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Double reduction of second order Benjamin-Ono equation via conservation laws and the exact solutions

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Abstract

In this study, the Benjamin-Ono equation which was first introduced to describe internal waves in stratified fluids are considered. Using the association between Lie point symmetries and local conserved vectors, a reduction in both the number of variables and the order of the equation is achieved. The auxiliary equation method successfully applied to the reduced equation and different types of solutions are obtained. Moreover, some graphical representations for special values of the parameters in solutions are presented.

Keywords: *Double reduction method, conservation vectors, Benjamin-Ono equation.*

İkinci mertebeden Benjamin-Ono denkleminin korunum kanunları yardımıyla çift indirgemesi ve tam çözümleri

Öz

Bu çalışmada, ilk kez tabakalı sıvılardaki iç dalgaları tanımlamak için sunulan Benjamin-Ono denklemini ele alınmıştır. Lie nokta simetrileri ve yerel korunum vektörleri arasındaki ilişkiyi kullanarak hem değişken sayısında hem de denklemin mertebesinde bir indirgeme elde edilmiştir. İndirgenen denkleme yardımcı denklem metodu başarılı bir şekilde uygulanmş ve farklı tipte çözümler elde edilmiştir. Ayrıca çözümlerdeki parametrelerin özel değerleri için bazı grafik temsilleri verilmiştir.

Anahtar kelimeler: *Çift indirgeme yöntemi, korunum vektörleri, Benjamin-Ono denklemi.*

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1. Introduction

Nonlinear evolution equations occur not only in many areas of mathematics but also in other disciplines such as biology, engineering sciences, space sciences, physics,quantum mechanics,chemistry and materials science. There are many nonlinear evolution equations naturally arising from various branches of science such as the nonlinear Sine-Gordon equation in quantum mechanics[1], the Cahn-Hilliard equation in the study of phase transitions of binary alloys [2], the Navier-Stokes equations in the study of the flow of viscous incompressible fluids [3].

Thanks to advances in computer science, using software programs (e.g. Maple, Mathematica), a number of useful methods and theories have been developed and implemented to find solutions to nonlinear evolution equations [4-23].

In this paper, we consider the second order Benjamin-Ono equation [24, 25]

$$u_{tt} + \beta(u^2)_{xx} + \gamma u_{xxxx} = 0, \tag{1}$$

which is presented to model the percolation of water on the porous surface of a horizontal layer of material, as well as the analysis of long waves in shallow water. In Eq. (1), dependent variable is the elevation of the free surface of the fluid; the vertical deflection or the quadratic nonlinearity accounts for the curvature of the bending beam, γ is the fluid depth, β is a constant controlling nonlinearity and the characteristic speed of the long waves [26]. Many researches have been conducted on this equation, which has attracted the attention of researchers for many years [27-31].

This paper is structured as follows: In Section 2 and 3, we introduce some properties of the double reduction and auxiliary equation method, respectively. In Section 4, we apply these methods to find the solutions of underlying equation. Our discussions and conclusions are given in Sections 5 and 6, respectively.

2. Overview of double reduction method

Here, the relationships between Lie symmetries and conservation laws of systems of partial differential equations (PDEs) will be presented. Then, how to perform double reduction of the equation under consideration will be introduced.

2.1 Fundamental theorems

Let's examine the sth order system of PDEs of m independent variables $x = (x^1, x^2, \dots, x^m)$ and n dependent variables $u = (\nu_1, \nu_2, \dots, \nu_n)$

$$P^\alpha(x, \nu, \nu_{(1)}, \dots, \nu_{(s)}) = 0, \quad \alpha = 1, \dots, n, \tag{2}$$

where $\nu_{(1)}, \nu_{(2)}, \dots, \nu_{(s)}$ symbolize the first, second, . . . , sth order partial derivatives, i.e., $\nu_i^\alpha = D_i(\nu^\alpha)$,$\nu_{ij}^\alpha = D_j D_i(\nu^\alpha)$,... respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + \nu_i^\alpha \frac{\partial}{\partial \nu^\alpha} + \nu_{ij}^\alpha \frac{\partial}{\nu_j^\alpha} + \dots, \quad i = 1, \dots, n, \tag{3}$$

where the Einstein's summation convention is utilised. The following definitions are acknowledged (see, e.g. [32-34]). The variational operator given by

$$\frac{\delta}{\delta v^{\alpha}} = \frac{\partial}{\partial v^{\alpha}} + \sum_{s\geq 1} (-1)^s D_{i_1 \ldots} D_{i_s} \frac{\partial}{\partial v^{\alpha}_{i_1 i_2 \ldots i_s}}, \quad \alpha = 1, \ldots, m. \tag{4}$$

The Lie-Bäcklund operator is given as

$$\Gamma = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial v^{\alpha}}, \quad \xi^i, \eta^{\alpha} \in \mathcal{S}, \tag{5}$$

where $\mathcal{S}$ is the space of differential functions. The operator (5) is an abbreviated version of the infinite formal sum

$$\Gamma = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial v^{\alpha}} + \sum_{s\geq 1} \zeta^{\alpha}_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial v^{\alpha}_{i_1 i_2 \ldots i_s}}, \tag{6}$$

where the extension coefficients are given by the extension formulae

$$\zeta^{\alpha}_{i_1 \ldots i_s} = D_{i_1 \ldots i_s}(\mathcal{W}^{\alpha}) + \xi^j v^{\alpha}_{j i_1 \ldots i_s} \quad s > 1, \tag{7}$$

where $\mathcal{W}^{\alpha}$ is the Lie characteristic function

$$\mathcal{W}^{\alpha} = \eta^{\alpha} - \xi^j v^{\alpha}_j. \tag{8}$$

The N^i Noether operator is presented in terms of Γ operator as

$$N^i = \xi^i + \mathcal{W}^{\alpha} \frac{\delta}{\delta v^{\alpha}_i} + \sum_{s\geq 1} D_{i_1 \ldots i_s}(\mathcal{W}^{\alpha}) \frac{\delta}{\delta v^{\alpha}_{i i_1 \ldots i_s}}, \quad i = 1, \ldots, m, \tag{9}$$

where the variational operators w.r.t. derivatives of v^{α} are obtained from (4) by replacing v^{α} by the corresponding derivatives. The m -tuple vector $T = (T^1, T^2, \ldots, T^m)$, $T^j \in S$, $j = 1, \ldots, m$ is a conserved vector of (2) if T^i satisfies

$$D_i T^i|_{(2)} = 0. \tag{10}$$

We now give the relevant results used in this study below.

Definition [35]: If the T^i conserved vectors and Γ operator of the equation (2) satisfy the following expression

$$\Gamma(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \quad i = 1, \ldots, m, \tag{11}$$

then it is said to be they are associated.

Theorem [34, 36]: Assume that Γ is any Lie-Bäcklund operator of Eq. (2) and the components of conserved vector of (2) are given by T^i. Then

$$T^{*i} = [T^i, \Gamma] = \Gamma(T^i) + T^i D_j \xi^j - T^j D_j \xi^i, \quad i = 1, \ldots, m, \tag{12}$$

construct the components of a conserved vector of (2), i.e., $D_i T^{*i}|_{(2)} = 0$.

Theorem [37] : Assume that $D_i T^i = 0$ is a conservation law of the PDE system (2). Then under a similarity transformation, there exists functions $\tilde{T}^i$ such that $J D_i T^i = \tilde{D}_i \tilde{T}^i$ where $\tilde{T}^i$ is given by

$$\begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^m \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \quad J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^m \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^m \end{pmatrix} \tag{13}$$

in which

$$A = \begin{pmatrix} \tilde{D}_1 x^1 & \tilde{D}_1 x^2 & & \tilde{D}_1 x^m \\ \tilde{D}_2 x^1 & \tilde{D}_2 x^2 & & \tilde{D}_2 x^m \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_m x^1 & \tilde{D}_m x^2 & & \tilde{D}_m x^m \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}^1 & D_1 \tilde{x}^2 & & D_1 \tilde{x}^m \\ D_2 \tilde{x}^1 & D_2 \tilde{x}^2 & & D_2 \tilde{x}^m \\ \vdots & \vdots & \vdots & \vdots \\ D_m \tilde{x}^1 & D_m \tilde{x}^2 & & D_m \tilde{x}^m \end{pmatrix} \tag{14}$$

and $J = det(A)$.

Theorem([37]): Assume that $D_i T^i = 0$ is a conservation law of (2). Then under a similarity transformation of a symmetry Γ (6), there exist functions $\tilde{T}^i$ such that the symmetry Γ is still a symmetry for the PDE $\tilde{D}^i \tilde{T}^i$ and

$$\begin{pmatrix} \Gamma\tilde{T}^1 \\ \Gamma\tilde{T}^2 \\ \vdots \\ \Gamma\tilde{T}^m \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} [T^1, \Gamma] \\ [T^2, \Gamma] \\ \vdots \\ [T^m, \Gamma] \end{pmatrix}. \tag{15}$$

If the conservation laws of the equation (6) are associtaed with the Lie symmetries of the equation in the sense of (11), then conservation laws $D_i T_i = 0$ of (6) can be reduced $\tilde{D}_i \tilde{T}^i = 0$ under the similarity transformations corresponding to Γ Lie symmetries [25].

Therefore, generalization can be clearly made. If $s\ th$ order equation (cf. Eq. (6)) has a non-trivial conserved form and this conserved form is asociated with Lie symmetries (for the m number of reductions where m is the number of independent variables of (6)) then the equation can be reduced to a $(s-1)$ th ordinary differential equation (ODE) [37].

3. Recapitulation of auxiliary equation method

The principal steps of auxiliary equation method are summarized in this section [38]. Suppose that a nonlinear evolution equation is expressed as

$$\mathfrak{K}(u, u_x, u_t, u_{xx}, u_{tt}, \dots) = 0, \tag{16}$$

where $\mathfrak{K}$ is a polynomial in $u(x,t)$ and its partial derivatives involve the highest order derivatives and nonlinear terms. After algebraic operations, Eq. (16) is transformed into an ODE with the transformation $\xi = x - \mu t$

$$\mathfrak{O}(u, u', u', u'', \dots) = 0. \tag{17}$$

Suppose that the solution of Eq. (17) has the form

$$u(\xi) = S(\phi) = \sum_{j=0}^{M} n_j \phi(\zeta)^j, \tag{18}$$

where the integer M can be obtained by balancing procedure appearing in Eq. (17) and $n_j (j = 0,1,\dots,M)$ are constants that need to be determined. Here, $\phi(\zeta)$ fulfills the following auxiliary ODE:

$$\left(\frac{d\phi}{d\zeta}\right)^2 = m_1 \phi(\zeta)^2 + m_2 \phi(\zeta)^4 + m_3 \phi(\zeta)^6, \tag{19}$$

where $m_1,\ m_2,\ m_3$ are real parameters. Eq. (19) admits several types of solutions:

Case 1. For $m_1 > 0,\ \phi_1(\zeta) = \sqrt{-\frac{m_1\, m_2\, \left(sech(\sqrt{m_1}\xi)\right)^2}{{m_2}^2 - m_1\, m_3\, \left(1+\epsilon \tanh(\sqrt{m_1}\xi)\right)^2}}$,

Case 2. For $m_1 > 0,\ \phi_2(\zeta) = \sqrt{\frac{m_1\, m_2\, \left(csch(\sqrt{m_1}\xi)\right)^2}{{m_2}^2 - m_1\, m_3\, \left(1+\epsilon \coth(\sqrt{m_1}\xi)\right)^2}}$,

Case 3. For $m_1 > 0,\ \Delta > 0,\ \phi_3(\zeta) = \sqrt{2}\sqrt{\frac{m_1}{\epsilon\, \sqrt{\Delta}\cosh(2\, \sqrt{m_1}\xi) - m_2}}$,

Case 4. For $m_1 < 0,\ \Delta > 0,\ \phi_4(\zeta) = \sqrt{2}\sqrt{\frac{m_1}{\epsilon\, \sqrt{\Delta}\cos(2\, \sqrt{-m_1}\xi) - m_2}}$,

Case 5. For $m_1 > 0,\ \Delta < 0,\ \phi_5(\zeta) = \sqrt{2}\sqrt{\frac{m_1}{\epsilon\, \sqrt{\Delta}\cos(2\, \sqrt{-m_1}\xi) - m_2}}$,

Case 6. For $m_1 < 0,\ m_3 > 0,\ \phi_6(\zeta) = \sqrt{2}\sqrt{\frac{m_1}{\epsilon\, \sqrt{\Delta}\sin(2\, \sqrt{-m_1}\xi) - m_2}}$,

Case 7. For $m_1 > 0,\ m_3 > 0,\ \phi_7(\zeta) = \sqrt{-\frac{m_1\, \left(sech(\sqrt{m_1}\xi)\right)^2}{{m_2}^2 - 2\, \epsilon\, \sqrt{m_1\, m_3}\tanh(\sqrt{m_1}\xi)}}$,

Case 8. For $m_1 < 0,\ m_3 > 0,\ \phi_8(\zeta) = \sqrt{-\frac{m_1\, \left(\sec(\sqrt{-m_1}\xi)\right)^2}{{m_2}^2 + 2\, \epsilon\, \sqrt{-m_1\, m_3}\tan(\sqrt{-m_1}\xi)}}$,

Case 9. For $m_1 > 0,\ m_3 > 0,\ \phi_9(\zeta) = \sqrt{\frac{m_1\, \left(csch(\sqrt{m_1}\xi)\right)^2}{{m_2}^2 + 2\, \epsilon\, \sqrt{m_1\, m_3}\coth(\sqrt{m_1}\xi)}}$,

Case 10. For $m_1 < 0,\ m_3 > 0,\ \phi_{10}(\zeta) = \sqrt{-\frac{m_1 \csc(\sqrt{-m_1}\xi)}{{m_2}^2+2\,\epsilon\,\sqrt{-m_1\,m_3}\tanh(\sqrt{-m_1}\xi)}}$,

Case 11. For $m_1 > 0,\ \Delta = 0,\ \phi_{11}(\zeta) = \sqrt{-\frac{m_1\left(1+\epsilon\tanh(1/2\sqrt{m_1}\xi)\right)}{m_2}}$,

Case 12. For $m_1 > 0,\ \Delta = 0,\ \phi_{12}(\zeta) = \sqrt{-\frac{m_1\left(1+\epsilon\coth(1/2\sqrt{m_1}\xi)\right)}{m_2}}$,

Case 13. For $m_1 > 0,\ \phi_{13}(\zeta) = 4\sqrt{\frac{m_1\,\mathrm{e}^{2\,\epsilon\,\sqrt{m_1}\xi}}{\left(\mathrm{e}^{2\,\epsilon\,\sqrt{m_1}\xi}-4\,m_2\right)^2-64\,m_1\,m_3}}$,

Case 14. For $m_1 > 0,\ m_2 = 0,\ \phi_{14}(\zeta) = 4\sqrt{\frac{m_1\,\mathrm{e}^{2\,\epsilon\,\sqrt{m_1}\xi}}{1-64\,m_1\,m_3\,\mathrm{e}^{4\,\epsilon\,\sqrt{m_1}\xi}}}$,

where $\Delta = m_2^2 - 4m_1m_2$ and $\epsilon = \pm 1$.

4. Solutions of the Benjamin-Ono equation

The symmetry group of the Benjamin-Ono equation (1) will be generated by the vector field of the form

$$\Gamma = \tau(x,t,u)\frac{\partial}{\partial t} + \zeta(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}. \qquad (20)$$

We obtain an overdetermined system of linear PDEs implementing the fourth prolongation $\Gamma^{[4]}$ to Eq. (1). Then, solving the obtained system, we get Lie point symmetries of (1) with the help of SADE (in Maple) [39]:

$$\Gamma_1 = \frac{\partial}{\partial x}, \qquad (21)$$
$$\Gamma_2 = \frac{\partial}{\partial t},$$
$$\Gamma_3 = u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t} - \frac{x}{2}\frac{\partial}{\partial x}.$$

It is shown by Kaplan et al. [40] that (1) accepts the following conserved vectors

$$T_1^t = xu_t, \qquad (22)$$
$$T_1^x = 2\beta xuu_x - \beta u^2 - \gamma u_{xx} + \gamma xu_{xxx},$$

$$T_2^t = -u + tu_t,$$
$$T_2^x = 2\beta tuu_x + \gamma tu_{xxx},$$

$$T_3^t = -xu + xtu_t,$$
$$T_3^x = 2\beta txu_xu - \beta tu^2 - \gamma tu_{xx} + \gamma tuu_{xxx},$$

$$T_4^t = u_t,$$

$T_4^x = 2\beta u u_x + \gamma u_{xxx},$

with the corresponding multipliers

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (x, t, xt, 1). \tag{23}$$

With the help of the double reduction theory, we compute the exact solutions of Eq. (1). If the following expression is satisfied

$$T^* = \Gamma \begin{pmatrix} T^t \\ T^x \end{pmatrix} - \begin{pmatrix} D_t \xi^t & D_x \xi^t \\ D_t \xi^x & D_x \xi^x \end{pmatrix} \begin{pmatrix} T^t \\ T^x \end{pmatrix} + (D_t \xi^t + D_x \xi^x) \begin{pmatrix} T^t \\ T^x \end{pmatrix} = 0, \tag{24}$$

then the Lie-Bäcklund symmetry generator Γ is associated with a conserved vector T of Eq. (1).

4.1 A double reduction of (1) by $\langle \boldsymbol{\Gamma_1}, \boldsymbol{\Gamma_2} \rangle$

We now show that Γ_1 and Γ_2 are associated with T_4. We obtain

$$\begin{pmatrix} T_4^{*t} \\ T_4^{*x} \end{pmatrix} = \Gamma_1^{[3]} \begin{pmatrix} T_4^t \\ T_4^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_4^t \\ T_4^x \end{pmatrix} + (0) \begin{pmatrix} T_4^t \\ T_4^x \end{pmatrix} \tag{25}$$

from (24). Here $\Gamma_1^{[3]} = \frac{\partial}{\partial x}$. (25) shows that

$T_4^{*t} = 0, T_4^{*x} = 0.$

Thus, Γ_1 is associated with T_4 [35].

Similarly for Γ_2, we obtain

$$\begin{pmatrix} T_4^{*t} \\ T_4^{*x} \end{pmatrix} = \Gamma_2^{[3]} \begin{pmatrix} T_4^t \\ T_4^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_4^t \\ T_4^x \end{pmatrix} + (0) \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix}, \tag{26}$$

where $\Gamma_2^{[3]} = \frac{\partial}{\partial t}$. (26) shows that

$T_4^{*t} = 0, T_4^{*x} = 0.$

Thus, Γ_2 is associated with T_4 in the sense of Kara and Mahomed's definition [35].

We investigate a linear combination of Γ_1 and Γ_2:

$$\Gamma = \alpha \Gamma_1 + \Gamma_2 = \alpha \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \tag{27}$$

and this generator is then transformed into its canonical form $Y = \frac{\partial}{\partial s}$ where we suppose that Y has the following form

$$Y = \frac{\partial}{\partial s} + 0 \frac{\partial}{\partial r} + 0 \frac{\partial}{\partial \omega}. \tag{28}$$

Thus, we get

$$\frac{dx}{\alpha}=\frac{dt}{1}=\frac{du}{0}=\frac{ds}{1}=\frac{dr}{0}=\frac{d\omega}{0}. \tag{29}$$

The invariants of (27) from (29) are given by

$$\begin{cases} \frac{dt}{1}=\frac{dx}{\alpha},\frac{dt}{1}=\frac{ds}{1}, \\ \frac{d\omega}{0},\frac{du}{0},\frac{dr}{0}, \end{cases} \tag{30}$$

and

$$b_1=\alpha t-x, b_2=s-t, b_3=r, b_4=\omega, b_5=u. \tag{31}$$

By choosing $b_1=b_3,\ b_2=0, b_4=b_5,$ we obtain the canonical coordinates

$$r=\alpha t-x, s=t, \omega=u, \tag{32}$$

where $w=w(r)$. The inverse canonical coordinates are presented below

$$x=\alpha s-r, t=s, u=\omega. \tag{33}$$

The matrices A and A^{-1} can be computed using the canonical coordinates above

$$A=\begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix}=\begin{pmatrix} 0 & -1 \\ 1 & \alpha \end{pmatrix}$$

and

$$(A^{-1})^T=\begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix}=\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}$$

where $J=detA=1.$ The reduced conserved form is given by

$$\begin{pmatrix} T_4^r \\ T_4^s \end{pmatrix}=J(A^{-1})^T\begin{pmatrix} T_4^t \\ T_4^x \end{pmatrix}=\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} T_4^t \\ T_4^x \end{pmatrix}=\begin{pmatrix} \alpha T_4^t-T_4^x \\ T_4^x \end{pmatrix}. \tag{34}$$

Substituting (32) and the partial derivatives of u, into (34), we get

$$\begin{aligned} T_4^r &= \alpha^2\omega_r+2\beta w w_r+\gamma\omega_{rrr}, \\ T_4^s &= -2\beta w w_r-\gamma\omega_{rrr}, \end{aligned} \tag{35}$$

where the reduced conserved form (35) satisfies

$$D_r T_4^r+D_s T_4^s=0. \tag{36}$$

The reduced form (36) also satisfies $D_r T_4^r=0$. This yields

$$\alpha^2\omega_r+2\beta w w_r+\gamma\omega_{rrr}=k_1. \tag{37}$$

4.2. Application of the auxiliary equation method

We seek solutions of (37) by the auxiliary equation method while setting the constant k_1 to zero. Balancing w_{rrr} and ww_r in Eq.(37), we have $M = 2$ and then proceeded as

$$w(r) = n_0 + n_1\phi(r) + n_2\phi(r)^2, \tag{38}$$

with n_0, n_1 and n_2 which are constants need to be determined. Substituting Eq. (38) into Eq. (37) and equating the coefficients $\phi^j(r)$ for $(j = 0,1,2,\dots,M)$ to zero, a system of algebraic equations was obtained. We recovered solutions for the obtained system as

$$\left\{m_2 = -\frac{\beta\, n_2}{6\gamma}, m_3 = 0, n_0 = -\frac{4\,\gamma\, m_1+\alpha^2}{2\beta}, n_1 = 0,\right\}. \tag{39}$$

With the help of inverse canonical coordinates, the solutions of Eq. (1) are obtained as follows when $m_1 > 0$

$$u_1(x,t) = -\frac{\alpha^2+4\,\gamma\, m_1}{2\beta} + 6\,\frac{m_1\,\gamma\left(sech\left(\sqrt{m_1}(\alpha\, t-x)\right)\right)^2}{\beta}, \tag{40}$$

$$u_2(x,t) = -\frac{\alpha^2+4\,\gamma\, m_1}{2\beta} - 6\,\frac{m_1\,\gamma\left(csch\left(\sqrt{m_1}(\alpha\, t-x)\right)\right)^2}{\beta}, \tag{41}$$

$$u_3(x,t) = -\frac{\alpha^2+4\,\gamma\, m_1}{2\beta} + 2\,n_2\,m_1\left(\frac{\epsilon}{\sqrt{36}}\sqrt{\frac{\beta^2{n_2}^2}{\gamma^2}}\cosh\left(2\sqrt{m_1}(\alpha\, t-x)\right) + \frac{\beta\, n_2}{6\gamma}\right)^{-1}, \tag{42}$$

$$u_4(x,t) = -\frac{\alpha^2+4\,\gamma\, m_1}{2\beta} + 16\,n_2\,m_1\,\mathrm{e}^{2\,\epsilon\sqrt{m_1}(\alpha\, t-x)}\left(\mathrm{e}^{2\,\epsilon\sqrt{m_1}(\alpha\, t-x)} + 2\frac{\beta\, n_2}{3\gamma}\right)^{-2}, \tag{43}$$

and when $m_1 < 0$

$$u_5(x,t) = -\frac{4\,\gamma\, m_1+\alpha^2}{2\beta} + 2\,n_2\,m_1\left(\frac{\epsilon}{\sqrt{36}}\sqrt{\frac{\beta^2{n_2}^2}{\gamma^2}}\cos\left(2\sqrt{-m_1}(\alpha\, t-x)\right) + \frac{\beta\, n_2}{6\gamma}\right)^{-1}, \tag{44}$$

$$u_6(x,t) = -\frac{4\,\gamma\, m_1+\alpha^2}{2\beta} + 2\,n_2\,m_1\left(\frac{\epsilon}{\sqrt{36}}\sqrt{\frac{\beta^2{n_2}^2}{\gamma^2}}\sin\left(2\sqrt{-m_1}(\alpha\, t-x)\right) + \frac{\beta\, n_2}{6\gamma}\right)^{-1}. \tag{45}$$

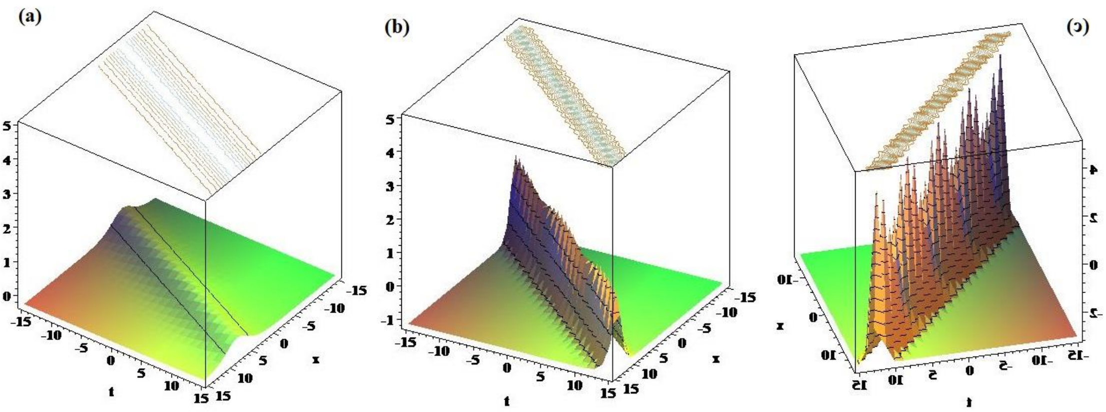

Figure 1. Profile of solution (40) where $\gamma = 1, \beta = 1.2$ with $(a)\ \alpha = 0.5, m_1 = 0.1$, $(b)\ \alpha = 0.9, m_1 = 0.5$ and $(c)\ \alpha = 1.2, m_1 = 1.5$.

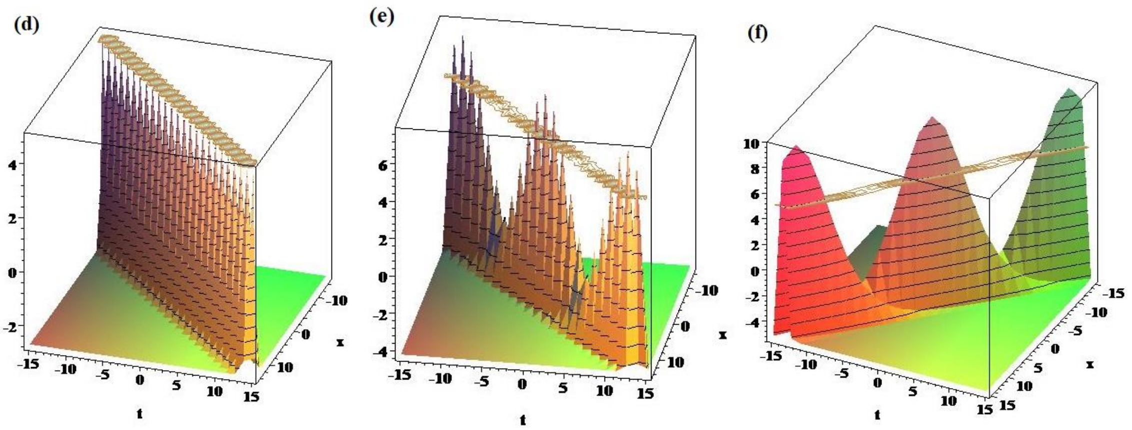

Figure 2. Profile of solution (40) where $\gamma = 1, \beta = 2$ with $(d)\ \alpha = 1, m_1 = 2.5$, $(e)\ \alpha = -1.1, m_1 = 5$ and $(f)\ \alpha = 1.1, m_1 = 4$.

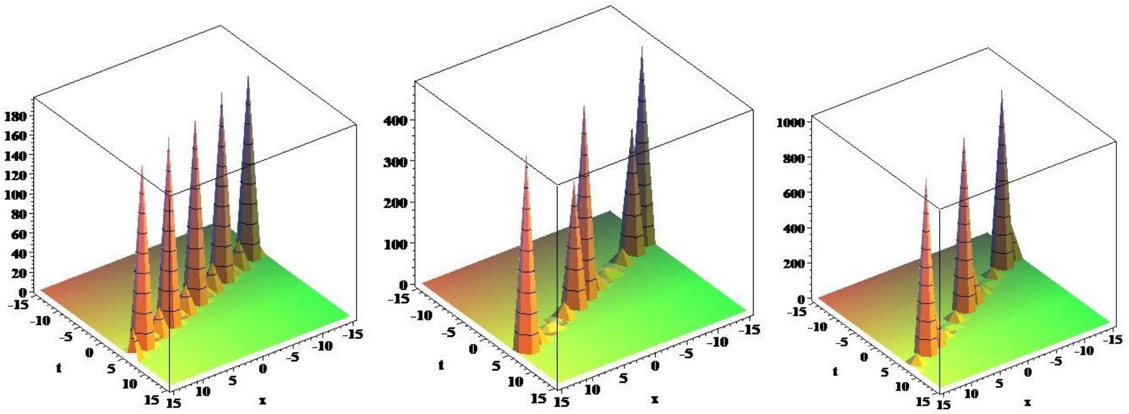

Figure 3. Profile of solution (43) where $\gamma = 2, \beta = -2, \epsilon = -1, n_2 = 1.2$ with $\{\alpha = 2.5, m_1 = .4\}$, $\{\alpha = 2.2, m_1 = 1\}$ and $\{\alpha = 1.8, m_1 = 2.1\}$, respectively.

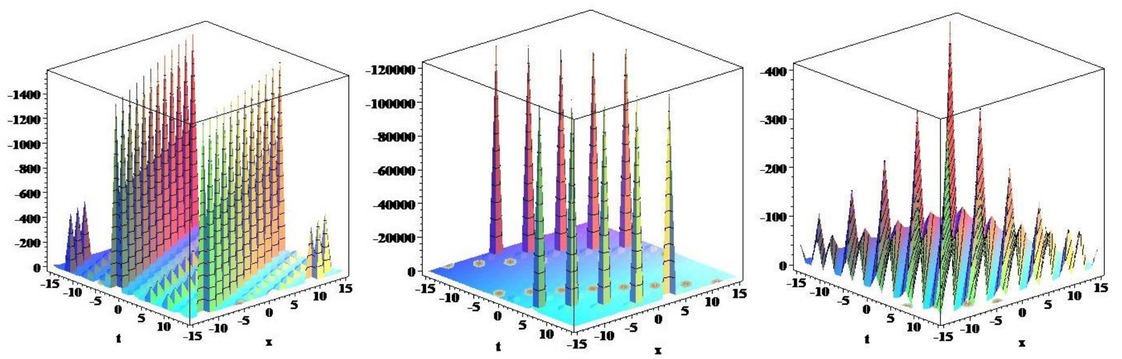

Figure 4. Profile of solution (44) where $\gamma = 1, \beta = 1, \epsilon = 1, n_2 = 0.6$ with $\{\alpha = -2, m_1 = -0.2\}$, $\{\alpha = 1.5, m_1 = -0.5\}$ and $\{\alpha = 1.5, m_1 = -1\}$, respectively.

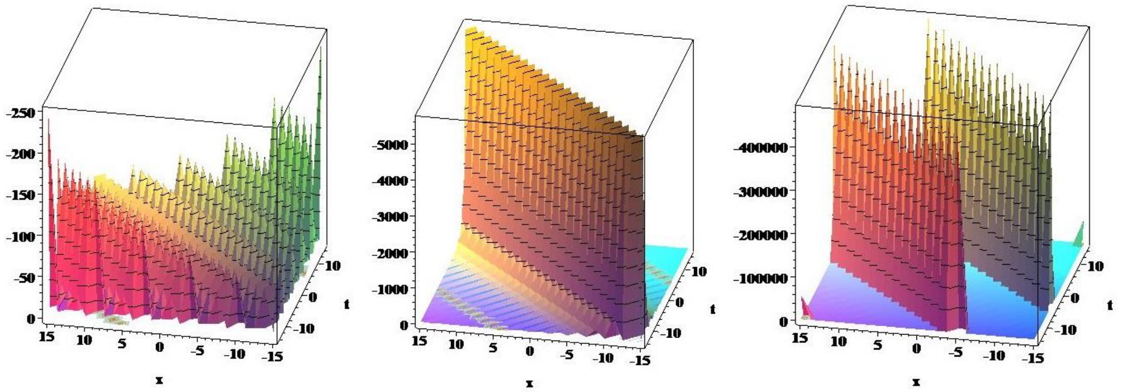

Figure 5. Profile of solution (44) where $\gamma = 1, \beta = 1, \epsilon = 1, n_2 = 0.6, \alpha = 1$ with $m_1 = -1, m_1 = -1.5$ and $m_1 = -2$, respectively.

Remark 1 *The accuracy of all the solutions obtained was examined by placing them in their original equations using Maple.*

6. Discussions

This work provides a new way of constructing various exact solutions for PDEs by establishing a relationship of the current symmetry with the conserved vectors of the equation. In order to find the solutions of the reduced equation obtained as a result of double reduction theory which has been applied after establishing the conserved vectors association with the Lie symmetries, the auxiliary equation method, which is an effective method, was used. We have achieved various traveling wave solutions including trigonometric, hyperbolic, and exponential solutions. In Figs. 1-5, a few graphic representations are given by giving special values to the parameters in the solutions obtained and the behavior caused by small changes in parameters is shown in 3D graphics. Figs. 1 and 2 represent solitary wave solution, Fig. 3 represent hyperbolic function solution, and Figs. 4-5 demonstrate periodic wave solutions which are traveling wave solutions that repeat its values in regular intervals or periods.

5. Concluding remarks

In this work, we considered Benjamin-Ono equation and we used the double reduction theory and the auxiliary equation method to investigate underlying equation. Double reduction theory is a powerful mathematical tool for obtaining reduced forms and exact solutions of partial differential equations or systems. This theory provides not only transformations that provide traveling wave solutions, but also a systematic way of finding other types of transformations. These transformations reduce a nonlinear system of qth-order PDEs with n independent and m dependent variables to a nonlinear system of $(q-1)$th-order ODEs provided that in every reduction at least one symmetry is associated with a nontrivial conserved vector; otherwise, reduction is not possible. The reduced ODE can be solved analytically or numerically to obtain exact or approximate solutions. Interestingly, the transformations that give traveling wave solutions can sometimes give more than one reduced form and the simple one can be used to find the exact solution [41]. Using the association between Lie point symmetry generators and conservation law of Benjamin-Ono equation, we obtained a reduction in the number of both orders and variables of the underlying equation. Therefore, we reduced the number of variables from two to one and the order of the equation from four to three, at the same time. The application of the double reduction method, which is the main purpose of this study, has been successfully completed. To obtain solutions of the reduced equation, we have successfully applied auxiliary equation method. These solutions include a periodic, parabolic, and exponential solutions confirming the effectiveness of the method. When some of the solutions obtained are compared with the studies in the literature and when parameters are given arbitrary values in works in which used different methods, it can be observed that solutions with similar form are obtained [29]. According to our best knowledge, the remaining solutions are new. The physical properties of some obtained results have been illustrated using suitable parameter values in Figure. $1-5$. We hope that the results obtained will be used for important physical practices and guide new research.

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