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# New results for infinite functional differential inclusions with impulses effect and sectorial operators in Banach spaces

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**Abstract**

This article aims to use Bohnenblust-Karlin's fixed point theorem to obtain new results for the impulsive inclusions with infinite delay in Banah space given by the form

$$(P)\begin{cases} {}^cD_t^\alpha x(t) - Ax(t) \in F(t, x_t), & t \in J = [0, b],\ t \neq t_i, \\ x(t) = \Psi(t), & t \in (-\infty, 0], \\ \Delta x(t_i) = I_i(x(t_i^-)), & i = 1, ..., m, \end{cases}$$

where ${}^cD^\alpha$ is the Caputo derivative. We examine the case when the multivalued function $F$ is a Carathéodory and the linear part is sectorial operator defined on Banach space. Also, we provide an example to elaborate the outcomes.

*Keywords:* Fractional impulsive differential inclusions; fixed point theorems; mild solutions; fractional derivative.
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## 1. Introduction

Throughout the last two decades, the topic of equations and inclusions with fractional oreder has been considered as an interesting aspect of investigation for many researchers. Not only because of many mathematical branches involved in this topic but rather its increasing significance in applications as modeling tool in various disciplines; technology, physics, optimal control, etc. See, [11, 15, 19, 22] for more details.

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In particular, fractional equations and inclusions problems with undergoing impulses have gained a tremendous attention. That is since such problems can efficiently describe procedures which at particular moments change their state abruptly which cannot be modeled by classical differential schemes. One can find some applications in [6, 17]. With regard to the background of general theory on the subject as well as applied developments, we suggest Benchohra's book [7], the papers [8, 10] and within references.

Nevertheless, special concern has been dictated to those differential inclusions in which the past can arbitrary affect the time progress of the state parameter. Such differential equations and inclusions, namely differential equations and inclusions with delay, often occur in most significant areas involving hereditary phenomena like populations model, model of blood cell productions, electrodynamics models etc. For further specifics and some recent work, one can see[9, 10, 14, 16, 17, 21, 23, 30, 31, 32]. For instance, Shu et al. [27] introduced a different concept of mild solutions for $(P)$ without delay when $F$ is a completely continuous single-valued function and $A$ is a sectorial operator with $\{\mathcal{S}_\alpha(t) : t \geq 0\}$ and $\{\mathcal{T}_\alpha : t \geq 0\}$ are compact. Thereafter, in [28], Shu et al. proved that the solutions obtained in [27] was not correct and presented the right form of the solutions when $0 < \alpha < 1$ and $1 < \alpha < 2$. Agarwal et al. [1] proved the results of $(P)$ with the absence of impulses and delay in case when the dimension of $E$ is finite and $A$ is a sectorial operator. They determined the dimension for mild solutions set. While Ouahab [25] investigate a version of Fillippov's Theorem for $(P)$ in the case when $A$ is an almost sectorial operator and with the absence of impulses and delay. Alsarori et al. [3] proved the existence of solution for $(P)$ without delay when $F$ is upper semicontinuous and convex and $A$ is not necessarily compact. Alsarori et al. [4] established new results for $(P)$ without delay when $F$ is lower semicontinuous, nonconvex and $A$ is not compact. Shu et al. [29] considered the mild solutions to a class of impulsive fractional evolution equations of order $0 < \alpha < 1$. They proposed a more appropriate new definition of mild solutions for impulsive fractional evolution equations by replacing the impulse term operator $\mathcal{T}_\alpha(t - t_i)$ with $\mathcal{T}_\alpha(t)\mathcal{T}_\alpha^{-1}(t_i)$, where $\mathcal{T}_\alpha^{-1}(t_i)$ denotes the inverse of the fractional solution operator $\mathcal{T}_\alpha(t)$ at $t = t_i$, $i = 1, \cdots, m$.

Motivated by the previous papers and work, we study a case differs from aforementioned cases. We examine the following system:

$$(P)\begin{cases} {}^cD_t^\alpha x(t) - Ax(t) \in F(t, x_t), & t \in J = [0, b],\ t \neq t_i, \quad (0 < \alpha < 1), \\ x(t) = \Psi(t), & t \in (-\infty, 0], \\ \Delta x(t_i) = I_i(x(t_i^-)), & i = 1, ..., m, \end{cases}$$

where ${}^cD^\alpha$ is the Caputo derivative , $E$ is separable Banach space, $A : D(A) \subset E \to E$ is sectorial operator, $F : J \times \Theta \to 2^E$ is a Carathéodory multifunction, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, for every $1 \leq i \leq m$, $I_i : E \to E$, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $x(t_i^-) = \lim_{s \to t_i^-} x(s)$, and $x(t_i^+) = \lim_{s \to t_i^+} x(s)$. For any $t \in J$, the element $x_t$ of $\Theta$ represents the history of the state from $-\infty$ to the present time $t$ defined by $x_t(\theta) = x(t + \theta)$, $\theta \in (-\infty, 0]$, $\Psi \in \Theta$. $\Theta$ will be specified later.

The article starts by addressing some preliminaries and notation with respect to topics of fractional calculus and the set-valued analysis. Section 3 outlines the essential discussion of the paper, that is; the proof of existence results for $(P)$. Finally, section 4 proceeds to interpret already proved results in the previous section through giving practical example.

## 2. Preliminaries

In this section, we present some primary concepts, definitions and initial facts which are useful for the development of this article.

Let $(E, \|\cdot\|)$ be a Banach space, $C(J, E) = \{\mu : J \to E,\ \mu \ \textit{is continuous}\}$, $L^1(J, E) = \{\mathcal{G} : J \to E,\ \mathcal{G}\ \textit{is Bochner integrable}\}$, $\mathcal{L}(E) = \{f : E \to E,\ f\ \textit{is bounded and linear operator}\}$,
$P_b(E) = \{X : X \subset E,\ X \neq \varnothing, X \textit{is bounded}\}$, $P_{cl}(E) = \{X : X \subset E,\ X \neq \varnothing, X \textit{is closed}\}$, $P_k(E) = \{X : X \subset E,\ X \neq \varnothing, X \textit{is compact}\}$, $P_c(E) = \{X : X \subset E,\ X \neq \varnothing, X \textit{is convex}\}$ and $P_{ck}(E) = \{X : X \subset E,\ X \neq \varnothing, X \textit{is convex and compact}\}$.

Let $\Theta = \{\Psi : (-\infty, 0] \to E\}$ be a linear space of functions from $(-\infty, 0]$ into $E$ endowed with a seminorm $\|\cdot\|_\Theta$ and satisfies the following axioms:

1. If $x : (-\infty, b] \to E$, such that $x_0 \in \Theta$, then $\forall\, t \in J, \quad x_t \in \Theta$, and
$$\|x(t)\| \leq a\|x_t\|_\Theta,$$
where $a$ is a positive constant.

2. There are two functions $\mu_1, \mu_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$\|x_t\|_\Theta \leq \mu_1(t) \sup_{s\in[0,t]} \|x(s)\| + \mu_2(t)\|x_0\|_\Theta, \qquad \textit{for each } t \in [0, b],$$
where $\mu_1$ is continuous and $\mu_2$ is locally bounded.

3. The space $\Theta$ is complete.

Let $J_0 = [0, t_1], J_i = (t_i, t_{i+1}]$, $i = 1, \cdots, m$, we define the set of functions:

$$\Lambda = \{x : (-\infty, b] \to E : x_{|_{J_i}} \in C(J_i, E) \ \ \textit{and there exist } x(t_i^+) \textit{ and } x(t_i^-)$$
$$\textit{such that } x(t_i) = x(t_i^-)\ , \quad x_0 = \Psi \in \Theta, \quad i = 1, \cdots, m\},$$

endowed with the seminorm

$$\|x\|_\Lambda = \sup\{|x(s)| : s \in [0, b]\} + \|\Psi\|_\Theta, \quad x \in \Lambda.$$

Now, let us recall some important definitions and lemmas on multivalued functions and fractional calculus.

**Definition 2.1.** *([13], [18]). Let $F : X \to P(Y)$, where $X$ and $Y$ are topological spaces. We say that*

1. *$F$ is Upper semicontinuous (u.s.c) if $F_+^{-1}(W) = \{x \in X : F(x) \subset W\}$ is an open subset of $X$ for every open set $W \subset Y$.*

2. *$F$ is Completely continuous if $F(V)$ is relatively compact for every bounded subset $V$ of $X$.*

3. *$F$ possess a fixed point if $\exists\, x \in X$ with $x \in F(x)$.*

**Remark 2.2.** *For any closed subset $U \subset X$, if $\overline{F(U)}$ is compact and $F(u)$ is closed for every $u \in U$, then $F$ is u.s.c. iff $F$ is closed.*

**Definition 2.3.** *Let $G : J \times E \to P(E)$. We say that $G$ is Carathéodory if*

1. *$t \to G(t, u)$ is measurable for every $u \in E$.*

2. *$u \to G(t, u)$ is u. s. c. for a.e. $t \in J$.*

**Lemma 2.4.** *([20]). Let $G : J \times E \to P_{ck}(E)$ be Carathéodory multivalued map, for each $u \in E$ the set $S_G = S^1_{G,u} = \{f \in L^1(J, E) : f(t) \in G(t, u(t))$ a.e. $t \in J\} \neq \emptyset$ and $\mathcal{F} : L^1(J, E) \to C(J, E)$ be a continuous linear map. Then the operator*

$$\mathcal{F} \circ S_G : C(J, E) \to P_{ck}(C(J, E)),$$

$$u \to (\mathcal{F} \circ S_G)(u) = \mathcal{F}(S_G)$$

*is a closed graph operator in $C(J, E) \times C(J, E)$.*

**Definition 2.5.** *([19]). Let $f \in L^1(J,E)$. Fractional integration of the order $\alpha > 0$ with lower limit zero for $f$ is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t > 0.$$

*provided that the right-hand side is point-wise defined on $[0, \infty)$.*

**Definition 2.6.** *([19]). Let $f \in C^n([0,\infty), \mathbb{R})$. The Caputo derivative of the order $\alpha > 0$ for $f$ is defined as*

$$\begin{aligned} {}^cD^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s)ds \\ &= I^{n-\alpha} f^n(t), \quad t > 0, \; n = [\alpha] + 1, \end{aligned}$$

*where $[\alpha]$ is the integer part of the real number $\alpha$.*

For further details on fractional calculus, we refer to [19, 22, 24].

**Definition 2.7.** *Let $A : D(A) \subset E \to E$ be linear closed operator. We say that $A$ is sectorial if $\exists \quad \omega \in \mathbb{R}, \quad \theta \in [\frac{\pi}{2}, \pi]$ and $M > 0$, with*

1. *$\rho(A) \subset \Sigma_{(\theta,\omega)} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}$.*

2. *$\|R(\lambda, A)\|_{\mathcal{L}(E)} \leq \dfrac{M}{|\lambda - \omega|}, \quad \lambda \in \Sigma_{(\theta,\omega)}$.*

*For more details on sectorial, we refer to [5, 12].*

**Definition 2.8.** *([5]). Let $(P_1)$ define as*

$$(P_1) \begin{cases} {}^cD_t^\alpha x(t) = Ax(t), \; \alpha \in (0,1), \\ x(0) = x_0, \end{cases}$$

*where $A$ is closed and linear and $D(A)$ is dense. Then, we call the family $\{\mathcal{T}_\alpha(t) : t \geq 0\} \subset \mathcal{L}(E)$ is a solution operator for $(P_1)$ iff*

1. *$\mathcal{T}_\alpha(t)$ is strongly continuous $\forall \; t \geq 0$ and $\mathcal{T}_\alpha(0) = I$.*

2. *$\mathcal{T}_\alpha(t)D(A) \subset D(A)$ and $A\mathcal{T}_\alpha(t)x = \mathcal{T}_\alpha(t)Ax \quad \forall \; x \in D(A), \; t \geq 0$.*

3. *$\mathcal{T}_\alpha(t)x$ is solution for $(P_1)$ $\forall \; x \in D(A), \; t \geq 0$.*

**Definition 2.9.** *([5]). Let $\mathcal{T}_\alpha(\cdot)$ be the solution operator for $(P_1)$ such that $\|\mathcal{T}_\alpha(t)\|_{\mathcal{L}(E)} \leq Me^{\omega t}$, then, we say that the operator $A$ is belong to $e^\alpha(M, \omega)$, where $t \geq 0, \quad M \geq 1$ and $\omega \geq 0$.*

**Lemma 2.10.** *([27]). Let $(P_2)$ define as*

$$(P_2) \begin{cases} {}^cD_t^\alpha x(t) = Ax(t) + h(t), \; \alpha \in (0,1), \\ x(0) = x_0. \end{cases}$$

*If $A$ is sectorial operator and $h$ satisfies the uniform Hölder condition with exponent $\sigma \in (0,1]$, then $(P_2)$ has unique solution $x(t)$ defined as:*

$$x(t) = \mathcal{T}_\alpha(t)x_0 + \int_0^t \mathcal{S}_\alpha(t-s)h(s)ds,$$

*where $\mathcal{T}_\alpha(t) = \frac{1}{2\pi i} \int_\Upsilon e^{\lambda t} \lambda^{\alpha-1} \; R(\lambda^\alpha, A)d\lambda, \qquad \mathcal{S}_\alpha(t) = \frac{1}{2\pi i} \int_\Upsilon e^{\lambda t} \; R(\lambda^\alpha, A)d\lambda$, and $\Upsilon$ is a suitable path.*

**Lemma 2.11.** *([5]). Let $\alpha \in (0,1)$ and $A \in A^{\alpha}(\theta_0,\omega_0)$ with $\omega_0 \in \mathbb{R}$ and $\theta_0 \in (0,\frac{\pi}{2}]$, then*

$$\|\mathcal{T}_\alpha(t)\|_{\mathcal{L}(E)} \le Me^{\omega t} \quad \text{and} \quad \|\mathcal{S}_\alpha(t)\|_{\mathcal{L}(E)} \le Ce^{\omega t}(1+t^{\alpha-1}) \ , \quad \text{for every } \ t>0,\ \omega>\omega_0.$$

*Let*

$$M_{\mathcal{T}_\alpha} = \sup_{0\le t\le b} \|\mathcal{T}_\alpha(t)\|_{\mathcal{L}(E)}, \qquad M_{\mathcal{S}_\alpha} = \sup_{0\le t\le b} Ce^{\omega t}(1+t^{\alpha-1}).$$

*Then,*

$$\|\mathcal{T}_\alpha(t)\|_{\mathcal{L}(E)} \le M_{\mathcal{T}_\alpha}, \qquad \|\mathcal{S}_\alpha(t)\|_{\mathcal{L}(E)} \le t^{\alpha-1}M_{\mathcal{S}_\alpha},$$

*where $A^{\alpha}(\theta_0,\omega_0)=\{A\in e^{\alpha} : A$ generates analytic solution operators $\mathcal{T}_\alpha$ of type $(\theta_0,\omega_0)\}$ and $e^{\alpha}=\cup\{e^{\alpha}(\omega):\omega\ge 0\}$.*

**Definition 2.12.** *Let $x:(-\infty,b]\to E$. The function $x(t)$ is called solution for $(P)$ if*

$$x(t)=\begin{cases}\Psi(t), & t\in(-\infty,0]\\ \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t\in J_0,\\ \sum_{k=1}^{k=i}\mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-))+\int_0^t\mathcal{S}_\alpha(t-s)f(s)ds, & t\in J_i,\end{cases} \tag{1}$$

*where $i=1,\cdots,m$ and $f\in S_F^1$.*

**Theorem 2.13.** *([20]). Let $E$ be Banach space and $D\in P_{cl,c}(E)$. If $G:D\to P_{cl,c}(D)$ is upper semicontinuous and $G(D)$ is relatively compact in $E$, then $G$ has fixed point in $D$.*

## 3. Main Results

This section aims to prove the existence results for the problem $(P)$.

**Theorem 3.1.** *Let $A\in A^{\alpha}(\theta_0,\omega_0)$ such that $\theta_0\in(0,\frac{\pi}{2}]$ and $\omega_0\in\mathbb{R}$.*
*We assume the following conditions:*

*($H_1$) The semigroup $\{\mathcal{S}_\alpha(t):t>0\}$ is compact.*

*($H_2$) The multivalued function $F: J\times\Theta\to P_{ck}(E)$ is Carathéodory and for every $x\in\Theta$, the set $S^1_{F,x}=\{f\in L^1(J,E): f(t)\in F(t,x_t)$ a.e. $t\in J\}$ is nonempty.*

*($H_3$) There is $\vartheta\in L^1(J,\mathbb{R}^+)$ with*

$$\|F(t,x)\|=\sup\{|u|:u\in F(t,x)\}\le\vartheta(t)(1+\|x\|_\Theta), \qquad t\in J,\ x\in\Theta.$$

*($H_4$) $I_i:E\to E$ is continuous, compact and $\|I_i(x)\|\le h_i\|x\|$ $\forall x\in E$, where $h_i>0$, $i=1,\cdots,m$.*

*Then (P) has a mild solution on $(-\infty,b]$ provided that there is $r>0$ such that*

$$M_{\mathcal{T}_\alpha}hr+M_{\mathcal{S}_\alpha}(1+\delta_2\|\Psi\|+\delta_1 r)\frac{b^{\alpha}}{\alpha}\int_0^t\vartheta(s)ds<r,$$

*where $\delta_1=\sup_{t\in J}\mu_1(t)$, $\delta_2=\sup_{t\in J}\mu_2(t)$, and $h=\sum_{i=1}^m h_i$.*

*Proof.* We will turn problem (P) into fixed point problem. Define the multivalued function $\Pi:\Lambda\to P(\Lambda)$ such that $\Pi(y)=\{y\in\Lambda\}$ with

$$y(t)=\begin{cases}\Psi(t), & t\in(-\infty,0]\\ \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t\in J_0,\\ \sum_{k=1}^{k=i}\mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-))+\int_0^t\mathcal{S}_\alpha(t-s)f(s)ds, & t\in J_i,\end{cases}$$

where $i = 1, \cdots, m$ and $f \in S_F^1$. Clearly, mild solutions of $(P)$ are just fixed points of $\Pi$. Let $\Omega : (-\infty, b] \to E$ such that

$$\Omega(t) = \begin{cases} \Psi(t), & t \in (-\infty, 0]; \\ 0, & t \in J. \end{cases}$$

This means, $\Omega_0 = \Psi$. For all $v \in C(J, E)$ with $v(0) = 0$, let us define the function $\overline{v}$ as

$$\overline{v}(t) = \begin{cases} 0, & t \in (-\infty, 0]; \\ v(t), & t \in J. \end{cases}.$$

Let $x_t = \Omega_t + \overline{v_t}$, $t \in (-\infty, b]$. Then, $x(\cdot)$ satisfies (1) iff $v_0 = 0$, and for each $t \in J$ we have

$$v(t) = \begin{cases} \int_0^t \mathcal{S}_\alpha(t-s) f(s) ds, & t \in J_0, \\ \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t - t_k) I_k(\Omega(t_k^-) + \overline{v}(t_k^-)) + \int_0^t \mathcal{S}_\alpha(t-s) f(s) ds, & t \in J_i, \end{cases}$$

where $i = 1, \cdots, m$ and $f \in S^1_{F(\cdot, \Omega+\overline{v})}$. Let $\overline{\Lambda} = \{v \in \Lambda : v_0 = 0\}$, for each $v \in \overline{\Lambda}$ we have

$$\|v\|_{\overline{\Lambda}} = \sup_{t \in J} \|v(t)\| + \|v_0\|_\Theta = \sup_{t \in J} \|v(t)\|.$$

Therefore, $(\overline{\Lambda}, \|\cdot\|_{\overline{\Lambda}})$ is Banach space.
Now, let $G : \overline{\Lambda} \to P(\overline{\Lambda})$ be an operator define as follows: $G(v) = \{y \in \overline{\Lambda}\}$ such that

$$y(t) = \begin{cases} \int_0^t \mathcal{S}_\alpha(t-s) f(s) ds, & t \in J_0, \\ \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t - t_k) I_k(\Omega(t_k^-) + \overline{v}(t_k^-)) + \int_0^t \mathcal{S}_\alpha(t-s) f(s) ds, & t \in J_i, \end{cases} \tag{2}$$

where $i = 1, \cdots, m$ and $f \in S^1_{F(\cdot, \Omega+\overline{v})}$. Clearly, $\Pi$ has fixed point iff $G$ has fixed point. So, we show that $G$ has fixed point by using Theorem 2.13. For better readability, we break the proof into a sequence of steps.
**Step 1**. $G(v) \subset \overline{\Lambda}$ is convex for each $v \in \overline{\Lambda}$.
Let $v \in \overline{\Lambda}$, $y_1, y_2 \in G(v)$, and $\lambda \in (0, 1)$. If $t \in J_0$ from (2), we have

$$\lambda y_1(t) + (1 - \lambda) y_2(t) = \int_0^t \mathcal{S}_\alpha(t-s)[\lambda f_1(s) + (1 - \lambda) f_2(s)] ds.$$

where $f_1, f_2 \in S^1_{F(\cdot, \Omega+\overline{v})}$. Since $F$ has convex values, $S^1_{F(\cdot, \Omega+\overline{v})}$ is convex. Then, $[\lambda f_1 + (1-\lambda) f_2] \in S^1_{F(\cdot, \Omega+\overline{v})}$.
Thus, $\lambda y_1(t) + (1 - \lambda) y_2(t) \in G(v)$, $\forall t \in J_0$.
Similarly, we can prove that $\lambda y_1(t) + (1 - \lambda) y_2(t) \in G(v)$ for all $t \in J_i$, $i = 1, \cdots, m$. This means that $G(v)$ is convex for each $v \in \overline{\Lambda}$.
**Step 2**. Let $D = \{v \in \overline{\Lambda} : v(0) = 0, \|v\|_{\overline{\Lambda}} \le r\}$. Obviously, $D$ is bounded, convex and closed set in $\overline{\Lambda}$. We show that $G(D) \subset D$. Let $y \in G(v)$ , $v \in D$, by using Lemma (3.5) in [2], Lemma 2.11 with $H_3$, for $t \in J_0$, we get

$$\begin{aligned} \|y(t)\| &\le \| \int_0^t \mathcal{S}_\alpha(t-s) f(s) ds\| \le M_{\mathcal{S}_\alpha} \int_0^t (t-s)^{\alpha-1} \vartheta(s)(1 + \|\Omega + \overline{v}\|_\Theta) ds \\ &\le M_{\mathcal{S}_\alpha}(1 + \delta_2\|\Psi\| + \delta_1 r) \int_0^t (t-s)^{\alpha-1} \vartheta(s) ds \\ &\le M_{\mathcal{S}_\alpha}(1 + \delta_2\|\Psi\| + \delta_1 r) \frac{b^\alpha}{\alpha} \int_0^t \vartheta(s) ds < r. \end{aligned}$$

Similarly, if $t \in J_i$, $i = 1, 2, \cdots, m$ by using $H_4$ in addition we get

$$
\begin{aligned}
\|y(t)\| &\leq \|\sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(\Omega(t_k^-) + \overline{v}(t_k^-)\| + \|\int_0^t \mathcal{S}_\alpha(t-s) f(s) ds\| \\
&\leq M_{\mathcal{T}_\alpha} hr + M_{\mathcal{S}_\alpha} \int_0^t (t-s)^{\alpha-1} \vartheta(s)(1 + \|\Omega + \overline{v}\|_\Theta) ds \\
&\leq M_{\mathcal{T}_\alpha} hr + M_{\mathcal{S}_\alpha} (1 + \delta_2 \|\Psi\| + \delta_1 r) \frac{b^\alpha}{\alpha} \int_0^t \vartheta(s) ds < r.
\end{aligned}
$$

Which follows that $y \in D$. Then, $G(D) \subset D$.

**Step 3**. $G$ maps bounded sets into equicontinuous sets in $\overline{\Lambda}$. Let $v \in D$ with $y \in G(v)$, from the definition of $G$, there is $f \in F(\cdot, \Omega + \overline{v})$ such that

$$
y(t) = \begin{cases} \int_0^t \mathcal{S}_\alpha(t-s) f(s) ds, & t \in J_0, \\ \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(\Omega(t_k^-) + \overline{v}(t_k^-)) + \int_0^t \mathcal{S}_\alpha(t-s) f(s) ds, & t \in J_i, \end{cases}
$$

If $t \in J_0$. Let $\iota > 0$ with $t + \iota \in J_0$. Then

$$
\begin{aligned}
\|y(t+\iota) - y(t)\| &\leq \|\int_0^{t+\iota} \mathcal{S}_\alpha(t+\iota-s) f(s) ds - \int_0^t \mathcal{S}_\alpha(t-s) f(s) ds\| \\
&\leq \int_0^t \|\mathcal{S}_\alpha(t+\iota-s) f(s) - \mathcal{S}_\alpha(t-s) f(s)\| ds \\
&\quad + \int_t^{t+\iota} \|\mathcal{S}_\alpha(t+\iota-s)\| \|f(s)\| ds \\
&\leq R_1 + R_2,
\end{aligned}
$$

where

$$
\begin{aligned}
R_1 &= \int_0^t \|\mathcal{S}_\alpha(t+\iota-s) f(s) - \mathcal{S}_\alpha(t-s) f(s)\| ds, \\
R_2 &= \int_t^{t+\iota} \|\mathcal{S}_\alpha(t+\iota-s)\| \|f(s)\| ds.
\end{aligned}
$$

Since $f \in F(\cdot, \Omega + \overline{v})$, hence $f$ dependant of $\Omega + \overline{v}$, so by the definition of $\mathcal{S}_\alpha$ and Theorem of Lebesgue Dominated Convergence, we get

$$
\begin{aligned}
\lim_{\iota \to 0} R_1 &\leq \lim_{\iota \to 0} \int_0^t \|\mathcal{S}_\alpha(t+\iota-s) f(s) - \mathcal{S}_\alpha(t-s) f(s)\| ds \\
&\leq \int_0^t \lim_{\iota \to 0} \|\mathcal{S}_\alpha(t+\iota-s) f(s) - \mathcal{S}_\alpha(t-s) f(s)\| ds = 0.
\end{aligned}
$$

For $R_2$ we have

$$
\lim_{\iota \to 0} R_2 \leq \lim_{\iota \to 0} M_{\mathcal{S}_\alpha} \frac{\iota^\alpha}{\alpha} (1 + \delta_2 \|\Psi\| + \delta_1 r) \int_t^{t+\iota} \vartheta(s) ds = 0.
$$

If $t \in J_i = (t_i, t_{i+1}]$, $i = 1, \cdots, m$. Let $t, t+\iota \in J_i$, we have

$$
\begin{aligned}
&\|y(t+\iota) - y(t)\| \\
&\leq \sum_{k=1}^{k=i} \|\mathcal{T}_\alpha(t+\iota-t_k) I_k(\Omega(t_k^-) + \overline{v}(t_k^-)) - \mathcal{T}_\alpha(t-t_k) I_k(\Omega(t_k^-) + \overline{v}(t_k^-))\| \\
&\quad + \|\int_0^{t+\iota} \mathcal{S}_\alpha(t+\iota-s) f(s) ds - \int_0^t \mathcal{S}_\alpha(t-s) f(s) ds\|.
\end{aligned}
$$

Similar discussion as before with consider that $\mathcal{T}_\alpha$ is strongly continuous, we get

$$\lim_{\iota\to 0}\|y(t+\iota)-y(t)\|=0.$$

Therefore, $G(D)$ is equicontinuous.

**Step 4**. $(GD)(t)=\{y(t):y\in G(D)\}$ is relatively compact in $E$ for each $t\in J$.

For $t\in J_0=[0,t_1]$, let $0<t\le s\le t_1$ and $\varepsilon\in(0,t)$. For $v\in D$ we define

$$y_\varepsilon(t)=\int_0^{t-\varepsilon}\mathcal{S}_\alpha(t-s)f(s)ds,$$

where $f\in F(\cdot,\Omega+\overline{v})$. Since $\mathcal{S}_\alpha(t)$ is compact for $t>0$ , the set $Y_\varepsilon=\{y_\varepsilon(t):y_\varepsilon\in G(D)\}$ is relatively compact in $E$. Moreover,

$$\|y(t)-y_\varepsilon(t)\|\le\|\int_{t-\varepsilon}^{t}\mathcal{S}_\alpha(t-s)f(s)ds\| \tag{3}$$

Similarly, for $t\in J_i=(t_i,t_{i+1}]$, $i=1,\cdots,m$. Let $t_i<t\le s\le t_{i+1}$ and $\varepsilon\in(0,t)$. For $v\in D$, we define

$$y_\varepsilon(t)=\sum_{k=1}^{k=i}\mathcal{T}_\alpha(t-t_k)I_k(\Omega(t_k^-)+\overline{v}(t_k^-))+\int_0^{t-\varepsilon}\mathcal{S}_\alpha(t-s)f(s)ds,$$

where $f\in F(\cdot,\Omega+\overline{v})$. Since $\mathcal{S}_\alpha(t)$ is compact for $t>0$ and the functions $I_k$, $k=1,\cdots,m$ are compact, the set $Y_\varepsilon=\{y_\varepsilon(t):y_\varepsilon\in G(D)\}$ is relatively compact in $E$. Furthermore,

$$\|y(t)-y_\varepsilon(t)\|\le\|\int_{t-\varepsilon}^{t}\mathcal{S}_\alpha(t-s)f(s)ds\|. \tag{4}$$

Obviously, the right hand side of (3) and (4) tend to zero as $\varepsilon\to 0$. Hence, there exists a relatively compact set that can arbitrary close to $(GD)(t)=\{y(t):y\in G(D)\}$ for $t\in J$. Therefore, $(GD)(t)$ is relatively compact in $E$ for $t\in J$.

As a consequence of Step 2 to 4 together with the Arzela-Ascoli theorem we conclude that $G$ is completely continuous.

**Step 5**. $G$ has closed graph. Let $v_n\to v^*$, $y_n\in G(v_n)$, $y_n\to y^*$, as $n\to\infty$. We claim that $y^*\in G(v^*)$. Because, $y_n\in G(v_n)$, $n\ge 1$, from the the definition of $G$, there exists $f_n\in S^1_{F(\cdot,\Omega_n+\overline{v_n})}$ with

$$y_n(t)=\begin{cases}\int_0^t\mathcal{S}_\alpha(t-s)f_n(s)ds, & t\in J_0,\\ \sum_{k=1}^{k=i}\mathcal{T}_\alpha(t-t_k)I_k(\Omega_n(t_k^-)+\overline{v_n}(t_k^-))+\int_0^t\mathcal{S}_\alpha(t-s)f_n(s)ds, & t\in J_i.\end{cases}$$

For $t\in J_0$, we prove that there is $f^*\in S^1_{F(\cdot,\Omega^*+\overline{v}^*)}$ with

$$y^*(t)=\int_0^t\mathcal{S}_\alpha(t-s)f^*(s)ds.$$

Let $\mathcal{F}:L^1(J_0,E)\to C(J_0,E)$ defined by

$$\mathcal{F}(f)(t)=\int_0^t\mathcal{S}_\alpha(t-s)f(s)ds.$$

Obviously, $\mathcal{F}$ is continuous linear operator. From Lemma 2.4, $\mathcal{F}\circ S^1_F$ is closed graph operator. Also, for all $t\in J_0$, we get

$$y_n(t)\in\mathcal{F}(S^1_{F(\cdot,\Omega_n+\overline{v_n})}).$$

Since $v_n\to v^*$ and $y_n\to y^*$, $\forall t\in J_0$ we have

$$y^*(t)=\int_0^t\mathcal{S}_\alpha(t-s)f^*(s)ds,$$

for some $f^* \in S^1_{F(\cdot,\Omega^*+\overline{v^*})}$.
Similarly, for any $t \in J_i,\ \ i = 1, \cdots, m$, we have

$$y_n(t) = \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(\Omega_n(t_k^-) + \overline{v_n}(t_k^-)) + \int_0^t \mathcal{S}_\alpha(t-s) f_n(s) ds.$$

We prove that for $t \in J_i$, there is $f^* \in S^1_{F(\cdot,\Omega^*+\overline{v^*})}$ with

$$y^*(t) = \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(\Omega^*(t_k^-) + \overline{v^*}(t_k^-)) + \int_0^t \mathcal{S}_\alpha(t-s) f^*(s) ds.$$

For each $t \in J_i$, $i = 1, 2, \cdots, m$, we have

$$\|[y_n(t) - \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(\Omega_n(t_k^-) + \overline{v_n}(t_k^-))]$$

$$-[y^*(t) - \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(\Omega^*(t_k^-) + \overline{v^*}(t_k^-))]\| \to 0 \quad as \quad n \to \infty.$$

Now, let us define the linear continuous operator $\mathcal{F} : L^1(J_i, E) \to C(J_i, E)$ such that

$$\mathcal{F}(f)(t) = \int_0^t \mathcal{S}_\alpha(t-s) f(s) ds.$$

By Lemma 2.4 and definition of $\mathcal{F}$ we have $\mathcal{F} \circ S^1_F$ is a closed graph operator. Moreover, for every $t \in J_i$,

$$[y_n(t) - \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(\Omega_n(t_k^-) + \overline{v_n}(t_k^-))] \in \mathcal{F}(S^1_{F(\cdot,\Omega_n+\overline{v_n})}).$$

Since $v_n \to v^*$, for some $f^* \in S^1_{F(\cdot,\Omega^*+\overline{v^*})}$ , it follows that, for each $t \in J_i$, $i = 1, \cdots, m$, we have

$$y^*(t) = \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(\Omega^*(t_k^-) + \overline{v^*}(t_k^-)) + \int_0^t \mathcal{S}_\alpha(t-s) f^*(s) ds.$$

Then, $G$ is closed. By Remark 2.2, $G$ is u. s. c.. Hence, by Theorem 2.13, $G$ has fixed point $v \in \overline{\Lambda}$ which is mild solution for the problem $(P)$.

□

## 4. Example

For all $z \in [0, \pi]$, and $i = 1, 2, \cdots, m$, consider the problem:

$$\begin{cases} \partial_t^\alpha u(t,z) - \partial_z^2 u(t,z) \in R(t, u(v,z)), & t \in [0,1],\ t \neq t_i, \\ u(t,0) = 0, & \\ u(t,\pi) = 0, & \\ \Delta u(t_i)(z) = \int_{-\infty}^{t_i} \beta_i(t_i - \mu) d\mu\ \cos(u(t_i)(z)), & \\ u(t,z) = u_0(v,z), & -\infty < v \leq 0, \end{cases} \tag{5}$$

where $\partial_t^\alpha$ is the Caputo fractional partial derivative $(0 < \alpha < 1)$, $t \in [0,1]$, $z \in [0,\pi]$ and $\beta_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, m$. Put $E = L^2([0,\pi]])$, and let $A : D(A) \subset E \to E$ defined by $Ay = y''$, such that $D(A) = \{y \in E : y, y'$ are absolutely continuous, $y'' \in E, y(0) = y(\pi) = 0\}$.

Then,

$$Ay = \sum_{n=1}^{\infty} n^2 (y, y_n) y_n, \quad y \in D(A),$$

where $y_n(s) = \sqrt{2}\sin(ns)$, $n \in \mathbb{N}$, is the orthogonal set of eigenvectors of $A$. From [26], $A$ generates analytic semigroup $\{T(t) : t \geq 0\}$ in $E$ given by

$$T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t}(y, y_n) y_n, \quad \forall\, y \in E, \forall\, t > 0.$$

So, $\{T(t) : t > 0\}$ is uniformly bounded compact. Therefore, $R(\lambda, A) = (\lambda - A)^{-1}$ is a compact operator for each $\lambda \in \rho(A)$. This means $A \in A^{\alpha}(\theta_0, \omega_0)$.

Set

$$I_i(x(t_i^-))(z) = \int_{-\infty}^{0} \beta_i(t_i - \mu) d\mu \; \cos(u(t_i)(z)), \quad i = 1, \cdots, m.$$

Also, we define $\Psi : (-\infty, 0] \to E$ by

$$\Psi(t) = u_0(v, z), \quad -\infty < v \leq 0, \quad z \in [0, \pi].$$

If $F(t, x_t)(z) = R(t, u(t, z))$, $z \in [0, \pi]$. Then, we can rewrite (5) as:

$$\begin{cases} {}^cD^{\alpha}x(t) - Ax(t) \in F(t, x_t), & t \in J = [0,1], t \neq t_i, i = 1, ..., m, \\ x(t) = \Psi(t), & -\infty < t \leq 0, \\ \Delta x(t_i) = I_i(x(t_i^-)), & i = 1, ..., m. \end{cases}$$

If we put stable conditions on $F$ as in Theorem 3.1, the system (5) has a mild solution on $(-\infty, 1]$.

## Conclusion

The present article succeeded to capture sufficient conditions with which the given system $(P)$ has mild solutions. Beside various techniques and methods, we mainly relied on Bohnenblust-Karlin's fixed point theorem. In essence, the results drawn by this paper extended and improved some previous related studies. To enhance our findings, we manage to provide a numerical example in the last section.

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