on 3 + 5 special cases, he/she reached the general judgment and proved it. The preservice teacher who stated that there was a change, did not generalize by expressing that they were the modeling of 3 + 5 both in the pre- (Figure 12b) and in the post -assesment (Figure 12 a) as seen in Figure 12.

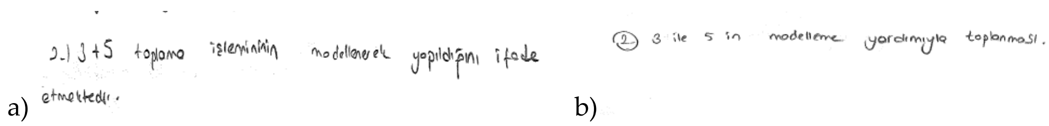


Figure 12. The answers of preservice teachers expressed the general rule in pre- and post-assesment

This preservice teacher stated that the given image was only valid for the special case, but the general rule was not stated. The biggest difficulty in the visuals given in this style was to judge the general situation based on the special cases and to support this finding in the observations. Preservice teachers generally stated that one situation could not express a general judgment, and if it was to comment on its general rule, it should be given in a few visuals like this. As seen from Table 3, 8 preservice teachers answered in the pre-assesment as forming rectangle, class placement or replacement of the balls, while they gave answers in the post-assesment as the sum of the odd numbers was even. These preservice teachers were coded as change since they could comment on what the visual given in the post assesment could be. 2 preservice teachers in the pre-assesment stated that I could not see anything and forming rectangle in the post assesment could be related to operation of addition. They did not mention it was related to even numbers. Since they expressed as operation of addition, they perceived that the balls were given altogether although they were given seperately in accordance with their color, so they were interpreted as they thought what was expected and they were coded as change. In the post-assesment, 18 preservice teachers gave answers as the sum of the area or areas would not change, conservation, replacement, having double group, combination, effectivate the visual. When the answers of these preservice teachers were examined, it was seen that they paid attention to the change in every step in the given visual and they tried to explain it.

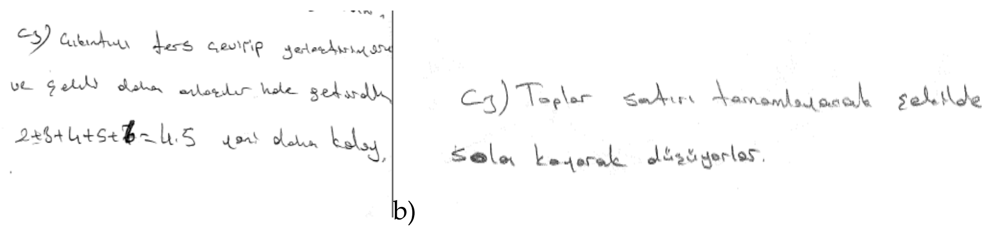
3.3. Findings from the third question

The answers of the preservice teachers to the third visual were summarized in Table 4.

Table 4. Findings from the third question

Post-assesment	Change Yes/No	Pre-assesment		f	∑f
2+3+4+5+6=4.5	Yes	Number of balls	2	8	12
		Calculating the area of right trapezium with the area of the rectangle	1		
		Forming rectangle	3		
		Replacement	2		
Operation of addition		Replaceme	1	1	
Triangular numbers		The sum of the consecutive numbers beginning with even number, multiplied by two consecutive numbers	1	3	
		Forming rectangle	2		
No answer	No	Forming rectangle	3	4	19
		Area	1		
Replacement		Forming rectangle	3	4	
		Repclacement	1		
Forming rectangle		Area	1	7	
		Forming rectangle	6		
Area		Arithmetical operation	1	4	
		Forming square	1		
		Area of trapezium	2		

When Table 3 was examined, it was seen that 12 preservice teachers associate the given visual with the sum of the consecutive numbers, that is in the special case, "the sum of 5 consecutive numbers, is the five times of the number in the middle" However, only 11 preservice teachers were coded as change between the pre- and the post-assesment. In the pre-assesment, while 8 preservice teachers associated relation between number of balls, the replacement of the points, the formation of the rectangle or the area of trapezoid and the rectangle in the pre-assesment; they wrote 2 + 3 + 4 + 5 + 6 = 4.5 in the post-assesment. The pre-(Figure 13b) and post-(Şekil 13a) answers of one preservice teacher were given. As seen in Figure 13, all the preservice teachers especially wrote 2 + 3 + 4 + 5 + 6 = 4.5 in the post- assesment. But none of them wrote as 5 consecutive numbers is five times of the number in the middle. This was interpreted as although preservice teachers expressed the given visual in the special case, they did not make generalization, or it did not express a general rule. Indeed, in the observations in the process, it was observed that the same perception was widespread in many examples of nonverbal proof.

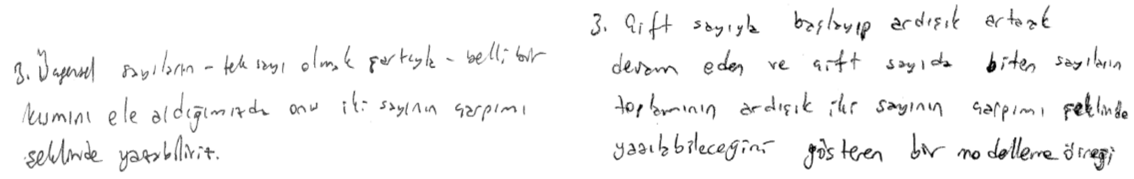


a) b)

Figure 13. The pre- and post-assesment answers of a preservice teacher

One preservice teacher stated that it expresses the formula of the sums with nonverbal proof in the post assesment, the answer of this preservice teacher wass coded as acceptable in a separate category. 3 preservice teachers likened to the triangular numbers in the post-assesment, since they saw consecutive numbers. While two of these preservice teachers stated that formin rectangle in the pre-assesment, they stated as triangular numbers in the post-assesment, so it was coded as change. Only a preservice teacher was coded as no change here. In fact, when the answer given in the pre-assesment was examined in

Figure 14b, he/she stated that the sum of the consecutive numbers starting with even number and ending with the even number is expressed as the multiplication of two numbers.



a) **b)**

Figure 14. A preservice teacher's answers in pre- and post-assesment

Generalizations are reached from special cases, but this preservice teacher used as starting with even and ending with even, without expressing even the special case, that was starting with 2 and ending with 6. When the answer of the post-assesment (Figure 14 a) was examined, it was seen that he/she stated as the multiplication of two numbers. Although the remaining 19 preservice teachers answered in the pre-assesment, they did not answer in the post-assesment, or they expressed the given visual as area, forming rectangle, replacement. These answers were coded as no change. In fact, when the answers in the post-assesment compared to the answers given in the pre-assesment, it was seen that they formed same sentences. This was interpreted as there was a need to incresing the number and type of such applications to be a change, and to give instant feedback and to make immediate corrections since the given visuals were easier to remember.

4. Discussion

Although the process of proving is complex, it is among the objectives of the curriculum that students gain. If visualization is an effective method for students with different learning styles in the classroom (Thornton, 2001), nonverbal proofs are an effective tool for both understanding proofs and producing new mathematical ideas. Hence, providing an experience with nonverbal proofs especially to preservice teachers, and researching the results will contribute to both the other studies and to the field. In this study, the effects of nonverbal proof-based education are limited to three visuals (Figures 15a, b and c) at the beginning of the process and at the end of the process. In this respect, firstly, the findings obtained with the thinking stages in three visuals are tried to be explained and discussed. These three visuals given in Figure 13 are both similar and different in many respects. Although the visual given in Figure 15a is similar to the images related to the pattern, it is given as the sum of the numbers from 1 to n in many sources, but the visual in b and c is characterized as not familiar. On the other hand, the visual shown in Fig. 15a grows to be one point more in each step. However, since only the replacement of the points is clearly shown without any point increase, b and c are similar but different from a in the sense. There is one increase in points, that is they are consecutive numbers, the visuals in a and c are similar in the sense.

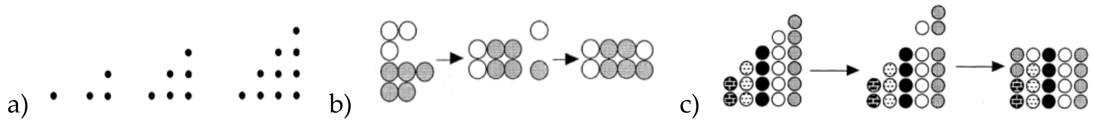


Figure 15. Visuals in the data collection tool

When the responses of the preservice teachers on the visuals in the pre- and post-assesment were compared, it was seen that there were changes in their answers. In the 1st question, 28 preservice teachers from 31 preservice teachers participating in the study, stated that the visual given in the post-assesment was either the triangular number or the sum of the numbers from 1 to n. In this question, only 22 preservice teachers answers was coded as "change" when the answers of the pre-and post-assesment were compared. These preservice teachers generally likened to geometric shapes in the pre-assesment. Miller (2012) states that since geometry usually deals with visuals there is a tendency towards visuals, and Karras (2012) states that there is no area other than geometry to develop diagrammatic reasoning skills in mathematics curriculum. Therefore, it can be said that the reason why preservice teachers try to interpret the given visuals as geometrically when they first meet is because visualization is mostly in geometry. On the other hand, in the visual in Figure 13a, after the preservice teachers stated at the 1st step 1 = 1 point, at the 2nd step 1 + 2 = 3 points, at the 3rd step 1 + 2 + 3 = 6 points, at 4th step 1 + 2 + 3 + 4 =10 points, they need to generalize as 1 + 2 + 3 + 1 + n at n step and need to associate with the 1+2+3+…+n=$\frac{n(n+1)}{2}$ rule, whose accuracy was proved with induction. In addition to seeing this relationship, preservice teachers have also linked the post-assesment to the triangular numbers. This finding supports Gierdien's (2017) view that nonverbal proofs can help to relate the topics of mathematics. In the second question, 13 of the 31 preservice teachers who participated in the study stated that the image given in the post-assesment was related to the operation of addition of the sum of the odd numbers. The pre- and the post-assesment of this question were compared, only 10 preservice teachers answers were coded as "change". Miyazaki (2000) made the formal proof of the general rule as shown in Figure 16. According to this proof, as x and y any two odd numbers and x = 2n + 1 and y = 2m + 1 was written.

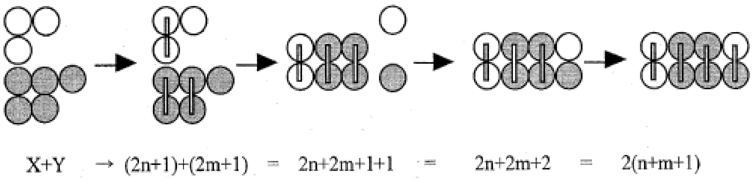


Figure 16. Proof given by Miyazaki (2000)

Based upon 3 + 5, it is a difficult skill to express and prove the general rule for any two odd numbers. However, in Figure 13b and c, generalization is expected from the special case. In the visual given in Figure 13c, 2 + 3 + 4 + 5 + 6 = 4.5 is expected to be seen and generalized and the rule that the sum of 5 consecutive numbers is 5 times of number of the middle number is expected to be reached. As stated by Kulpa (2009), the biggest difficulty in all three questions is the problem of generalization or general rule expression. However, as Alsina and Nelsen (2010) stated, the purpose of the nonverbal proof is not to make proof, but to provide an understanding by visualizing the mathematical expression for any situation. Similarly, Flores (2000) states that pictures contain elements that facilitate general judgment, although they depict a particular situation. In fact, Miller (2012) has the view that nonverbal proof demonstrating a single situation is generalizable to all numbers, but the findings of the study indicate that this requires a process or experience. In the 3rd question, 12 of the 31 preservice teachers participating in the study were found to associate the visual given in the post-assesment with the sum of the consecutive numbers, that is in the special case "5 consecutive numbers is 5 times of the number in the middle". When the answers of the pre- and the post-assesment in this question were compared, only 11 preservice teachers' answers coded as "change". These findings can be said to be preservice teachers can recognize the visuals, interpreting or can express a suggestion based upon given visual. It is not enough to say that the expected changes in the answers in the pre-and the post-assesment have an effect on the ability to prove. As Healy and Hoyles (2000) stated, the process of proof is a complex process and requires a lot of knowledge and competence. In the studies to be done, the effects on proof skills can also be examined. In fact, as Bell (2011) stated, nonverbal proofs are especially effective in understanding the proof process. While preservice teachers likened given visuals to geometric figures in the pre-assesment, they suggested a mathematical expression in the post-assesment. One of the reasons for this is that the visualization is more involved in geometry. The ability of students to use visualization is possible with facing with the visualization activities in the courses and encouraging them to use visualization (Rodd, 2000). Alsina and Nelsen (2010) also mention the necessity of visualization ability for mathematical success. Bardelle (2009) attributes the reason why students are not aware of the techniques, tools and theorems that they can use when they encounter any topic, because the students work very little with visualization. In future studies, the relationship between visualization skills and nonverbal proof can be examined.

5. Conclusion

This study investigated the effects of nonverbal proof-based education on preservice mathematics teachers. The results showed that the preservice mathematics teachers' experience of nonverbal proof effect on the recognition of the given image and establishing a relationship with different mathematics subjects. However, the findings of this study can be said to be preservice teachers can recognize the visuals, interpreting or can express a suggestion based upon given visual. It is not enough to say that the expected changes in the answers in the pre-and the post-assesment have an effect on the ability to prove. In future studies, the relationship between visualization skills and nonverbal proof and the effects on proof skills can also be examined.

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