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Dynamics and Bifurcation of $x_{n+1}=\frac{\alpha+\beta x_{n-2}}{A+Bx_n+Cx_{n-2}}$

Batool Raddad[1] **and Mohammad Saleh**[1*]

[1]*Department of Mathematics, Birzeit University, P.O.Box 14, West Bank, Palestine*
[*]*Corresponding author*

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Abstract

In this paper, we study dynamics and bifurcation of the third order rational difference equation

$$x_{n+1}=\frac{\alpha+\beta x_{n-2}}{A+Bx_n+Cx_{n-2}},\quad n=0,1,2,\dots$$

with positive parameters α,β,A,B,C and non-negative initial conditions $\{x_{-k},x_{-k+1},\dots,x_0\}$. We study the dynamic behavior, the sufficient conditions for the existence of the Neimark-Sacker bifurcation, and the direction of the Neimark-Sacker bifurcation. Then, we give numerical examples with figures to support our results.

1. Introduction

The study of dynamical system is the focus of dynamical systems theory, which has application to a wide variety of fields such as mathematics, physics, chemistry, biology, medicine, engineering and economics. Dynamical systems are a fundamental part of bifurcation theory which studies the changes in the qualitative or topological structure of systems. A bifurcation occurs when a small change made to the bifurcation parameter of a system causes a qualitative or topological change in its behavior.

In this paper, we will study the third order rational difference equation

$$x_{n+1}=\frac{\alpha+\beta x_{n-2}}{A+Bx_n+Cx_{n-2}},\quad n=0,1,2,\dots. \tag{1.1}$$

We focus on the dynamic behavior of the positive fixed points and the type of bifurcation exists where the change of stability occurs. Then, numerical examples are treated to support our results.

Local and global stability, period two solutions, boundedness, invariant intervals and semicycles of

$$x_{n+1}=\frac{\alpha+\beta x_{n-k}}{A+Bx_n+Cx_{n-k}},\quad n=0,1,2,\dots \tag{1.2}$$

were studied by Guo-Mei Tang, Lin-Xia Hu, and Gang Ma in [1]. Also, it was shown that (1.2) has no nonnegative prime period-two solutions for even integer k. Equation (1.1) was studied by Ladas in [2].

The aim of this paper is to study the bifurcation of the third order rational difference equation (1.1). The change of variables $x_n=\frac{A}{B}y_n$ convert the rational difference equation (1.1) with five positive parameters into $y_{n+1}=\frac{p+qy_{n-2}}{1+y_n+ry_{n-2}},\ \ n=0,1,2,\dots$ with three positive parameters p, q, and r, where $p=\frac{B}{A^2}\alpha$, $q=\frac{B}{A}$ and $r=\frac{C}{B}$. Recent studies on dynamics and bifurcation can be found in [3], [4], [5], [6], [7].

Email address and ORCID number**: brdad@birzeit.edu, 0000-0002-2716-5740 (B. Raddad), msaleh@birzeit.edu, 0000-0002-4254-2540 (M. Saleh)**

2. Dynamics of $y_{n+1}=\frac{p+qy_{n-2}}{1+y_n+ry_{n-2}}$

In this section we will study the dynamics of the third order rational difference equation

$$y_{n+1}=\frac{p+qy_{n-2}}{1+y_n+ry_{n-2}} \tag{2.1}$$

with positive parameters p,q, and r, and non-negative initial conditions y_{-2},y_{-1} and y_0. Note that equation (2.1) has the unique positive fixed point $\bar{y}=\frac{q-1+\sqrt{(q-1)^2+4p(1+r)}}{2(1+r)}$.

In order to convert equation (2.1) to a third dimensional system, let $z_n=y_n, x_n=y_{n-1}$ and $t_n=y_{n-2}$. We have the following system

$$z_{n+1}=\frac{p+qt_n}{1+z_n+rt_n}$$

$$x_{n+1}=z_n$$

$$t_{n+1}=x_n \tag{2.2}$$

which has the positive fixed point $(\bar{y},\bar{y},\bar{y})$. In order to shift this fixed point to the origin, let $w_n=z_n-\bar{y}$, $v_n=x_n-\bar{y}$ and $u_n=t_n-\bar{y}$. System (2.2) corresponds

$$w_{n+1}=\frac{p+q(u_n+\bar{y})}{1+(w_n+\bar{y})+r(u_n+\bar{y})}-\bar{y}$$
$$v_{n+1}=w_n$$

$$u_{n+1}=v_n \tag{2.3}$$

System (2.3) has $(0,0,0)$ as a fixed point.

The Jacobian matrix of system (2.3) is

$$J(w,v,u)=\begin{pmatrix} -\frac{p+q(w_n+\bar{y})}{(1+w_n+\bar{y}+r(u_n+\bar{y}))^2} & 0 & \frac{q(1+w_n+\bar{y})-rp}{(1+w_n+\bar{y}+r(u_n+\bar{y}))^2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J(0,0,0)=\begin{pmatrix} -\frac{p+q\bar{y}}{(1+\bar{y}+r\bar{y})^2} & 0 & \frac{q+q\bar{y}-rp}{(1+\bar{y}+r\bar{y})^2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}=\begin{pmatrix} -\frac{\bar{y}}{1+\bar{y}+r\bar{y}} & 0 & \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of the Jacobian matrix J is

$$p(\lambda)=-\lambda^3-\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}. \tag{2.4}$$

Let $p_1=\frac{\bar{y}}{1+\bar{y}+r\bar{y}}$, $p_2=0$ and $p_3=-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$.

We will use the following theorem to determine the stability of the zero solution.

Theorem 2.1. *[8] For the third-order difference equation*

$$x(n+3)+p_1x(n+2)+p_2x(n+1)+p_3x(n)=0, \tag{2.5}$$

the characteristic polynomial is

$$p(\lambda)=\lambda^3+p_1\lambda^2+p_2\lambda+p_3.$$

A necessary and sufficient condition for the zero solution to be asymptotically stable is

$$|\, p_1+p_3 \,|<1+p_2 \ \ \textit{and} \ \ |\, p_2-p_1p_3 \,|<1-p_3^2. \tag{2.6}$$

Theorem (2.1) implies that the zero solution is asymptotically stable if condition (2.6) holds which is equivalent to

$$|\, \frac{\bar{y}}{1+\bar{y}+r\bar{y}}-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \,|<1 \tag{2.7}$$

and

$$|\, \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\times\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \,|<1-(\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})^2. \tag{2.8}$$

Inequality (2.7) is equivalent to

$$1+\frac{\bar{y}}{1+\bar{y}+r\bar{y}}-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}>0, \tag{2.9}$$

and

$$1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}>0, \tag{2.10}$$

and inequality (2.8) is equivalent to

$$1+\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\times\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}-(\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})^2>0, \tag{2.11}$$

and

$$1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\times\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}-(\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})^2>0. \tag{2.12}$$

Inequality (2.9) always holds since

$$1+\frac{\bar{y}}{1+\bar{y}+r\bar{y}}-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}=\frac{1-q+2(1+r)\bar{y}}{1+(1+r)\bar{y}}>0.$$

Also, inequality (2.10) holds for all values of the parameters p,q and r since,

$$1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}=\frac{1+q}{1+(1+r)\bar{y}}>0.$$

Inequality (2.11) is equivalent to

$$1+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}[\frac{\bar{y}}{1+\bar{y}+r\bar{y}}-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}]>0. \tag{2.13}$$

Note that we take $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$ as a common factor. Now, add -1 to both sides of inequality (2.13), we have

$$\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}[\frac{(1+r)\bar{y}-q}{1+\bar{y}+r\bar{y}}]>-1 \tag{2.14}$$

Multiply both sides of (2.14) by $\frac{1+\bar{y}+r\bar{y}}{(1+r)\bar{y}-q}$, for $(1+r)\bar{y}-q<0$, we have

$$\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}<\frac{1+\bar{y}+r\bar{y}}{q-(1+r)\bar{y}}.$$

Inequality (2.12) is equivalent to

$$\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}[\frac{-\bar{y}}{1+\bar{y}+r\bar{y}}-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}]>-1$$

or,

$$\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}[\frac{-\bar{y}-q+r\bar{y}}{1+\bar{y}+r\bar{y}}]>-1. \tag{2.15}$$

Note that for $(1+r)\bar{y}-q<0$, $r\bar{y}-\bar{y}-q<0$. So, if we multiply both sides of (2.15) by $\frac{1+\bar{y}+r\bar{y}}{r\bar{y}-\bar{y}-q}$, we have

$$\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}<\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}.$$

Note that for $r\bar{y}-\bar{y}-q<r\bar{y}+\bar{y}-q<0$,

$$0<q-r\bar{y}-\bar{y}<q-r\bar{y}+\bar{y},$$

and hence,

$$\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}<\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}-\bar{y}}.$$

So for $q-(1+r)\bar{y}>0$, if $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}<\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$, then $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}<\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}-\bar{y}}$, and hence, for $q-(1+r)\bar{y}>0$ if inequality (2.12) holds, then inequality (2.11) holds.
Note that if $q-(1+r)\bar{y}>0$, we have

$$q-(1+r)(\frac{q-1+\sqrt{(q-1)^2+4p(1+r)}}{2(1+r)}>0$$

or

$$q-\frac{q-1+\sqrt{(q-1)^2+4p(1+r)}}{2}>0$$

$$q+1-\sqrt{(q-1)^2+4p(1+r)}>0$$

$$q+1>\sqrt{(q-1)^2+4p(1+r)}$$

take the square of both sides, we get

$$q^2+2q+1>q^2-2q+1+4p(1+r)$$

or,

$$4q>4p(1+r)$$

$$p < \frac{q}{1+r}.$$

So for $p < \frac{q}{1+r}$, the zero solution is asymptotically stable if

$$\frac{q - r\bar{y}}{1+\bar{y}+r\bar{y}} < \frac{1+\bar{y}+r\bar{y}}{q - r\bar{y}+\bar{y}}. \tag{2.16}$$

Note that if we fix q and r and choose p as a parameter where $p < \frac{q}{1+r}$, then the stability exchanges at the value of p that satisfies equation $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$. Name this value as p^*.

3. Existence of Neimark-Sacker bifurcation of $y_{n+1} = \frac{p+qy_{n-2}}{1+y_n+ry_{n-2}}$

In this section we study Neimark-Sacker bifurcation of (2.1) which occurs at $p = p^*$ as p is the bifurcation parameter. Note that equation (2.1) has no positive distinct periodic solutions of prime period two. Hence, we focus our attention on Neimark-Sacker bifurcation.

Theorem 3.1. *The characteristic polynomial (2.4) $p(\lambda)$ has two complex conjugate roots if one of the following cases holds*

1. $q - r\bar{y} < 0$
2. $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} > \frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3$

Proof.

$$p(\lambda) = -\lambda^3 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2 + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$$
$$\acute{p}(\lambda) = -3\lambda^2 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda$$

$\acute{p}(\lambda) = 0$ at $\lambda_1^* = -\frac{2}{3}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})$ or $\lambda_2^* = 0$.
Since, $\bar{y} > 0$, $\lambda_1^* < \lambda_2^*$. $p(\lambda)$ has local minimum value at $\lambda = \lambda_1^*$ and local maximum value at $\lambda = \lambda_2^*$. Note that $\lim_{\lambda \to -\infty} p(\lambda) = \infty$ and $\lim_{\lambda \to \infty} p(\lambda) = -\infty$.
So, $p(\lambda)$ has only one real root if one of the following cases holds

1. $p(\lambda_1^*) > 0$ and hence $p(\lambda_2^*) > p(\lambda_1^*) > 0$.
2. $p(\lambda_2^*) < 0$ and hence $p(\lambda_1^*) < p(\lambda_2^*) < 0$.

So, $p(\lambda)$ has two conjugate complex roots if one of the following holds

1. $p(\lambda_1^*) = -\frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3 + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} > 0$.
2. $p(\lambda_2^*) = \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < 0$.

Consider case one. Note that $p(0) = \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} > 0$ and $p(1) = -1 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}} + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$. Substitute the value of $\bar{y}$, we have $p(1) = -2\frac{\sqrt{(q-1)^2+4p(1+r)}}{q+1+\sqrt{(q-1)^2+4p(1+r)}} < 0$. So $p(\lambda)$ has a real root ξ such that $\xi \in (0,1)$.
In the second case, by similar argument we can show that $p(\lambda)$ has a real root of modulus less than one. Note that $p(0) < 0$ and $p(-1) > 0$ in this case.
Consider the case where $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} > \frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3$. We will find where the conditions of Neimark-Sacker conditions hold. □

Theorem 3.2. *For $p < \frac{q}{1+r}$, the characteristic polynomial $p(\lambda)$ has two complex conjugate roots of modulus one and a real root of modulus less than one at $p = p^*$ if $q > 3$.*
Moreover, if $p^* > \frac{\left(2(1+r)\left(\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}\right)-(q-1)\right)^2-(q-1)^2}{4(1+r)}$, then Neimark-Sacker conditions hold.

To prove this theorem we need *Viète formula*.

Theorem 3.3. *[9](Viète formula) Given any polynomial of degree n, say*

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

with roots $r_1, r_2, \ldots, r_n$. Vièteformula say that
$r_1 + r_2 + \ldots + r_n = -\frac{a_{n-1}}{a_n}$,
$(r_1r_2 + r_1r_3 + \ldots + r_1r_n) + (r_2r_3 + r_2r_4 + \ldots + r_2r_n) + \ldots + r_{n-1}r_n = \frac{a_{n-2}}{a_n}$,
$(r_1r_2r_3 + r_1r_2r_4 + \ldots + r_1r_2r_n) + (r_1r_3r_4 + r_1r_3r_5 + \ldots + r_1r_3r_n) + \ldots + r_{n-2}r_{n-1}r_n = -\frac{a_{n-3}}{a_n}$,

$\vdots$

$r_1r_2r_3 \ldots r_n = (-1)^n \frac{a_0}{a_n}$.

Proof of theorem (3.2): Consider that $q > 3$ and $p < \frac{q}{1+r}$. Note that for $p < \frac{q}{1+r}$, we have $q-(1+r)\bar{y} > 0$ and hence $q - r\bar{y} > \bar{y}$. Recall that $1 > (\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2$ so $1 > \frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2$ and hence, $\bar{y} > \frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2\bar{y}$. So, $q - r\bar{y} > \bar{y} > \frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2\bar{y}$. Multiply by $\frac{1}{1+\bar{y}+r\bar{y}}$, we get $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} > \frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3$.
So, in this case the characteristic polynomial has two complex conjugate roots and another real root of modulus less than one as we have shown in the proof of theorem (3.1).
Now we will show that the modulus of the conjugate roots equals one at $p = p^*$. Let λ_1, λ_2 and λ_3 be the roots of $p(\lambda)$ where, λ_1 and λ_2 are the conjugate roots and λ_3 is the real root. Recall that $\lambda_3 = \xi$ has modulus less than one. By *Viète theorem*, we have

$$\lambda_1 + \lambda_2 + \lambda_3 = -\frac{\bar{y}}{1+\bar{y}+r\bar{y}} \tag{3.1}$$

$$\lambda_1\lambda_2\lambda_3 = \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \tag{3.2}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 0. \tag{3.3}$$

If λ_1 and λ_2 has modulus equal one, then $\lambda_1\lambda_2 = 1$. From (3.2), we get $\lambda_3 = \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$.
Substitute λ_3 in equation (3.1), we get $\lambda_1 + \lambda_2 + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = -\frac{\bar{y}}{1+\bar{y}+r\bar{y}}$

$$\lambda_1 + \lambda_2 = -\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}}. \tag{3.4}$$

Also, substitute λ_3 in equation (3.3), we get $\lambda_1 + \lambda_2 = -\frac{1}{\lambda_3} = -\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}}$.
That implies $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$. This shows that at $p = p^*$ where p^* satisfies $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$, $p(\lambda)$ has two complex conjugate roots of modulus one and a real root of modulus less than one for $p < \frac{q}{1+r}$.
As p is the bifurcation parameter and q and r are fixed, the bifurcation point is p^* which satisfies

$$\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$$

$$(1+\bar{y}+r\bar{y})^2 = (q-r\bar{y}+\bar{y})(q-r\bar{y})$$

$$(1+r)^2\bar{y}^2 + 2(1+r)\bar{y} + 1 = q^2 - qr\bar{y} + q\bar{y} - qr\bar{y} + r^2\bar{y}^2 - r\bar{y}^2$$

$$(1+3r)\bar{y}^2 + (2(1+r)+q(2r-1))\bar{y} - (q^2-1) = 0. \tag{3.5}$$

Equation (3.5) is a quadratic equation with the following roots

$$\bar{y} = \frac{-(2(1+r)+q(2r-1)) \pm \sqrt{(2(1+r)+q(2r-1))^2 + 4(q^2-1)(1+3r)}}{2(1+3r)}.$$

Since, $\bar{y} > 0$, for $q^2 > 1$

$$\bar{y} = \frac{-(2(1+r)+q(2r-1)) + \sqrt{(2(1+r)+q(2r-1))^2 + 4(q^2-1)(1+3r)}}{2(1+3r)}.$$

Substitute the value of $\bar{y}$, we have
$\frac{q-1+\sqrt{(q-1)^2+4p^*(1+r)}}{2(1+r)} = \frac{-(2(1+r)+q(2r-1))+\sqrt{(2(1+r)+q(2r-1))^2+4(q^2-1)(1+3r)}}{2(1+3r)}$
$\sqrt{(q-1)^2+4p^*(1+r)} = 1-q+2(1+r)\left(\frac{-(2(1+r)+q(2r-1))+\sqrt{(2(1+r)+q(2r-1))^2+4(q^2-1)(1+3r)}}{2(1+3r)}\right)$

$$p^* = \frac{\left(1-q+2[1+r][\frac{-(2(1+r)+q(2r-1))+\sqrt{(2(1+r)+q(2r-1))^2+4(q^2-1)(1+3r)}}{2(1+3r)}]\right)^2 - (q-1)^2}{4(1+r)}.$$

To check if Neimark-Saker bifurcation exists at p^*, we must show that $e^{ik\theta^*} \neq 1$ for $k = 1,2,3,4$ and $\acute{r}(p^*) \neq 0$ where $\lambda_{1,2} = \cos\theta^* \pm i\sin\theta^*$. To show that $e^{i\theta^*} \neq 1$, let $\lambda = \cos\theta + i\sin\theta$ and $\bar{\lambda} = \cos\theta - i\sin\theta$ be the complex roots of $p(\lambda)$ at p^*. Substitute λ in $p(\lambda)$, we have $-\lambda^3 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2 + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = 0$ or

$$\lambda^3 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2 - \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = 0. \tag{3.6}$$

Recall that at p^*, $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$. So equation (3.6) becomes

$$\lambda^3 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2 - \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}} = 0. \tag{3.7}$$

By similar argument, substitute $\bar{\lambda}$ in $p(\lambda)$ we get

$$\bar{\lambda}^3 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2 - \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}} = 0. \tag{3.8}$$

Multiply equation (3.7) by $\bar{\lambda}^2$, we have

$$\lambda + \frac{\bar{y}}{1+\bar{y}+r\bar{y}} - \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}\bar{\lambda}^2 = 0. \tag{3.9}$$

Also, multiply equation (3.8) by λ^2, we have

$$\bar{\lambda} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}} - \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}\lambda^2 = 0. \tag{3.10}$$

Add (3.9) to (3.10), we get

$$\lambda + \bar{\lambda} + 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}}) - \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}(\lambda^2+\bar{\lambda}^2) = 0 \tag{3.11}$$

Note that $\lambda+\bar{\lambda} = 2\cos\theta$ and $\lambda^2+\bar{\lambda}^2 = 4\cos^2\theta - 2$.
Equation (3.11) becomes $2\cos\theta + 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}} - \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}(4\cos^2\theta-2) = 0$ or

$$-4(\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}})\cos^2\theta + 2\cos\theta + 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}}) + 2(\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}) = 0. \tag{3.12}$$

From equation (3.4), we have $\lambda+\bar{\lambda} = -\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}}$ and hence, $2\cos\theta = -\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}}$. That implies $\cos\theta = -\frac{1}{2}(\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}})$. Note that this is a root of equation (3.12) since, $-4(\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}})(-\frac{1}{2}\times\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 2(-\frac{1}{2}\times\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}}) + 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}}) + 2(\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}) = -2\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}} + 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + 2\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}} = 2(-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} + \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}) = 2(0) = 0$.
Note that $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < 1$ or $q-r\bar{y} < 1+\bar{y}+r\bar{y}$. To show this, note that $0 < 4p(1+r)$, add $(q-1)^2$ to the both sides, we get $(q-1)^2 < (q-1)^2+4p(1+r)$. Now, take the square root of the both sides, since we assume $q>3$, we get $q-1 < \sqrt{(q-1)^2+4p(1+r)}$ or, $q-1 < \frac{q-1+\sqrt{(q-1)^2+4p(1+r)}}{2}$. That is equivalent to $q-1 < (1+r)\bar{y}$. So $q-r\bar{y} < 1+\bar{y}+r\bar{y}$ and hence, $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < 1$.
Since, $\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}} = \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < 1$, $\cos\theta < -\frac{1}{2}$.

Also, note that $\frac{1}{2} < \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$. To show that we will use that for $q>3$, we have $2-q<0$ and then $\frac{2-q}{1+r} < p$, multiply both sides with 4, we have $8-4q < 4p(1+r)$. Add $(q-1)^2$ to the both sides, we get $q^2-6q+9 < (q-1)^2+4p(1+r)$ or $(q-3)^2 < (q-1)^2+4p(1+r)$. Take the square root of both sides. Since we take $q>3$, we get $q-3 < \sqrt{(q-1)^2+4p(1+r)}$. Add $q-1$ to the both sides, we have $2q-4 < q-1+\sqrt{(q-1)^2+4p(1+r)}$ or, $q-2 < (1+r)\bar{y}$ and hence, $q-2 < (1+3r)\bar{y}$ or, $q-r\bar{y}+\bar{y} < 2+2\bar{y}+2r\bar{y}$ or $\frac{1}{2} < \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$.
Since, $\frac{1}{2} < \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$, $\cos\theta > -1$.
So, at p^* where $\frac{1}{2} < \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}} = \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < 1$, there exists $\theta^* \in (\frac{\pi}{2},\pi)$ such that $-1 < \cos\theta^* = -\frac{1}{2}(\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}}) < -\frac{1}{2}$. Note that $e^{ik\theta^*} \neq 1$ for $k = 1,2,3,4$.

To check if $\acute{r}(p^*) \neq 0$, it is enough to show that $\frac{d|\lambda|^2}{dp}\,|_{p=p^*} \neq 0$.

$$p(\lambda) = -\lambda^3 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2 + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}.$$

$\frac{d|\lambda|^2}{dp}\,|_{p=p^*} = \frac{d(\lambda\bar{\lambda})}{dp}\,|_{p=p^*} = [\lambda\frac{d\bar{\lambda}}{dp} + \bar{\lambda}\frac{d\lambda}{dp}]\,|_{p=p^*} = \lambda(\frac{dp(\bar{\lambda})}{dp}.\frac{d\bar{\lambda}}{dp(\bar{\lambda})}) + \bar{\lambda}(\frac{dp(\lambda)}{dp}.\frac{d\lambda}{dp(\lambda)})$.

$$\frac{d\mid\lambda\mid^2}{dp}\,|_{p=p^*} = \bar{\lambda}\Bigg(\frac{-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\lambda^2 - \frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}} - \frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}}{-3\lambda^2 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda}\Bigg)$$

$$+\lambda\Bigg(\frac{-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\bar{\lambda}^2 - \frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}} - \frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}}{-3\bar{\lambda}^2 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}}\Bigg)$$

At p^* $\lambda\bar{\lambda} = 1$, so we have

$$\frac{d\mid\lambda\mid^2}{dp}\,|_{p=p^*} = \Bigg(\frac{-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\lambda^2 - \frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}} - \frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}}{-3\lambda^3 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2}\Bigg)$$

$$+\Bigg(\frac{-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\bar{\lambda}^2 - \frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}} - \frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}}{-3\bar{\lambda}^3 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2}\Bigg)$$

$$= \Bigg(\frac{(-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\lambda^2 - \frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}} - \frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}})(-3\bar{\lambda}^3 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2)}{(-3\lambda^3 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2)(-3\bar{\lambda}^3 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2)}\Bigg)$$
$$+\Bigg(\frac{(-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\bar{\lambda}^2 - \frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}} - \frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}})(-3\lambda^3 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2)}{(-3\lambda^3 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2)(-3\bar{\lambda}^3 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2)}\Bigg).$$

The denominator is non zero term since

$$(-3\lambda^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2)(-3\bar{\lambda}^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2)=9+6\frac{\bar{y}}{1+\bar{y}+r\bar{y}}(\lambda+\bar{\lambda})+4(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2.$$

At p^* $\lambda+\bar{\lambda}=-\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}}$, so the denominator becomes

$$9-6\frac{\bar{y}(q-r\bar{y}+\bar{y})}{(1+\bar{y}+r\bar{y})^2}+4(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2=9-6\frac{\bar{y}(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}-2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2.$$

Note that $\frac{\bar{y}}{1+\bar{y}+r\bar{y}}<1$ so $-(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2>-1$ and $-\frac{\bar{y}(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}>-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$.
So,

$$9-6\frac{\bar{y}(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}-2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2>9-6\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}-2$$

and since at p^* $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}<1$,

$$9-6\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}-2>9-6-2=1>0.$$

It remains to show that the numerator is non zero term.
The numerator is

$$\Big(-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\lambda^2-\frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}-\frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\Big)\Big(-3\bar{\lambda}^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2\Big)+\Big(-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\bar{\lambda}^2-\frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}-\frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\Big)\Big(-3\lambda^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2\Big)$$

$$=\frac{3}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}(\lambda+\bar{\lambda})+\Big(\frac{3r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}+\frac{3(1+r)(q-ry)}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\Big)$$
$$(\lambda^3+\bar{\lambda}^3)+\Big(\frac{2r\bar{y}}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}+\frac{2(1+r)(q-ry)y}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}\Big)(\lambda^2+\bar{\lambda}^2)+\frac{4\bar{y}}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}.$$

Recall that at p^* $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}=\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$. Also, at p^* $\lambda+\bar{\lambda}=2\cos\theta_0$, $\lambda^2+\bar{\lambda}^2=4\cos^2\theta_0-2$ and $\lambda^3+\bar{\lambda}^3=8\cos^3\theta_0-6\cos\theta_0$ where $\cos\theta_0=-\frac{1}{2}(\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}})$.
The numerator at p^* is

$$-\frac{3(q-r\bar{y}+\bar{y})}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}-\frac{3r(q-r\bar{y}+\bar{y})^3}{(1+\bar{y}+r\bar{y})^4\sqrt{(q-1)^2+4p(1+r)}}-\frac{3(1+r)(q-r\bar{y}+\bar{y})^2}{(1+\bar{y}+r\bar{y})^4\sqrt{(q-1)^2+4p(1+r)}}$$
$$+\frac{9r(q-r\bar{y}+\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}+\frac{9(1+r)}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}+\frac{2r\bar{y}(q-r\bar{y}+\bar{y})^2}{(1+\bar{y}+r\bar{y})^4\sqrt{(q-1)^2+4p(1+r)}}$$
$$+\frac{2(1+r)(q-r\bar{y}+\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^4\sqrt{(q-1)^2+4p(1+r)}}-\frac{4r\bar{y}}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}-\frac{4(1+r)\bar{y}}{(1+\bar{y}+r\bar{y})^2(q-r\bar{y}+\bar{y})\sqrt{(q-1)^2+4p(1+r)}}+\frac{4\bar{y}}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}.$$

Note that $-1<\cos\theta^*<-\frac{1}{2}$ which implies that $1<\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}}<2$ and $-\frac{1}{q-r\bar{y}+\bar{y}}>-\frac{1}{1+\bar{y}+r\bar{y}}$.
The numerator is greater than

$$-\frac{6}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}-\frac{12(1+r)}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}+\frac{9r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}+\frac{9(1+r)}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}+\frac{2r\bar{y}}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}+$$
$$\frac{2(1+r)\bar{y}}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}-\frac{4r\bar{y}}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}-\frac{4(1+r)\bar{y}}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}+\frac{4\bar{y}}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}$$

$$=\frac{2(1-r)\bar{y}}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}-\frac{6+3(1+r)+2r\bar{y}}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}$$

$$+\frac{9r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}. \tag{3.13}$$

Term (3.13) is positive if $2(1-r)\bar{y}-(6+3(1+r)+2r\bar{y})(1+\bar{y}+r\bar{y})+9r(1+\bar{y}+r\bar{y})^2>0$.
That is equivalent to $(9r^3+16r^2+7r)\bar{y}^2+(13r^2+16r-7)\bar{y}+6r-9>0$
or

$$\bar{y}>\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}.$$

Substitute the value of $\bar{y}$, we get

$$\frac{q-1+\sqrt{(q-1)^2+4p^*(1+r)}}{2(1+r)}>\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}$$

multiply both sides by $2(1+r)$ and then add $-(q-1)$ for both sides, we get

$$\sqrt{(q-1)^2+4p^*(1+r)}>2(1+r)\Big(\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}\Big)-(q-1)$$

take the square of both sides, we obtain

$$(q-1)^2+4p^*(1+r)>\Big(2(1+r)(\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)})-(q-1)\Big)^2$$

add $-(q-1)^2$ for the both sides and then multiply by $\frac{1}{4(1+r)}$, we get

$$p^* > \frac{\left(2(1+r)\left(\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}\right)-(q-1)\right)^2-(q-1)^2}{4(1+r)}$$

If term (3.13) is grater than zero, then $\frac{d|\lambda|^2}{dp}\mid_{p=p^*}>0$, and then Neimark-Sacker bifurcation conditions are satisfied.

4. Direction of Neimark-Sacker bifurcation

In this section we will use the normal form theory of discrete systems to determine the direction and the stability of the invariant closed curve bifurcating from the positive fixed point (see [9]). System (2.3) can be written as

$$Y_{n+1} = JY_n + G(Y_n) \tag{4.1}$$

where, $J = \begin{pmatrix} -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & 0 & \frac{\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and, $Y_n = \begin{pmatrix} w_n \\ v_n \\ u_n \end{pmatrix}$

$G(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y) + O(\| Y \|^3)$

$B(Y,Y) = \begin{pmatrix} B_1(Y,Y) \\ 0 \\ 0 \end{pmatrix}$ and, $C(Y,Y,Y) = \begin{pmatrix} C_1(Y,Y,Y) \\ 0 \\ 0 \end{pmatrix}$

$B_i(x,y) = \sum_{j,k=1}^{n} \frac{\partial^2 X_i(\xi)}{\partial\xi_j\partial\xi_k}\mid_{\xi=0}(x_jy_k)$ and $C_i(x,y,z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 X_i(\xi)}{\partial\xi_j\partial\xi_k}\mid_{\xi=0}(x_jy_kz_l)$

$B_1(\phi,\psi) = \frac{2(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}\phi_1\psi_1 - \frac{2r\bar{y}}{(1+\bar{y}+r\bar{y})^2}\phi_3\psi_3 + \frac{qr-(r^2+1)\bar{y}}{(1+\bar{y}+r\bar{y})^2}[\phi_3\psi_1+\phi_1\psi_3]$,

$C_1(\phi,\psi,\eta) = -\frac{6(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3}\phi_1\psi_1\eta_1 + \frac{2\bar{y}-4r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3}\Big(\phi_1\psi_1\eta_3+\phi_3\psi_1\eta_1+\phi_1+\psi_3\eta_1\Big) + \frac{2r^3\bar{y}+4r\bar{y}-2r^2q}{(1+\bar{y}+r\bar{y})^3}\Big(\phi_1\psi_3\eta_3+\phi_3\psi_1\eta_3+\phi_3\psi_3\eta_1\Big) + \frac{6r^2\bar{y}}{(1+\bar{y}+r\bar{y})^3}\phi_3\psi_3\eta_3$,

Recall that $\theta_0 = \cos^{-1}(-\frac{q-r\bar{y}+\bar{y}}{2(1+\bar{y}+r\bar{y})})$. Let q and p^* be the eigenvectors corresponding to the eigenvalues $\lambda = \cos\theta_0 + i\sin\theta_0 = e^{i\theta_0}$ and $\bar{\lambda} = \cos\theta_0 - i\sin\theta_0 = e^{-i\theta_0}$, respectively, where $q \sim \begin{pmatrix} e^{i\theta_0} \\ 1 \\ e^{-i\theta_0} \end{pmatrix}$ and $p^* \sim \begin{pmatrix} \frac{1+\bar{y}+r\bar{y}}{\bar{y}}e^{-i\theta_0} \\ e^{i\theta_0} \\ 1 \end{pmatrix}$. Note that $\sim$ means that the vector can differ from that given by a non-zero complex multiplier. To normalize q and p^*, we must find ζ such that $<\zeta p^*, q>=1$, where $<.,.>$ is the standard scalar product in $\mathbb{C}^3$.

$$<\zeta p^*, q> = \zeta\sum_{i=1}^{3}\bar{p}_i^* q_i = \zeta(\frac{1+\bar{y}+r\bar{y}}{\bar{y}}e^{-i\theta_0} + 2e^{-i\theta_0})$$

Set $\zeta = \frac{1}{\frac{1+\bar{y}+r\bar{y}}{\bar{y}}e^{2i\theta_0}+2e^{-i\theta_0}}$. So take $p = \zeta * p^*$. We have $<p,q>=1$.

The critical real eigenspace T^c corresponding to $\lambda_{1,2}$ is two-dimensional and is spanned by $\{Re(q), Im(q)\}$. The real eigenspace T^s corresponding to the real eigenvalues of J is one-dimensional. Any vector $x \in \mathbb{R}^3$ can be decomposed as

$$x = zq + \bar{z}\bar{q} + y$$

where, $z \in \mathbb{C}^1$, $\bar{z}\bar{q} \in T^c$ and $y \in T^s$. The complex variable z is a coordinate on T^c. We have

$$z = <p,x>,$$

$$y = x - <p,x>q - <\bar{p},x>\bar{q}.$$

In these coordinates, the map (4.1) takes the form

$$\tilde{z} = e^{i\theta_0}z + <p, G(zq+\bar{z}\bar{q}+y)>,$$

$$\tilde{y} = Jy + G(zq+\bar{z}\bar{q}+y) - <p, G(zq+\bar{z}\bar{q}+y)>q - <\bar{p}, G(zq+\bar{z}\bar{q}+y)>\bar{q}.$$

The previous system can be written as

$$\tilde{z} = e^{i\theta_0}z + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}G_{21}z^2\bar{z} + <G_{10},y>z + <G_{01},y>\bar{z},$$

$$\tilde{y} = Jy + \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2 + \frac{1}{2}H_{21}z^2\bar{z}$$

where,

$$G_{20} = <p, B(q,q)>, G_{11} = <p, B(q,\bar{q}>, G_{02} = <p, B(\bar{q},\bar{q})>, G_{21} = <p, C(q,q,\bar{q})>$$

and

$$H_{20} = B(q,q) - <p, B(q,q)>q - <\bar{p}, B(q,q)>\bar{q},$$

$$H_{11} = B(q,\bar{q}) - <p, B(q,\bar{q})>q - <\bar{p}, B(q,\bar{q})>\bar{q}$$

and

$$< G_{10}, y >=< p, B(q,y) >, < G_{01}, y >=< p, B(\bar{q},y) >$$

where the scaler product is in $\mathbb{C}^3$.
From the center manifold theorem, there exists a center manifold W^c which can be approximated as

$$Y = V(z,\bar{z}) = \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2$$

where $< q, w_{ij} >= 0$. The vectors $w_{ij} \in \mathbb{C}^3$ can be found from the linear equations

$$(e^{2i\theta_0}I_3 - J)w_{20} = H_{20},$$

$$(I_3 - J)w_{11} = H_{11},$$

$$(e^{-2i\theta_0}I_3 - J)w_{02} = H_{02}.$$

These equations has unique solutions. Note that the matrices $(I_3 - J)$ and $(e^{\pm 2i\theta_0}I_3 - J)$ are invertible in $\mathbb{C}^3$ since 1 and $e^{\pm 2i\theta_0}$ are not eigenvalues of J. Recall that $e^{i\theta_0} \neq 1$. So, z can be written as
$\tilde{z} = e^{i\theta_0}\bar{z} + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}[G_{21} + 2 < p, B(q,(I-J)^{-1}H_{11}) > + < p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1}H_{20}) >]z^2\bar{z} + \ldots$.
Taking into account the identities

$$(I-J)^{-1}q = \frac{1}{1-e^{i\theta_0}}q, \ (e^{2i\theta_0}I-J)^{-1}q = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}q, \ (i-j)^{-1}\bar{q} = \frac{1}{1-e^{i\theta_0}}\bar{q}$$

and

$$(e^{2i\theta_0}I-J)^{-1}\bar{q} = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}\bar{q}.$$

Also, z can be written using the map

$$\tilde{z} = e^{i\theta_0}z + \sum_{k,l\geq 2}\frac{1}{k!j!}g_{kj}z^k\bar{z}^j \tag{4.2}$$

where, $g_{20} =< p, B(q,q) >, \ g_{11} =< p, B(q,\bar{q}) >, \ g_{02} =< p, B(\bar{q},\bar{q}) >$
and $g_{21} =< p, C(q,q,\bar{q}) > +2 < p, B(q,(I-J)^{-1}B(q,\bar{q})) > + < p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q)) > + \frac{e^{-i\theta_0}(1-2e^{i\theta_0})}{1-e^{i\theta_0}} < p, B(q,q) >< p, B(q,\bar{q}) >$
$-\frac{2}{1-e^{-i\theta_0}} |< p, B(q,\bar{q}) >|^2 - \frac{e^{i\theta_0}}{e^{3i\theta_0}-1} |< p, B(\bar{q},\bar{q}) >|^2$.
The map (4.2) can be transformed into the form

$$\tilde{z} = e^{i\theta_0}z(1 + d(p^*)) \mid z^2 \mid$$

where, p^* is the value of the bifurcation parameter p where the Neimark-Sacker bifurcation exists and the real number $a(p^*) = Re(d(p^*))$, that determines the direction of bifurcation of the closed invariant curve, can be computed by the following formula

$$a(p^*) = Re\Big(\frac{e^{-i\theta_0}g_{21}}{2}\Big) - Re\Big(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})}g_{20}g_{11}\Big) - \frac{1}{2} \mid g_{11} \mid^2 - \frac{1}{4} \mid g_{02} \mid^2.$$

Now, we compute $a(p^*)$. Recall that $g_{20} =< p, B(q,q) >$

where, $B(q,q) = \begin{pmatrix} \frac{2\left(q-r\bar{y}\right)e^{2i\theta_0} - 2r\bar{y}e^{-2i\theta_0} + 2qr - 2\left(r^2+1\right)\bar{y}}{(1+\bar{y}+r\bar{y})^2} \\ 0 \\ 0 \end{pmatrix}$.

$$g_{20} = \frac{1}{e^{3i\theta_0} + 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}}\Big(\frac{2qe^{2i\theta_0} + 2qr - 2(r^2+1)\bar{y} - 4r\bar{y}\cos 2\theta_0}{(1+\bar{y}+r\bar{y})^2}\Big).$$

$g_{11} =< p, B(q,\bar{q}) >$, where $B(q,\bar{q}) = \begin{pmatrix} \frac{2\left(q-r\bar{y}\right) - 2r\bar{y} + 2\left(qr-(r^2+1)\bar{y}\right)\cos 2\theta_0}{(1+\bar{y}+r\bar{y})^2} \\ 0 \\ 0 \end{pmatrix}$

so,

$$g_{11} = \frac{1}{e^{3i\theta_0} + 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}}\Big(\frac{2(q-r\bar{y}) - 2r\bar{y} + 2(qr-(r^2+1)\bar{y})\cos 2\theta_0}{(1+\bar{y}+r\bar{y})^2}\Big).$$

$g_{02} =< p, B(\bar{q},\bar{q}) >$ where $B(\bar{q},\bar{q}) = \begin{pmatrix} \frac{2qe^{-2i\theta_0} - 4r\bar{y}\cos 2\theta + 2\left(qr-(r^2+1)\right)}{(1+\bar{y}+r\bar{y})^2} \\ 0 \\ 0 \end{pmatrix}$.

So,

$$g_{02} = \zeta\frac{1+\bar{y}+r\bar{y}}{\bar{y}}e^{-i\theta_0}\Big(\frac{2qe^{-2i\theta_0} - 4r\bar{y}\cos 2\theta + 2(qr-(r^2+1))}{(1+\bar{y}+r\bar{y})^2}\Big)$$

or

$$g_{02}=\frac{1}{e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}}\Big(\frac{2qe^{-2i\theta_0}-4r\bar{y}\cos 2\theta+2(qr-(r^2+1))}{(1+\bar{y}+r\bar{y})^2}\Big).$$

$g_{21}=< p,C(q,q,\bar{q})> +2< p,B(q,(I-J)^{-1}B(q,\bar{q}))> + < p,B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q))> + \frac{e^{-i\theta_0}(1-2e^{i\theta_0})}{1-e^{i\theta_0}} < p,B(q,q)>< p,B(q,\bar{q})>$ $-\frac{2}{1-e^{-i\theta_0}}\ |< p,B(q,\bar{q})>|^2-\frac{e^{i\theta_0}}{e^{3i\theta_0}-1}\ |< p,B(\bar{q},\bar{q})>|^2.$

$$C(q,q,\bar{q})=\begin{pmatrix}\frac{(-6(q-r\bar{y})-4r^2(q-r\bar{y})+8r\bar{y})e^{i\theta_0}+(4\bar{y}-8r(q-r\bar{y})+6r^2\bar{y})e^{-i\theta_0}+(2\bar{y}-4r(q-r\bar{y}))e^{3i\theta_0}+(4r\bar{y}-2r^2(q-r\bar{y}))e^{-i\theta_0}}{(1+\bar{y}+r\bar{y})^3}\\0\\0\end{pmatrix}$$

$$\begin{aligned}< p,C(q,q,\bar{q})> &=\frac{1}{e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}}\Bigg(\frac{\Big(-6(q-r\bar{y})-4r^2(q-r\bar{y})+8r\bar{y}\Big)e^{i\theta_0}}{(1+\bar{y}+r\bar{y})^3}\\&+\frac{\Big(4\bar{y}-8r(q-r\bar{y})+6r^2\bar{y}\Big)e^{-i\theta_0}+\Big(2\bar{y}-4r(q-r\bar{y})\Big)e^{3i\theta_0}+\Big(4r\bar{y}-2r^2(q-r\bar{y})\Big)e^{-i\theta_0}}{(1+\bar{y}+r\bar{y})^3}\Bigg).\end{aligned}$$

The second term in g_{21} is $< p,B(q,(I-J)^{-1}B(q,\bar{q}))>$

$$(I-J)^{-1}=\begin{pmatrix}\frac{1+q+\bar{y}}{1+\bar{y}+r\bar{y}} & 0 & -\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\\-1 & 1 & 0\\0 & -1 & 1\end{pmatrix}^{-1}=\begin{pmatrix}\frac{1+\bar{y}+r\bar{y}}{q+1} & \frac{\bar{y}}{q+1} & \frac{\bar{y}}{q+1}\\\frac{1+\bar{y}+r\bar{y}}{q+1} & \frac{1+q+\bar{y}}{q+1} & \frac{\bar{y}}{q+1}\\\frac{1+\bar{y}+r\bar{y}}{q+1} & \frac{1+q+\bar{y}}{q+1} & \frac{1+q+\bar{y}}{q+1}\end{pmatrix}$$

$$(I-J)^{-1}B(q,\bar{q})=\begin{pmatrix}\frac{1}{q+1}\Big(\frac{2(q-r\bar{y})-2r\bar{y}-2(qr-(r^2+1)\bar{y})\cos 2\theta_0}{1+\bar{y}+r\bar{y}}\Big)\\\frac{1}{q+1}\Big(\frac{2(q-r\bar{y})-2r\bar{y}-2(qr-(r^2+1)\bar{y})\cos 2\theta_0}{1+\bar{y}+r\bar{y}}\Big)\\\frac{1}{q+1}\Big(\frac{2(q-r\bar{y})-2r\bar{y}-2(qr-(r^2+1)\bar{y})\cos 2\theta_0}{1+\bar{y}+r\bar{y}}\Big)\end{pmatrix}=\begin{pmatrix}S\\S\\S\end{pmatrix}$$

$$B(q,(I-J)^{-1}B(q,\bar{q}))=\begin{pmatrix}\frac{2\Big(q-r\bar{y}\Big)Se^{i\theta}-2r\bar{y}Se^{-i\theta}+2\Big(qr-(r^2+1)\bar{y}\Big)S\cos 2\theta_0}{(1+\bar{y}+r\bar{y})^2}\\0\\0\end{pmatrix}$$

$$\begin{aligned}< p,B(q,(I-J)^{-1}B(q,\bar{q}))> &=\frac{1}{e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}}\Bigg(\frac{2(q-r\bar{y})Se^{i\theta}-2r\bar{y}Se^{-i\theta}}{(1+\bar{y}+r\bar{y})^2}\\&+\frac{2\Big(qr-(r^2+1)\bar{y}\Big)S\cos 2\theta_0}{(1+\bar{y}+r\bar{y})^2}\Bigg)\\(e^{2i\theta_0}I-J)^{-1} &=\begin{pmatrix}e^{2i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & 0 & -\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\\-1 & e^{2i\theta_0} & 0\\0 & -1 & e^{2i\theta_0}\end{pmatrix}^{-1}\\&=\frac{1}{D}\begin{pmatrix}e^{4i\theta_0} & \frac{\bar{y}}{1+\bar{y}+r\bar{y}} & \frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{2i\theta_0}\\e^{2i\theta_0} & e^{4i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}e^{2i\theta_0} & \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\\1 & e^{2i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & e^{4i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}e^{2i\theta_0}\end{pmatrix}\end{aligned}$$

where D is the determinant of the matrix $(e^{2i\theta_0}I-J)$ such that $D=e^{4i\theta_0}(e^{2i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})-\frac{\bar{y}}{1+\bar{y}+r\bar{y}}$.

$$(e^{2i\theta_0}I-J)^{-1}B(q,q)=\begin{pmatrix}\frac{L}{D}e^{4i\theta_0}\\\frac{L}{D}e^{2i\theta_0}\\\frac{L}{D}\end{pmatrix}$$

where, $L=\frac{2\Big(q-r\bar{y}\Big)e^{2i\theta_0}-2r\bar{y}e^{-2i\theta_0}+2\Big(qr-(r^2+1)\bar{y}\Big)}{(1+\bar{y}+r\bar{y})^2}$.

$$B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q)=\begin{pmatrix}\frac{L}{D}\Big(\frac{2(q-r\bar{y})e^{3i\theta_0}-2r\bar{y}e^{i\theta_0}+(qr-(r^2+1)\bar{y})(e^{5i\theta_0}+e^{-i\theta_0})}{(1+\bar{y}+r\bar{y})^2}\Big)\\0\\0\end{pmatrix}.$$

$< p,B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q))>=\frac{L}{D}\Big(\frac{1}{e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}}\Big)\Big(\frac{2(q-r\bar{y})e^{3i\theta_0}-2r\bar{y}e^{i\theta_0}+(qr-(r^2+1)\bar{y})(e^{5i\theta_0}+e^{-i\theta_0})}{(1+\bar{y}+r\bar{y})^2}\Big).$

$a(p^*)=Re(\frac{e^{-i\theta_0}}{2}< p,c(q,q,\bar{q})>)+Re(e^{-i\theta_0}< p,B(q,(I-J)^{-1}B(q,\bar{q}))>)+Re(\frac{e^{-i\theta_0}}{2}< p,B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q)))>).$

Theorem 4.1. *If $a(p^*) < 0$ (respectively, > 0), then Neimark-Saker bifurcation of system (2.3) at $p = p^*$ is supercritical (respectively, subcritical) and there exists a unique invariant closed curve bifurcates from the positive fixed point $\bar{y}$ which is asymptotically stable (respectively, unstable).*

5. Numerical examples

In this section we give numerical examples which support our results in the previous sections.

Example 5.1. *Take*

$$y_{n+1} = \frac{p + 4y_{n-2}}{1 + y_n + 0.3y_{n-2}} \tag{5.1}$$

with the initial conditions $y_{-2} = y_{-1} = y_0 = 1$.

$$\bar{y} = \frac{3+\sqrt{9+5.2p}}{2.6}.$$

Note that for $p < \frac{q}{1+r} = 3.0769$, the bifurcation point p^* is satisfy

$$\frac{1+\frac{3+\sqrt{9+5.2p^*}}{2}}{4+.7(\frac{3+\sqrt{9+5.2p^*}}{2.6})} = \frac{4-.3(\frac{3+\sqrt{9+5.2p^*}}{2.6})}{1+\frac{3+\sqrt{9+5.2p^*}}{2}}$$

$$(.25+\frac{.21}{(2.6)^2})(3+\sqrt{9+5.2p^*})^2 + (1+\frac{1.2}{2.6}-\frac{2.8}{2.6})(3+\sqrt{9+5.2p^*}) - 15 = 0$$

$$3+\sqrt{9+5.2p^*} = \frac{-(1+\frac{1.2}{2.6}-\frac{2.8}{2.6})+\sqrt{(1+\frac{1.2}{2.6}-\frac{2.8}{2.6})^2+4\times 15\times(0.25+\frac{0.21}{(2.6)^2})}}{2(0.25+\frac{0.21}{(2.6)^2})}$$

$$p^* = \frac{([\frac{-(1+\frac{1.2}{2.6}-\frac{2.8}{2.6})+\sqrt{(1+\frac{1.2}{2.6}-\frac{2.8}{2.6})^2+4\times 15\times(0.25+\frac{0.21}{(2.6)^2})}}{2(0.25+\frac{0.21}{(2.6)^2})}]-3)^2-9}{5.2}$$

$$p^* = 0.83564585$$

Now, we will check if Neimark-Sacker bifurcation conditions hold. By theorem (3.2), it is enough to check if

$$p^* > \frac{\left(2(1+r)\left(\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}\right)-(q-1)\right)^2-(q-1)^2}{4(1+r)}.$$

Note that at $p = p^*$, $\bar{y} = 2.55889613$ and

$$\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}} = \frac{1+.3\times 2.55889613+2.55889613}{4-.3\times 2.55889613+2.55889613} = .74708948,$$

and

$$\frac{\left(2(1+r)\left(\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}\right)-(q-1)\right)^2-(q-1)^2}{4(1+r)}$$

$$= \frac{\left(2(1.3)(\frac{-(-13(0.3)^2+16\times 0.3-7)+\sqrt{(13(0.3)^2 16\times 0.3-7)^2+4(6\times 0.3-9)(9(0.3)^3+16(0.3)^2+7\times 0.3)}}{2(9(0.3)^3+16(0.3)^2+7\times 0.3)})-3\right)^2-(3)^2}{4(1.3)}$$

$$= -1.1334411 < 0.83564585 = p^*.$$

So the condition of Theorem 3.2 is satisfied. That implies equation (5.1) undergoes a Neimark-Sacker bifurcation at $p = p^* = 0.83564585$. The bifurcation diagram of equation (5.1) is shown in Figure 5.1. Figure 5.1 shows that the positive fixed point $\bar{y}$ is asymptotically stable for $p > p^*$ and change its stability at Neimark-Sacker bifurcation value p^* and an invariant simple closed curve appears on the plane $(x(n), x(n-2))$ for $p < p^*$. Figure 5.2 and Figure 5.3 shows the phase portraits associated with Figure 5.2 for $p = p^*$ and $p = 0.95$, respectively.

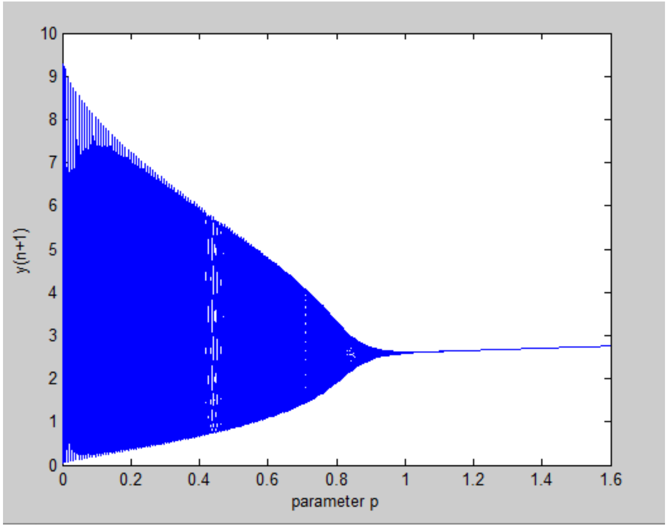

Figure 5.1: Neimark-Sacker bifurcation of the map $y_{n+1} = \frac{p+4y_{n-2}}{1+y_n+0.3yn-2}$, p is a parameter.

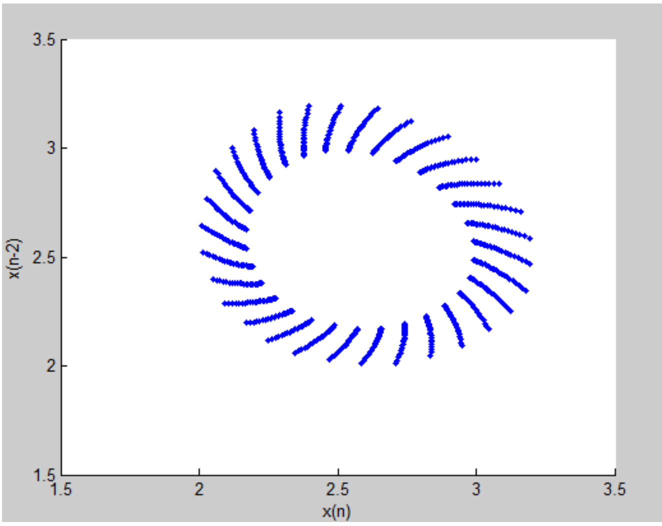

Figure 5.2: Phase portraits of the map $y_{n+1} = \frac{p+4y_{n-2}}{1+y_n+0.3yn-2}$ for $p = p^*$.

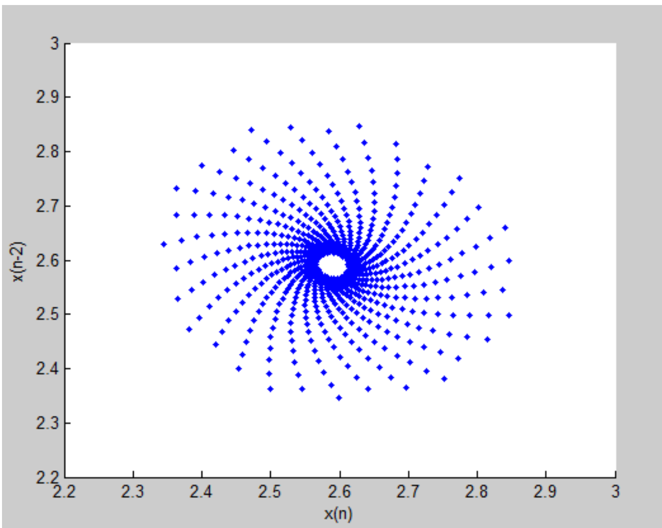

Figure 5.3: Phase portraits of the map $y_{n+1} = \frac{p+4y_{n-2}}{1+y_n+0.3yn-2}$ for $p = 0.95$.

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