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NEW REPRODUCING KERNELS AND HOMOGENIZING TRANSFORMS FOR SOME BOUNDARY VALUE PROBLEMS

BAZI SINIR DEĞER PROBLEMLERİ İÇİN YENİ ÜRETİCİ ÇEKİRDEKLER VE HOMOJENLEŞTİRME DÖNÜŞÜMLERİ

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Abstract

Nonlinear boundary value problems have a significant role in the science. The solution approximations are also important as much as problems. In this study, new reproducing kernel spaces are constructed and reproducing kernel functions have been obtained for some boundary value problems. In the reproducing kernel theory, it is higly important to study wih homogeneous differential equation with the homogeneous conditions. For this purpose, homogenizing transformation functions have been found and nonlinear nonhomogeneous problems transformed to the homogeneous form.

Keywords: Boundary value problems, nonhomogeneous ordinary differential equations, reproducing kernel functions, reproducing kernel method.

Öz

Lineer olmayan sınır değer problemleri fizikte ve matematikte önemli bir yer tutmaktadır. Problemlere dair çözüm yaklaşımları ise bir o kadar öneme haizdir. Bu çalışmada, bazı yeni üretici çekirdekli uzaylar inşa edilerek bu uzaylara ait üretici çekirdek fonskiyonları elde edildi. Üretici çekirdek teorisi gereği çalışılan denklemin ve denkleme ait sınırşartlarının muhakkak suretle homojen olması önemli olduğundan, homojen olmayan sınır değer problemleri özel dönüşüm fonksiyonları kullanılarak homojen hale getirildi.

Anahtar Kelimeler: Homojen olmayan adi diferansiyel denklemler, sınır değer problemleri, üretici çekirdek fonksiyonları, üretici çekirdek metodu.

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1.INTRODUCTION

In this study, by using reproducing kernel method we aim to find the reproducing kernel functions and homogenizing transforms the problems in the form :

$$(\mathbb{L}h)(x) = \rho(x)r^n(x) + \eta(x)r^{n-1}(x) = g(x), \qquad a \leq x \leq b$$

The reproducing kernel functions have a vital role to solve different types of differential equations. The theory of reproducing kernel was begin with the research of Aronszajn and Bergman (Aronszajn, 1950; Bergman, 1950). Since the method is very powerful, many researcher used the reproducing kernel functions for several kind of problems. For example Cui et al. (Cui and Lin, 2009) published a book about numerical analysis in the reproducing kernel space which is a very important study. Syam et al. (Syam et al., 2017) studied a class of fractional Sturm-Liouville eigenvalue problems. Jiang and Tian (Jiang and Tian, 2015) examined the Volterra integro-differential equations of fractional order by the reproducing kernel method. Li et al. (Li and Wu, 2014) applied the method for numerical solutions of fractional Riccati differential equations. For more details see (Akram and Rehman, 2013;Alvandi and Paripour, 2016; Freihat et al. 2016).

In many models and problems, the equations need to be solved numerically. Therefore many approaches have been used and there have been lots of efforts for solving non-linear higher order ordinary differential equations in researches. For instance, Homotopy perturbation method (Abbasbandy, 2006), Adomian decomposition method (Hasan and Zhu, 2009), Chebyshev collocation method (Daşçıoğlu and Yaslan, 2011) used. Adomian decomposition method for solving initial value problems in second-order ordinary differential equations is given in (Wazwaz, 2002). Lu et al. Furthermore Runge- Kutta method (Wazwaz,, 1983), Predictor-Corrector method (Fox and Mayers, 1987), decomposition method (Wazwaz, 2001), direct block method (Waeleh et al., 2012) have been used for solving IVP. For a further reading and more details one can see (Coddington and Levinson, 1971; Hoppensteadt, 1971; Sell, 1965; Yokuş and Kaya, 2020; Yokuş, 2020).

2. PRELIMINARIES

In this section, we present some essential definitions and theorems of reproducing kernel theory.

Definition 2.1. [*Reproducing Kernel*] (Bergman, 1950) Let X be a nonempty set. A function $Q: X \times X \to \mathbb{F}$ is called a reproducing kernel of the Hilbert space $\mathcal{H}$ if and only if

1. $Q(\cdot, s) \in \mathcal{H}, \quad \forall s \in X,$
2. $\langle \ \psi \ , Q(\cdot, s) \rangle = \psi(s).$

The item (b) is called "reproducing property" of kernel Q. The value of the function ψ at the point s is reproduced by the inner product of γ with $Q(\cdot, s)$.

Definitin 2.1. (Cui and Lin, 2009) The space $U_2^m[a,b]$ consist of the functions $r: [a,b] \to \mathbb{R}$ and define as follows:

$$U_2^m[a,b] = \{r(x) | r^{(m-1)}(x) \in AC[a,b], \quad r^{(m)}(x) \in L^2[a,b], \quad x \in [a,b]\}. \tag{1}$$

$U_2^m[a,b]$ equipped with the inner product

$$< r,k >_{U_2^m} = \sum_{i=0}^{m-1} r^{(i)}(a)k^{(i)}(a) + \int_a^b r^{(m)}(x)k^{(m)}(x)dx. \quad (2)$$

Here we denote the vector space of absolutely continuous (real-valued) functions with $AC[a,b]$ and the quadratically integrable functions on the interval $[a,b]$ with $L^2[a,b]$.

Lemma 2.1. If a Hilbert space has a reproducing kernel, it is called a reproducing kernel Hilbert space (RKHS).

Lemma 2.2. (Cui and Lin, 2009) $U_2^m[a,b]$ function space is a reproducing kernel space.

The reproducing kernel function of the space U_2^m can be written as:

$$Q_x(y) = \begin{cases} Q(x,y) = \sum_{i=1}^{2m} b_i(y)x^{i-1}, & x \le y, \\ Q(y,x) = \sum_{i=1}^{2m} d_i(y)x^{i-1}, & x > y. \end{cases} \quad (3)$$

For the proof of Lemma 2 one can see (Cui and Lin, 2009).

2.1. $U_2^3[0,1]$ Reproducing Kernel Space and Its Kernel Function

Let we define a function space for $m = 3$

$$U_2^3[1,2] = \{k(x) | k''(x) \in AC[1,2], \quad k'''(x) \in L^2[1,2], \quad x \in [1,2]\}$$

with the inner product

$$\langle k, Q_y \rangle_{U_2^3[1,2]} = k(1)Q_y(1) + k'(1)Q'_y(1) + k''(1)Q''_y(1) + \int_1^2 k^{(3)}(x)Q_y^{(3)}(x)dx.$$

We use integration by parts and obtain

$$\begin{aligned} < k, Q_y >_{U_2^3[1,2]} &= k(1)Q_y(1) + k'(1)Q'_y(1) + k''(1)Q''_y(1) + k''(2)Q_y^{(3)}(2) \\ &-k''(1)Q_y^3(1) - k'(2)Q^{(4)}(2) + k'(1)Q^{(4)}(1) + k(2)Q^{(5)}(2) \\ &-k(1)Q^{(5)}(2) - \int_1^2 k(x)Q^{(6)}(x)dx. \end{aligned}$$

We have $Q_y(1) = 0 = Q_y(2)$ by the conditions in 3. Therefore we get,

$$\begin{aligned} < k, Q_y >_{U_2^3[1,2]} &= k'(1)Q'_y(1) + k''(1)Q''_y(1) + k''(2)Q_y^{(3)}(2) - k''(1)Q_y^3(1) \\ &-k'(2)Q_y^4(2) + k'(1)Q_y^4(1) - \int_1^2 k(x)Q_y^{(6)}(x)dx. \end{aligned}$$

If we have the following equations:

1. $Q_y'(1) + Q_y^{(4)}(1) = 0.$
2. $Q_y''(1) - Q_y'''(1) = 0.$
3. $Q_y^{(3)}(2) = 0.$
4. $Q_y^{(4)}(2) = 0.$ (4)

We will get

$$< k, Q_y >_{U_2^3[1,2]} = -\int_1^2 k(x) Q_y^{(6)}(x) dx.$$

Note that property of the reproducing kernel is

$$< k, Q_y >_{U_2^3[1,2]} = k(y).$$

Thus, we reach

$$-\int_1^2 k(x) Q_y^{(6)}(x) dx = k(y).$$

This gives us the Dirac-Delta function

$$-Q_y^{(6)}(x) dx = \delta(x - y).$$

When $x \neq y$, we get

$$Q_y^{(6)}(x) = 0.$$

Therefore, we obtain the reproducing kernel function Q_y as:

$$Q_y(x) = \begin{cases} \sum_{i=1}^{6} b_i x^{i-1}, & x \leq y, \\ \sum_{i=1}^{6} d_i x^{i-1}, & x > y. \end{cases}$$

There are twelve unknown coefficients. So we need twelve equations to find these unknown coefficients. By Dirac-Delta function:

5. $Q_{y^+}(y) = Q_{y^-}(y).$
6. $Q'_{y^+}(y) = Q'_{y^-}(y).$
7. $Q''_{y^+}(y) = Q''_{y^-}(y).$
8. $Q'''^{y^+}(y) = Q'''_{y^-}(y).$
9. $Q_{y^+}^{(4)}(y) = Q_{y^-}^{(4)}(y).$
10. $Q_{y^+}^{(5)}(y) = Q_{y^-}^{(5)}(y).$

We have the following equations:

$$11.\, Q_y(1)=0 \quad , \qquad 12.\, Q_y(2)=0. \tag{5}$$

So we have twelve unknown coefficients and twelve equations. when we solve these equations by using Maple 17, we get the reproducing kernel function for $x \leq y$ as:

$$\begin{aligned} Q_y(x) = {} & \frac{3}{13}xy - \frac{1}{156}xy^5 + \frac{5}{156}xy^4 - \frac{5}{78}xy^3 - \frac{5}{26}xy^2 + \frac{21}{104}x^2y^2 - \frac{1}{624}x^2y^5 \\ & + \frac{5}{624}x^2y^4 - \frac{5}{312}x^2y^3 - \frac{5}{26}x^2y + \frac{7}{104}x^3y^2 - \frac{1}{1872}x^3y^5 + \frac{5}{1872}x^3y^4 \\ & - \frac{5}{936}x^3y^3 - \frac{5}{78}x^3y - \frac{1}{104}x^4y + \frac{1}{3744}x^4y^5 - \frac{5}{3744}x^4y^4 + \frac{5}{1872}x^4y^3 \\ & + \frac{5}{624}x^4y^2 - \frac{1}{18720}x^5y^5 + \frac{1}{3744}x^5y^4 - \frac{1}{1872}x^5y^3 - \frac{1}{624}x^5y^2 - \frac{1}{156}x^5y \\ & + \frac{1}{120}x^5. \end{aligned}$$

2.2. $U_2^4[0,1]$ Reproducing Kernel Space and Its Kernel Function

Because of the theory, in order to find the reproducing kernel function we first need to construct the space which is related to the derivative of the problem. If we choose $m=4$ in the equation (1) then we get space definition as follows:

$$U_2^4[0,1] = \{k(x) | k''(x) \in AC[0,1], \quad k'''(x) \in L^2[0,1], \quad x \in [0,1]\}$$

with the inner product

$$\begin{aligned} < k, Q_y >_{U_2^4[0,4\pi]} & = k(0)Q_y(0) + k'(0)Q'{}_y(0) + k''(0)Q''{}_y(0) - k^{(3)}(0)Q_y^{(3)}(0) \\ & + \int_0^1 \Big(k(x)^{(4)}(x)Q_y^{(4)}(x)\Big)dx. \end{aligned}$$

Integrating this equation by parts for four times, we have

$$\begin{aligned} < k, Q_y >_{U_2^4[0,1]} = {} & k(0)Q_y(0) + k'(0)Q'{}_y(0) + k''(0)Q''{}_y(0) - k^{(3)}(0)Q_y^{(3)}(0) \\ & + k^{(3)}(1)Q_y^{(4)}(1) - k^{(3)}(0)Q_y^{(4)}(0) - k''(1)Q_y^{(5)}(1) \\ & + k''(0)Q_y^{(5)}(0) + k'(1)Q_y^{(6)}(1) - k'(0)Q_y^{(6)}(0) \\ & - k(1)Q_y^{(7)}(1) + k(0)Q_y^{(7)}(0) + \int_0^1 k(x)Q_y^{(8)}(x)dx. \end{aligned}$$

Because of the conditions, we get the following equations:

$$1.\, Q_y(0)=0, \;\; 2.\; Q_y'(0)=0, \;\; 3.\; Q_y''(0)=0. \tag{6}$$

With these three functions being zero we obtain:

$$\begin{aligned} < k, Q_y >_{U_2^4[0,1]} = {} & k^{(3)}(0)Q_y^{(3)}(0) + k^{(3)}(1)Q_y^{(4)}(1) - k^{(3)}(0)Q_y^{(4)}(0) \\ & - k''(1)Q_y^{(5)}(1) + k'(1)Q_y^{(6)}(1) - k(1)Q_y^{(7)}(1) + \int_0^1 k(x)Q_y^{(8)}(x)dx. \end{aligned}$$

When the equation is rearranged we get the following equations:

4. $Q_y^{(3)}(0) - Q_y^{(4)}(0) = 0.$
5. $Q_y^{(4)}(1) = 0.$
6. $Q_y^{(5)}(1) = 0.$
7. $Q_y^{(6)}(1) = 0.$
8. $Q_y^{(7)}(1) = 0.$ (7)

Then we will get:

$$< k, Q_y >_{U_2^4[0,1]} = \int_0^1 k(x) Q_y^{(8)}(x) dx$$

With the knowledge of reproducing kernel property, the function $k(y)$ can be written in the form:

$$< k, Q_y >_{U_2^4[0,1]} = k(y).$$

For this reason, we reach

$$\int_0^1 k(x) Q_y^{(8)}(x) dx = k(y). \tag{8}$$

Because of the definition of Dirac-Delta function, it is obvious that the equation (8) is equal to the $\delta(x-y)$. That gives us the following equation:

$$Q_y^{(8)}(x) = \delta(x-y).$$

When $x \neq y$, the reproducing kernel function Q_y can be written in the form as:

$$Q_y(x) = \begin{cases} \sum_{i=1}^{8} b_i x^{i-1}, & x \leq y, \\ \sum_{i=1}^{8} d_i x^{i-1}, & x > y. \end{cases}$$

By using the feature of Dirac-Delta function, the following equations can be written:

9. $Q_{y^+}(y) = Q_{y^-}(y).$
10. $Q'_{y^+}(y) = Q'_{y^-}(y).$
11. $Q''_{y^+}(y) = Q''_{y^-}(y).$
12. $Q'''_{y^+}(y) = Q'''_{y^-}(y).$
13. $Q_{y^+}^{(4)}(y) = Q_{y^-}^{(4)}(y).$
14. $Q_{y^+}^{(5)}(y) = Q_{y^-}^{(5)}(y).$
15. $Q_{y^+}^{(6)}(y) = Q_{y^-}^{(6)}(y).$
16. $Q_{y^+}^{(7)}(y) - R_{y^-}^{(7)}(y) = 1.$ (9)

In order to find the reproducing kernel function of the given space, we need to solve the differential equation system above. For this purpose, we needed sixteen equation since the (2.2) has sixteen coefficients and we obtained them. If we solve the system thus we get the reproducing kernel function as:

$$
\begin{aligned}
Q_y(x) = {} & \frac{1}{288}x^3y^3 + \frac{1}{5760}x^3y^7 - \frac{7}{5760}x^3y^6 + \frac{7}{1920}x^3y^5 - \frac{7}{1152}x^3y^4 + \frac{1}{1152}x^4y^3 \\
& + \frac{1}{23040}x^4y^7 - \frac{2}{23040}x^4y^6 + \frac{7}{7680}x^4y^5 - \frac{7}{4608}x^4y^4 - \frac{1}{240}x^5y^2 \\
& - \frac{1}{38400}x^5y^7 + \frac{7}{38400}x^5y^6 - \frac{7}{12800}x^5y^5 + \frac{7}{7680}x^5y^4 + \frac{7}{1920}x^5y^3 \\
& + \frac{1}{720}x^6y + \frac{1}{115200}x^6y^7 - \frac{7}{115200}x^6y^6 + \frac{7}{38400}x^6y^5 - \frac{7}{23040}x^6y^4 \\
& - \frac{7}{5760}x^6y^3 - \frac{1}{806400}x^7y^7 + \frac{1}{115200}x^7y^6 - \frac{1}{38400}x^7y^5 + \frac{1}{5760}x^7y^3 \\
& - \frac{1}{5040}x^7.
\end{aligned}
$$

3. BOUNDARY VALUE PROBLEMS AND HOMOGENIZING TRANSFORMS

In this section we will determine the homogenizing transforms of the given problems.

Example 3.1. Consider the nonlinear boundary value problem (Hasan and Zhu, 2009)

$$y''' - \frac{2}{x}y'' - y - y^2 = g(x) \tag{10}$$
$$y(0) = 0, y'(0) = 0, y(1) = 2,71828182, \tag{11}$$

where $g(x) = 7x^2e^x + 6xe^x - 6e^x - x^6e^{2x}$. The exact solution of the problem is $y(x) = xe^x$.

It is needed to transform the differential equation to the homogeneous type for reproducing kernel method. For this reason we seek the function y of the form $y(x) = Y(x) + S(x)$. This will provide new boundary conditions which are homogeneous. We call transformation function with $Y(x)$ which satisfies the initial conditions and $Y(x)$ will be the new homogeneous initial value problem. For the equation (3), let we use the transformation function given follow as:

$$S(x) = ax^2.$$

Here $a = 2,71828182$. Let we write y function as:

$$y(x) = Y(x) + ax^2$$

With the help of the deravative calculations we get

$$
\begin{aligned}
y'(x) &= Y'(x) + 2ax, \\
y''(x) &= Y''(x) + 2a, \\
y'''(x) &= Y'''(x).
\end{aligned}
$$

By substituting these equations into the (3), the equation will transform to

$$Y'''(x) - \frac{2}{x}Y''(x) - Y(x)(1 + 2ax^2) = Y^2(x) + g(x) + \frac{4}{x}a + ax^2 + a^2x^4, \qquad x \in [0,1]$$

with the boundary conditions

$$Y(0) = Y'(0) = Y(1) = 0.$$

This is the new homogeneous boundary value problem.

Example 3.2. Consider the nonlinear boundary value problem (Hasan and Zhu, 2009):

$$\begin{aligned} & y'' - \frac{1}{x}y' = \frac{4x^2}{4+x^2}e^y, \\ & y(0) = ln\left(\frac{1}{4}\right), y(1) = ln\left(\frac{1}{5}\right) \end{aligned} \tag{12}$$

with the exact solution $y(x) = ln\left(\frac{1}{4+x^2}\right)$.

Similar to the previous example we choose the transformation function as:

$$T(x) = ln\left(\frac{4}{5}\right)x + ln\left(\frac{1}{4}\right).$$

Hence, the $y(x)$ will become

$$y(x) = Y(x) + ln\left(\frac{4}{5}\right)x + ln\left(\frac{1}{4}\right).$$

By calculating the necessary derivatives we obtain

$$\begin{aligned} & y'(x) = Y'(x) + ln\left(\frac{4}{5}\right), \\ & y''(x) = Y''(x). \end{aligned}$$

If we put these functions into the 3 we arrive

$$\begin{aligned} & Y''(x) - \frac{1}{4}\left(Y'(x) + ln\left(\frac{4}{5}\right)\right) = \frac{4x^2}{4+x^2}e^{[Y(x)+ln\left(\frac{4}{5}\right)x+ln\left(\frac{1}{4}\right)]} \\ & Y''(x) - \frac{1}{4}Y'(x) = \frac{x^2}{4+x^2}\left(\frac{4}{5}\right)^x e^{[Y(x)+ln\left(\frac{4}{5}\right)x+ln\left(\frac{1}{4}\right)]} + \frac{1}{4}ln\left(\frac{4}{5}\right). \end{aligned}$$

The last equation is the new boundary value problem with homogeneous boundary conditions below

$$Y(0) = 0, \qquad Y(1) = 0.$$

4. CONCLUSION

In this study, we presented the new reproducing kernel functions and their special reproducing kernel Hilbert spaces which belongs to the problems given section 3 . By homogenizing the given problems we obtained new boundary conditions and new boundary value problems which are ready to apply the reprocuding kernel method. This study is important and opens a door for the further studies to apply the kernel method easily.

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