Advances in the Theory of Nonlinear Analysis and its Applications **5** (2021) No. 1, 49–57.
https://doi.org/10.31197/atnaa.848928
Available online at www.atnaa.org

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Hilfer-Hadamard fractional differential equations; Existence and Attractivity

Fatima Si Bachir[a], Saïd Abbas[b], Maamar Benbachir[c], Mouffak Benchohra[d]

[a] *Laboratory of Mathematics and Applied Sciences, University of Ghardaia, 47000, Algeria.*
[b] *Department of Mathematics, University of Saïda–Dr. Moulay Tahar, P.O. Box 138, EN-Nasr, 20000 Saïda, Algeria.*
[c] *Department of Mathematics, Saad Dahlab Blida1, University of Blida, Algeria.*
[d] *Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria.*

Abstract

This work deals with a class of Hilfer-Hadamard differential equations. Existence and stability of solutions are presented. We use an appropriate fixed point theorem.

Keywords: Hilfer-Hadamard fractional derivative, Schauder fixed-point Theorem, uniformly locally attracting.
2010 MSC: 26A33, 34A08.

1. Introduction

The beginning of the fractional calculus in 1695, the fractional differential equation has been used in fields like mathematics, engineering, bioengineering, physics, etc.[16, 30], to see interesting results in the theory of fractional calculus and fractional differential equations, the reader may consult the monographs by; Abbas *et al.* [8, 9], Kilbas *et al.* [22], Oldham *et al.* [26], Podlubny [27], Samko *et al.* [28], Zhou *et al.* [33], and the papers by Abbas *et al.* [3, 5], Benchohra *et al.* [12], Lakshmikantham *et al.* [23, 24, 25]. Other recent results are provided in [11, 13, 17, 18, 19, 20, 21, 29, 31, 32]. Attractivity results for various classes of fractional differential equations are considered in [1, 2, 4, 6, 10].

Email addresses: sibachir.fatima@univ-ghardaia.dz (Fatima Si Bachir), abbasmsaid@yahoo.fr, said.abbas@univ-saida.dz (Saïd Abbas), mbenbachir2001@gmail.com (Maamar Benbachir), benchohra@yahoo.com (Mouffak Benchohra)

Received November 19, 2020 ; Accepted: December 29, 2020; Online: December 31, 2020.

In [7], Abbas *et al.* studied some existence and Ulam stability results of the following problem

$$\begin{cases} ({}^H D_{1^+}^{\tau,\theta} i)(t) = \chi(t, i(t)); \quad t \in [1, T], \\ ({}^H I_{1^+}^{1-\varrho} i)(1) = d, \quad \varrho = \tau + \theta(1-\tau). \end{cases}$$

This work is devoted to the existence and attractivity of solutions of the following problem

$$\begin{cases} ({}^H D_{c^+}^{\tau,\theta} i)(t) = \chi(t, i(t)); \quad t \in [c, +\infty),\ c > 0, \\ ({}^H I_{c^+}^{1-\varrho} i)(c) = d, \quad \varrho = \tau + \theta(1-\tau), \end{cases} \tag{1}$$

where $d \in \mathbb{R}$, $\chi : [c, +\infty) \times \mathbb{R} \to \mathbb{R}$, ${}^H I_{c^+}^{1-\varrho}$ is the left-sided Hadamard fractional of order $\tau > 0$ and ${}^H D_{c^+}^{\tau,\theta}$ is the Hilfer-Hadamard derivative operator of order τ $(0 < \tau < 1)$ and type θ $(0 \le \theta \le 1)$.

2. Preliminaries

We will introduce some spaces. We denote by $C_{\varrho,\log}[c,e]$, $(0 < c < e < \infty)$, the space $C_{\varrho,\log}[c,e] = \{\iota : (c,e] \to \mathbb{R} : (\log \frac{t}{c})^{1-\varrho}\ \iota(t) \in C[c,e]\}$, with the norm

$$\|\iota\|_{C_{\varrho,\log}} = \sup_{t\in[c,e]} \left| \left(\log \frac{t}{c}\right)^{1-\varrho} \iota(t) \right|.$$

$BC^* := BC([c,+\infty))$ denotes the space continuous and bounded functions $\iota : [c,+\infty) \to \mathbb{R}$. $BC_\varrho = \{\iota : (c,+\infty) \to \mathbb{R}$: $(\log \frac{t}{c})^{1-\varrho}\iota(t) \in BC^*\}$, with the norm

$$\|\iota\|_{BC_\varrho} := \sup_{t\in[c,+\infty)} \left| \left(\log \frac{t}{c}\right)^{1-\varrho} \iota(t) \right|.$$

Denote $\|\iota\|_{BC_\varrho}$ by $\|\iota\|_{BC^*}$.

Definition 2.1. *[22]. Let (c,e) $(0 \le c < e \le \infty)$ and $\tau > 0$. The Hadamard left-sided fractional integral ${}^H I_{c^+}^{\tau} j$ of order $\tau > 0$ is defined by*

$$\left({}^H I_{c^+}^{\tau} j\right)(x) := \frac{1}{\Gamma(\tau)} \int_c^x \left(\log \frac{x}{t}\right)^{\tau-1} \frac{j(t)dt}{t}, \quad c < x < e.$$

When $\tau = 0$, we set

$${}^H I_{c^+}^{0} j = j.$$

Definition 2.2. *[22] Let $(c,e)(0 \le c < e \le \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}_+$ and let $\tau > 0$. The Hadamard right-sided fractional integral ${}^H I_{e^-}^{\tau} j$ of order $\tau > 0$ is defined by*

$$\left({}^H I_{e^-}^{\tau} j\right)(x) := \frac{1}{\Gamma(\tau)} \int_x^e \left(\log \frac{t}{x}\right)^{\tau-1} \frac{j(t)dt}{t}, \quad c < x < e.$$

When $\tau = 0$, we set

$${}^H I_{e^-}^{0} j = j.$$

Example 2.3. *For each $\tau > 0$ and $\lambda \in \mathbb{R}$, we have*

$${}^H I_1^{\tau} (\log x)^{\lambda-1} := \frac{\Gamma(\lambda)}{\Gamma(\tau+\lambda)} (\log x)^{\tau+\lambda-1};\ x \ge 1.$$

Definition 2.4. *[22] The left-sided Hadamard fractional derivative of order $\tau(0 \leq \tau < 1)$ on (c,e) is defined by*

$$\left({}^{H}D^{\tau}_{c^+}j\right)(x) = \frac{1}{\Gamma(1-\tau)}\left(x\frac{d}{dx}\right)\int_c^x \left(\log\frac{x}{t}\right)^{-\tau}\frac{j(t)dt}{t}, \quad c < x < e.$$

In particular, when $\tau = 0$ we have

$${}^{H}D^{0}_{c^+}j = j.$$

Definition 2.5. *[22] The right-sided Hadamard fractional derivative of order $\tau(0 \leq \tau < 1)$ on (c,e) is defined by*

$$\left({}^{H}D^{\tau}_{e^-}j\right)(x) = -\left(x\frac{d}{dx}\right)\frac{1}{\Gamma(1-\tau)}\int_x^e \left(\log\frac{t}{x}\right)^{-\tau}\frac{j(t)dt}{t}.$$

In particular, when $\tau = 0$ we have

$${}^{H}D^{0}_{e^-}j = j.$$

Definition 2.6. *Let (c,e) be a finite interval of the half-axis $\mathbb{R}_+$. The fractional derivative ${}^{Hc}D^{\tau}_{c^+}j$ of order $\tau\,(0 < \tau < 1)$ on (c,e) defined by:*

$${}^{Hc}D^{\tau}_{c^+}j = {}^{H}I^{1-\tau}_{c^+}\delta j,$$

where $\delta = x(d/dx)$, is called the Hadamard-Caputo fractional derivative of order τ.

Lemma 2.7. *[22] Let $\tau > 0, \theta > 0$ and $0 \leq \mu < 1$. If $0 < c < e < \infty$, then for $j \in C_{\mu,\log}[c,e]$ the equality ${}^{H}I^{\tau}_{c^+}\,{}^{H}I^{\theta}_{c^+}j = {}^{H}I^{\tau+\theta}_{c^+}j$ holds.*

Theorem 2.8. *[22] Let $0 < \tau < 1$ and $0 < c < e < \infty$. If $j \in C_{\mu,\log}[c,e](0 \leq \mu < 1)$ and ${}^{H}I^{1-\tau}_{c^+}j \in C^1_{\delta,\mu}[c,e]$ then*

$$\left({}^{H}I^{\tau}_{c^+}\,{}^{H}D^{\tau}_{c^+}j\right)(x) = j(x) - \frac{\left({}^{H}I^{1-\tau}_{c^+}j\right)(c)}{\Gamma(\tau)}\left(\log\frac{x}{c}\right)^{\tau-1},$$

holds at any point $x \in (c,e]$. If $j \in C[c,e]$ and ${}^{H}I^{1-\tau}_{c^+}j \in$, $C^1_{\delta}[c,e]$, then the relation holds at any point $x \in [c,e]$.

Definition 2.9. *(Hilfer-Hadamard fractional derivative). The left sided fractional derivative of order τ $(0 < \tau < 1)$ and type $0 \leq \theta \leq 1$ with respect to x is defined by*

$$\left({}^{H}D^{\tau,\theta}_{c^+}j\right)(x) = \left({}^{H}I^{\theta(1-\tau)}_{c^+}\ {}^{H}D^{\tau+\theta-\tau\theta}_{c^+}j\right)(x).$$

Corollary 2.10. *[21] Let $\sigma \in C_{\varrho,\log}(I)$. Then the problem*

$$\begin{cases} ({}^{H}D^{\tau,\theta}_{c^+}i)(t) = \sigma(t), \quad t \in I := [c,e] \\ ({}^{H}I^{1-\varrho}_{c^+}i)(c) = d, \end{cases}$$

admits the following unique solution

$$i(t) = \frac{d}{\Gamma(\varrho)}\left(\log\frac{t}{c}\right)^{\varrho-1} + \left({}^{H}I^{\tau}_{c^+}\sigma\right)(t). \tag{2}$$

Lemma 2.11. *Let $\chi : (c,e] \times \mathbb{R} \to \mathbb{R}$ be a function such that $\chi(\cdot, i(\cdot)) \in BC_{\varrho}$ for any $i \in BC_{\varrho}$. Then the problem (1) is equivalent to the integral equation*

$$i(t) = \frac{d}{\Gamma(\varrho)}\left(\log\frac{t}{c}\right)^{\varrho-1} + \left({}^{H}I^{\tau}_{c^+}\chi(\cdot, i(\cdot))\right)(t). \tag{3}$$

Let $\varnothing \neq H \subset BC^*$ and let $T : H \to H$. Let the equation

$$(Ti)(t) = i(t). \tag{4}$$

Definition 2.12. *Solutions of equation (4) are locally attractive if there exists a ball $B(i_0, \delta)$ in the space BC^* such that, for any solutions $w = w(t)$ and $\Theta = \Theta(t)$ of equations (4) that belong to $B(i_0, \delta) \cap H$, we can write*

$$\lim_{t \to \infty} (w(t) - \Theta(t)) = 0. \tag{5}$$

If limit (5) is uniform with respect to $B(i_0, \delta) \cap H$, then (4) is uniformly locally attractive.

Lemma 2.13. *[14] Let $P \subset BC^*$. Then P is relatively compact in BC^* if the following conditions are satisfied:*

(a) *P is uniformly bounded in BC^*;*

(b) *the functions belonging to P are almost equicontinuous in $\mathbb{R}_+$, i.e., equicontinuous on every compact set in $\mathbb{R}_+$*

(c) *the functions from P are equiconvergent, i.e., given $\varsigma > 0$, there exists $M(\varsigma) > 0$ such that*

$$\left| i(t) - \lim_{t \to \infty} i(t) \right| < \varsigma,$$

for any $t \geq M(\varsigma)$ and $i \in P$.

Theorem 2.14. *(Schauder Fixed-Point Theorem [15]). Let X be a Banach space, let D be a nonempty bounded convex and closed subset of X, and let $L : D \to D$ be a compact and continuous map. Then L has at least one fixed point in D.*

3. Existence and Attractivity Results

Definition 3.1. *A measurable function $i \in BC_\varrho$ is a solution of (1) if it verifies $({}^H I_{c^+}^{1-\varrho} i)(c) = d$, and the equation $({}^H D_{c^+}^{\tau,\theta} i)(t) = \chi(t, i(t))$ on $[c, +\infty)$.*

We will give the following hypotheses:

(H_1) The function $t \mapsto \chi(t, i)$ is measurable on $[c, +\infty)$ for each $i \in BC_\varrho$, and $i \mapsto \chi(t, i)$ is continuous.

(H_2) There exists a continuous function $l : [c, +\infty) \to [0, +\infty)$ such that

$$|\chi(t, i)| \leq \frac{l(t)}{1 + |i|} \quad for\ a.e. \quad t \in [c, +\infty) \quad and \quad each \quad i \in \mathbb{R},$$

and

$$\lim_{t \to \infty} \left(\log \frac{t}{c} \right)^{1-\varrho} \left({}^H I_{c^+}^{\tau} l \right)(t) = 0.$$

Set

$$l^* = \sup_{t \in [c, +\infty)} \left(\log \frac{t}{c} \right)^{1-\varrho} \left({}^H I_{c^+}^{\tau} l \right)(t).$$

Theorem 3.2. *If (H_1) and (H_2) hold, then (1) has at least one solution which is uniformly locally attractive.*

Proof. Define the operator L by

$$(Li)(t) = \frac{d}{\Gamma(\varrho)} \left(\log \frac{t}{c} \right)^{\varrho - 1} + \frac{1}{\Gamma(\tau)} \int_c^t \left(\log \frac{t}{s} \right)^{\tau - 1} \chi(s, i(s)) \frac{ds}{s}.$$

We can prove that the operator L maps BC_ϱ into BC_ϱ. Indeed; the map $L(i)$ is continuous on $[c,+\infty)$, and for any $i \in BC_\varrho$ and, for each $t \in [c,+\infty)$, we have

$$\begin{aligned}\left|\left(\log\frac{t}{c}\right)^{1-\varrho}(Li)(t)\right| &\leq \frac{|d|}{\Gamma(\varrho)} + \frac{\left(\log\frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_c^t\left(\log\frac{t}{s}\right)^{\tau-1}|\chi(s,i(s))|\frac{ds}{s}\\ &\leq \frac{|d|}{\Gamma(\varrho)} + \frac{\left(\log\frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_c^t\left(\log\frac{t}{s}\right)^{\tau-1}l(s)\frac{ds}{s}\\ &\leq \frac{|d|}{\Gamma(\varrho)} + l^*\\ &:= R^*,\end{aligned}$$

so

$$\|L(i)\|_{BC_\varrho} \leq R^*. \tag{6}$$

Therefore, $L(i) \in BC_\varrho$, which proves that the operator $L\left(BC_\varrho\right) \subset BC_\varrho$. Equation (6) implies that L maps

$$B_{R^*} := B(0,R^*) = \left\{v \in BC_\varrho : \|v\|_{BC_\varrho} \leq R^*\right\}$$

into itself.

Step 1. L is continuous.

Let $\{i_n\}_{n\in N}$ be a sequence converging to i in B_{R^*}. Then,

$$\begin{aligned}&\left|\left(\log\frac{t}{c}\right)^{1-\varrho}\left(Li_n\right)(t) - \left(\log\frac{t}{c}\right)^{1-\varrho}(Li)(t)\right|\\ &\leq \frac{1}{\Gamma(\tau)}\int_c^t\left(\log\frac{t}{s}\right)^{\tau-1}\left|\left(\log\frac{t}{c}\right)^{1-\varrho}\chi\left(s,i_n(s)\right) - \left(\log\frac{t}{c}\right)^{1-\varrho}\chi(s,i(s))\right|\frac{ds}{s}.\end{aligned} \tag{7}$$

Case 1. If $t \in [c,T], T > 0$, then, since $i_n \to i$ as $n \to \infty$ and from the continuity of χ, we get

$$\|L\left(i_n\right) - L(i)\|_{BC_\varrho} \to 0 \quad \text{as} \quad n \to \infty.$$

Case 2. If $t \in (T,\infty), T > 0$, then (7) implies that

$$\begin{aligned}\left|\left(\log\frac{t}{c}\right)^{1-\varrho}\left(Li_n\right)(t) - \left(\log\frac{t}{c}\right)^{1-\varrho}(Li)(t)\right| &\leq 2\frac{\left(\log\frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\\ &\times\int_c^t\left(\log\frac{t}{s}\right)^{\tau-1}l(s)\frac{ds}{s},\end{aligned} \tag{8}$$

since $i_n \to i$ as $n \to \infty$ and $\left(\log\frac{t}{c}\right)^{1-\varrho}\left({}^H I_{c^+}^\tau l\right)(t) \to 0$ as $t \to \infty$, it follows from (8) that

$$\|L\left(i_n\right) - L(i)\|_{BC_\varrho} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2. $L\left(B_{R^*}\right)$ is uniformly bounded and equicontinuous.

Since $L(B_{R^*}) \subset B_{R^*}$ and B_{R^*} is bounded, then $L(B_{R^*})$ is uniformly bounded. Next let $t_1, t_2 \in [c, T], t_1 < t_2$, and let $i \in B_{R^*}$. This yields

$$\begin{aligned}
&\left|\left(\log\frac{t_2}{c}\right)^{1-\gamma}(Li)(t_2) - \left(\log\frac{t_1}{c}\right)^{1-\varrho}(Li)(t_1)\right| \\
&\leq \left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}\left[\frac{d}{\Gamma(\varrho)}\left(\log\frac{t_2}{c}\right)^{\varrho-1} + \frac{1}{\Gamma(\tau)}\int_c^{t_2}\left(\log\frac{t_2}{s}\right)^{\tau-1}\chi(s,i(s))\frac{ds}{s}\right]\right. \\
&\left.- \left(\log\frac{t_1}{c}\right)^{1-\varrho}\left[\frac{d}{\Gamma(\varrho)}\left(\log\frac{t_1}{c}\right)^{\varrho-1} + \frac{1}{\Gamma(\tau)}\int_c^{t_1}\left(\log\frac{t_1}{s}\right)^{\tau-1}\chi(s,i(s))\frac{ds}{s}\right]\right| \\
&\leq \left|\frac{\left(\log\frac{t_2}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_c^{t_2}\left(\log\frac{t_2}{s}\right)^{\tau-1}\chi(s,i(s))\frac{ds}{s}\right. \\
&\left.- \frac{\left(\log\frac{t_1}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_c^{t_1}\left(\log\frac{t_1}{s}\right)^{\tau-1}\chi(s,i(s))\frac{ds}{s}\right|.
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}(Li)(t_2) - \left(\log\frac{t_1}{c}\right)^{1-\varrho}(Li)(t_1)\right| \\
&\leq \frac{\left(\log\frac{t_2}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\tau-1}|\chi(s,i(s))|\frac{ds}{s} \\
&+ \frac{1}{\Gamma(\tau)}\int_c^{t_1}\left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}\left(\log\frac{t_2}{s}\right)^{\tau-1} - \left(\log\frac{t_1}{c}\right)^{1-\varrho}\left(\log\frac{t_1}{s}\right)^{\tau-1}\right||\chi(s,i(s))|\frac{ds}{s} \\
&\leq \frac{\left(\log\frac{t_2}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\tau-1}l(s)\frac{ds}{s} \\
&+ \frac{1}{\Gamma(\tau)}\int_c^{t_1}\left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}\left(\log\frac{t_2}{s}\right)^{\tau-1} - \left(\log\frac{t_1}{c}\right)^{1-\varrho}\left(\log\frac{t_1}{s}\right)^{\tau-1}\right|l(s)\frac{ds}{s}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}(Li)(t_2) - \left(\log\frac{t_1}{c}\right)^{1-\varrho}(Li)(t_1)\right| \\
&\leq \frac{l_*\left(\log\frac{t_2}{c}\right)^{1-\varrho}}{\Gamma(\tau)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\tau-1}\frac{ds}{s} \\
&+ \frac{l_*}{\Gamma(\tau)}\int_c^{t_1}\left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}\left(\log\frac{t_2}{s}\right)^{\tau-1} - \left(\log\frac{t_1}{c}\right)^{1-\varrho}\left(\log\frac{t_1}{s}\right)^{\tau-1}\right|\frac{ds}{s} \\
&\leq \frac{l_*\left(\log\frac{T}{c}\right)^{1-\varrho}}{\Gamma(\tau+1)}\left(\log\frac{t_2}{t_1}\right)^{\tau} \\
&+ \frac{l_*}{\Gamma(\tau)}\int_c^{t_1}\left|\left(\log\frac{t_2}{c}\right)^{1-\varrho}\left(\log\frac{t_2}{s}\right)^{\tau-1} - \left(\log\frac{t_1}{c}\right)^{1-\varrho}\left(\log\frac{t_1}{s}\right)^{\tau-1}\right|\frac{ds}{s}.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the inequality tends to zero.

Step 3. $L(B_{R^*})$ is equiconvergent.

Let $t \in [c, +\infty)$ and let $i \in B_{R^*}$. We have

$$\begin{aligned} \left| \left(\log \frac{t}{c} \right)^{1-\varrho} (Li)(t) \right| &\leq \frac{|d|}{\Gamma(\varrho)} + \frac{\left(\log \frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)} \int_c^t \left(\log \frac{t}{s} \right)^{\tau-1} |\chi(s, i(s))| \frac{ds}{s} \\ &\leq \frac{|d|}{\Gamma(\varrho)} + \frac{\left(\log \frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)} \int_c^t \left(\log \frac{t}{s} \right)^{\tau-1} l(s) \frac{ds}{s} \\ &\leq \frac{|d|}{\Gamma(\varrho)} + \left(\log \frac{t}{c} \right)^{1-\varrho} \left({}^H I_c^\tau l \right)(t). \end{aligned}$$

Since

$$\left(\log \frac{t}{c} \right)^{1-\varrho} \left({}^H I_{c^+}^\tau l \right)(t) \to 0 \, as \ t \to +\infty,$$

we find

$$|(Li)(t)| \leq \frac{|d|}{\left(\log \frac{t}{c}\right)^{1-\varrho} \Gamma(\varrho)} + \frac{\left(\log \frac{t}{c}\right)^{1-\varrho} \left({}^H I_{c^+}^\tau l \right)(t)}{\left(\log \frac{t}{c}\right)^{1-\varrho}} \to 0 \quad \text{as} \quad t \to +\infty.$$

Hence

$$|(Li)(t) - (Li)(+\infty)| \to 0 \quad \text{as} \quad t \to +\infty.$$

As a consequence of Steps $1-3$, we conclude that $L : B_{R^*} \to B_{R^*}$ is compact and continuous. Applying Schauder's fixed point theorem, we get that L has a fixed point i, which is a solution of problem (1) on $[c, +\infty)$.

Step 4. Assume that i_0 is solution of (1). Set $i \in B(i_0, 2l^*)$, we have

$$\begin{aligned} & \left| \left(\log \frac{t}{c} \right)^{1-\varrho} (Li)(t) - \left(\log \frac{t}{c} \right)^{1-\varrho} i_0(t) \right| \\ &= \left| \left(\log \frac{t}{c} \right)^{1-\varrho} (Li)(t) - \left(\log \frac{t}{c} \right)^{1-\varrho} (Li_0)(t) \right| \\ &\leq \frac{\left(\log \frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)} \int_c^t \left(\log \frac{t}{s} \right)^{\tau-1} |\chi(s, i(s)) - \chi(s, i_0(s))| \frac{ds}{s} \\ &\leq \frac{2\left(\log \frac{t}{c}\right)^{1-\varrho}}{\Gamma(\tau)} \int_c^t \left(\log \frac{t}{s} \right)^{\tau-1} l(s) \frac{ds}{s} \\ &\leq 2l^*. \end{aligned}$$

We get

$$\|L(i) - i_0\|_{BC_\varrho} \leq 2l^*.$$

So, we conclude that L is a continuous function such that

$$L(B(i_0, 2l^*)) \subset B(i_0, 2l^*).$$

Moreover, if i is a solution of problem (1), then

$$\begin{aligned} |i(t) - i_0(t)| &= |(Li)(t) - (Li_0)(t)| \\ &\leq \frac{1}{\Gamma(\tau)} \int_c^t \left(\log \frac{t}{s} \right)^{\tau-1} |\chi(s, i(s)) - \chi(s, i_0(s))| \frac{ds}{s} \\ &\leq 2 \left({}^H I_{c^+}^\tau l \right)(t). \end{aligned}$$

Therefore,

$$|i(t) - i_0(t)| \leq \frac{2\left(\log \frac{t}{c}\right)^{1-\varrho}\left({}^{H}I^{\tau}_{c^+}l\right)(t)}{\left(\log \frac{t}{c}\right)^{1-\varrho}}. \tag{9}$$

By (9) and

$$\lim_{t\to\infty}\left(\log \frac{t}{c}\right)^{1-\varrho}\left({}^{H}I^{\tau}_{c^+}l\right)(t) = 0,$$

we get

$$\lim_{t\to\infty}|i(t) - i_0(t)| = 0.$$

Hence, solutions of (1) are uniformly locally attractive.

4. An Example

Consider the problem

$$\begin{cases}({}^{H}D^{\frac{1}{2},\frac{1}{2}}_{1^+}i)(t) = \chi(t, i(t)); \quad t \in [1, +\infty),\\ ({}^{H}I^{\frac{1}{4}}_{1^+}i)(1) = 1,\end{cases} \tag{10}$$

where

$$\begin{cases}\chi(t,i) = \frac{(t-1)^2(\log t)^{-1}\cos t}{64(t^2+1)(1+|i|)}, \ t \in (1,\infty), \quad i \in \mathbb{R},\\ \chi(1,i) = 0, \ i \in \mathbb{R}.\end{cases} \tag{11}$$

Clearly, the function χ is continuous, and (H_2) is satisfied with

$$\begin{cases}l(t) = \frac{(t-1)^2(\log t)^{-1}|\cos t|}{64(t^2+1)}; \quad t \in (1,\infty),\\ l(1) = 0,\end{cases} \tag{12}$$

and

$$\begin{aligned}(\log t)^{\frac{1}{4}}\,{}^{H}I^{1/2}_{1}l(t) &= \frac{(\log t)^{1/4}}{\Gamma\left(\frac{1}{2}\right)}\int_1^t\left(\log\frac{t}{s}\right)^{-1/2}\frac{l(s)}{s}ds\\ &\leq \frac{(\log t)^{1/4}}{\Gamma\left(\frac{1}{2}\right)}\int_1^t\left(\log\frac{t}{s}\right)^{-1/2}\frac{(\log s)^{-1}}{s}ds\\ &\leq \frac{1}{\sqrt{\pi}}(\log t)^{-1/4} \to 0 \quad \text{as} \quad t\to\infty.\end{aligned}$$

Hence, problem (10) has at least one solution which is uniformly locally attractive.

References

[1] S. Abbas, W. Albarakati and M. Benchohra, Existence and attractivity results for Volterra type nonlinear multi-delay Hadamard-Stieltjes fractional integral equations, *PanAmer. Math. J.* **26** (2016), 1-17.
[2] S. Abbas and M. Benchohra, Existence and attractivity for fractional order integral equations in Fréchet spaces, *Discuss. Math. Differ. Incl. Control Optim.* **33** (2013), 1-17.
[3] S. Abbas and M. Benchohra, Existence and stability of nonlinear fractional order Riemann-Liouville, Volterra-Stieltjes multi-delay integral equations, *J. Integ. Equat. Appl.* **25** (2013), 143-158.
[4] S. Abbas, M. Benchohra, and T. Diagana, Existence and attractivity results for some fractional order partial integrodifferential equations with delay, *Afr. Diaspora J. Math.* **15** (2013), 87-100.
[5] S. Abbas, M. Benchohra and J. Henderson, Asymptotic attractive nonlinear fractional order Riemann-Liouville integral equations in Banach algebras, *Nonlin. Stud.* **20** (2013), 1-10.
[6] S. Abbas, M. Benchohra and J. Henderson, Existence and attractivity results for Hilfer fractional differential equations, *J. Math. Sci.* **243** (2019), 347-357.

[7] S. Abbas, M. Benchohra, J.-E. Lagreg, A. Alsaedi, Y. Zhou, Existence and Ulam stability for fractional differential equations of Hilfer-Hadamard type, *Adv. Difference Equ.* **180** (2017), 1-14.

[8] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Sci. Publ., New York, 2014.

[9] S. Abbas, M. Benchohra, and G. M.N' Guérékata, *Topics in Fractional Differential Equations*, Dev. Math., 27, Springer, New York, 2015.

[10] S. Abbas, M. Benchohra, and J. J. Nieto, Global attractivity of solutions for nonlinear fractional order Riemann-Liouville Volterra-Stieltjes partial integral equations, *Electron. J. Qual. Theory Differ. Equat* **81** (2012), 1-15.

[11] H. Afshari, E. Karapinar, A discussion on the existence of positive solutions of the boundary value problems via ψ-Hilfer fractional derivative on b-metric spaces. *Adv. Difference Equ.* **2020**, 616.

[12] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab, Existence results for functional differential equations of fractional order, *J. Math. Anal. Appl.* **338** (2008), 1340-1350.

[13] N. Bouteraa, S. Benaicha, The uniqueness of positive solution for higher-order nonlinear fractional differential equation with fractional multi-point boundary conditions, *Adv. Theory Nonl. Anal. Appl.* **2** (2) (2018), 74-84.

[14] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Acad. Press, New York, 1973.

[15] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[16] R. Hilfer, *Applications of Fractional Calculus in Physics* , World Scientific, Singapore, 2000.

[17] R. Hilfer, Threefold introduction to fractional derivatives, *R. Klages* et al. (editors), Anomalous Transp.: Found. Appl., WileyVCH, Weinheim , pp. (2008), 17-73.

[18] R. Kamocki and C. Obcz?nski, On fractional Cauchy-type problems containing Hilfer's derivative, *Electron. J. Qual. Theory Differ. Equ.* (2016), No. 50, 1-12.

[19] E. Karapinar, T. Abdeljawad, F. Jarad, Applying new fixed point theorems on fractional and ordinary differential equations. *Adv. Difference Equ.* **2019**, Paper No. 421, 25 pp.

[20] M.D. Kassim, K.M. Furati, N.-E. Tatar, On a differential equation involving Hilfer-Hadamard fractional derivative, *Abstr. Appl. Anal.* (2012), Article ID 391062.

[21] M.D. Kassim, N.E. Tatar, Well-posedness and stability for a differential problem with Hilfer-Hadamard fractional derivative, *Abstr. Appl. Anal.* **1** (2013), 1-12.

[22] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[23] V. Lakshmikantham and J. Vasundhara Devi, Theory of fractional differential equations in a Banach space, *Eur. J. Pure Appl. Math.* **1** (2008), 38-45.

[24] V. Lakshmikantham and A.S. Vatsala, Basic theory of fractional differential equations, *Nonlin. Anal.* **69** (2008), 2677-2682.

[25] V. Lakshmikantham and A. S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations, *Appl. Math. Lett.* **21** (2008), 828-834.

[26] K. Oldham, J. Spanier, *The Fractional Calculus*, Acad. Press, New York, 1974.

[27] Podlubny, *Fractional Differential Equations, in: Mathematics in Science and Engineering*, **198**, Acad. Press, New York, 1999.

[28] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications* , Gordon Breach, Tokyo-Paris-Berlin, 1993.

[29] S. Muthaiah, M. Murugesan, N. G. Thangaraj, Existence of solutions for nonlocal boundary value problem of Hadamard fractional differential equations, *Adv. Theory Nonl. Anal. Appl.* **3** (3) (2019), 162-173.

[30] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to the Dynamics of Particles, Fields and Media*, Springer, Beijing-Heidelberg, 2010.

[31] Z. Tomovski, R. Hilfer and H. M. Srivastava, Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler-type functions, *Integral Transforms Spec. Funct.* **21** (2010), No. 11, 797-814.

[32] J.-R. Wang and Y. Zhang, Nonlocal initial value problems for differential equations with Hilfer fractional derivative, *Appl. Math. Comput.* **266** (2015), 850-859.

[33] Y. Zhou, J.-R. Wang, L. Zhang, *Basic Theory of Fractional Differential Equations*, Second edition. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.