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Triangular numbers and graphs

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Abstract

Graphs have applications in all areas of science and therefore the interest in Graph Theory is increasing everyday. They have applications in Chemistry, Pharmacology, Anthropology, Biology, Network Sciences etc. In this paper, Graph theory is connected with algebra by means of a new graph invariant Ω and define triangular graphs as graphs with a degree sequence consisting of n successive triangular numbers and use Ω and its properties to give a characterization of them. We give the conditions for the realizability of a set D of n consecutive triangular numbers and also give all possible graphs for $1 \leq t \leq 4$.

Keywords: *Omega invariant, degree sequence, triangular number, triangular graph.*

Üçgensel sayılar ve graflar

Öz

Tüm bilim dallarındaki grafik uygulamaları Graf Teoriye olan ilgiyi her gün arttırmaktadır. Kimya, İlaç Sanayi, Fizik, Biyoloji, Sosyal Bilimler, Antropoloji ve Bilişimdeki uygulamaların yanında Graf Teori ile Matematiğin diğer alanları arasında yakın bir ilişki vardır. Ardışık n üçgensel sayı köşe mertebeleri olmak üzere elde edilen graflar üçgensel graflar olarak tanımlanmaktadır. Üçgensel grafların sınıflandırılmalarını vermek için Ω invaryantı ve özellikleri kullanılmaktadır. Ayrıca n ardışık üçgensel sayıdan oluşan bir D kümesinin bir graf olarak çizilebilmesi için gerek ve yeter şartlar belirlenmiş ve $1 \leq t \leq 4$ için tüm olası durumlar sınıflandırılmıştır.

Anahtar kelimeler: *Omega sabiti, derece dizisi, üçgensel sayı, üçgensel graf.*

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1. Introduction

We assume that G = (V,E) is a graph having *n* vertices and *m* edges. The largest vertex degree in a graph is usually denoted by Δ. Let *u* and *v* be two adjacent vertices in *G*. The edge *e* connecting these vertices will be denoted by $e = u.v$ and *u* and *v* are adjacent and *e* is incident with the vertices *u* and *v*. If there is a path between every pair of vertices, then the graph is connected, and disconnected if not. An edge connecting a vertex to itself is called a loop, and at least two edges connecting two vertices will be called multiple edges.

A degree sequence is

$$D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \ldots, \Delta^{(a_\Delta)}\right\},$$

where some of a_i's could be zero. Let $D = \{d_1, d_2, d_3, \ldots, \Delta\}$. If the degree sequence of a graph *G* is equal to *D*, then *D* is realizable and *G* is called a realization.

There is a lot of special and famous number sequences and one of them having geometrical meaning is the triangular numbers. A triangular number is denoted by T_n and it is the sum of the first *n* positive integers. The first few members of this sequence are 1, 3, 6, 10, 15, 21, 28, … It is well known that $T_n = \frac{n(n+1)}{2}$. Triangular numbers have a lot of properties. For example,

Lemma 1.1. The sum of the first *n* triangular numbers satisfy the following equality:

$$\sum_{i=1}^{n} T_i = \frac{n(n+1)(n+2)}{6}. \tag{1}$$

Lucas graphs are defined as graphs with all the vertex degrees are consecutive Lucas numbers. In [5], the existence conditions for all Lucas graphs have been determined. In [7], the same search is done for Fibonacci numbers. Similarly, a triangular graph is a graph with vertex degrees being successive triangular numbers. We shall look for the existence of triangular graphs. We shall study the existence of triangular graphs.

2. Ω Invariant

In [1], a new graph invariant $\Omega(G)$ or $\Omega(D)$ was defined for a given graph *G* or for a realizable degree sequence *D* having a realization *G* as follows:

Definition 2.1. Let $D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \ldots, \Delta^{(a_\Delta)}\right\}$ be a realizable set. The $\Omega(D)$ of this degree sequence is

$$\Omega(D) = a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 = \sum_{i=1}^{\Delta}(i-2)a_i. \tag{2}$$

For convenience, the omega invariant of a realization *G* of *D* is also denoted by $\Omega(G)$. Some properties of Ω can be found in [1, 2, 3, 4, 6]. We recall some very important properties of it here.

Theorem 2.1. [1] For a graph *G*,

$$\Omega(G) = 2(m - n). \tag{3}$$

The cyclomatic number of *G* is the number *r* of cycles and it can also be given by means of $\Omega(G)$:

Theorem 2.2. [1] Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \ldots, \Delta^{(a_\Delta)}\}$ be a realization of a graph *G having c* components. Then *r* is

$$r = \frac{\Omega(G)}{2} + c. \tag{4}$$

In [5], the highest number of components of *D* is given as

$$c_{max} = \sum_{d\ even} a_i + \sum_{d\ odd} a_i. \tag{5}$$

3. Existence conditions for triangular graphs

We now determine the conditions for the existence of graphs having *n* consecutive triangular numbers as the vertex degrees. We solve this problem by the graph invariant omega and related results given in [1].

As we shall use several special classes of graphs with one, two, three or four vertices in our proofs and illustrations, we need to introduce these classes first. A graph having *q* loops at one vertex is shown by L_q. For positive integers *r,s*, $B_{r,s}$ denotes the graph with *r* and *s* loops at its vertices. A connected graph with three vertices *u,v,w* with degrees $2a+1, 2b+2$ *and* $2c+1$ is a graph consisting of a path $P_3 = \{u, v, w\}$ such that *a* loops are incident to *u*, *b* loops are incident to *v* and *c* loops are incident to *w* and denoted by $T_{a,b,c}$. Finally, a connected graph having four vertices *u,v,w,z* with degrees $2a+1, 2b+2, 2c+2$ and $2d+1$ is a graph denoted by $Q_{a,b,c,d}$ consisting of a path $P_4 = \{u, v, w, z\}$ where *a* loops are at u, *b* loops are at *v*, *c* loops are at *w* and *d* loops are at *z*.

"$r[k]$" means that *k* loops are attached to a new vertex on one of the *r* loops incident to other vertices. "$r[k_1, k_2, \ldots, k_t]$" similarly means that k_1 loops are attached to a loop of degree *r* at a new vertex, k_2 loops are attached to another loop of degree *r* at another vertex and k_t loops are attached to another loop of degree *r*. Finally, the notation "$r[k_1, k_2, \ldots, k_t]$" means that k_1 loops, k_2 loops and k_t loops are attached to the same loop at some new vertices. $B_{2[3],1}$, $B_{2[3,1],1}$ and $T_{2[3;1],1}$ corresponds to graphs shown below:

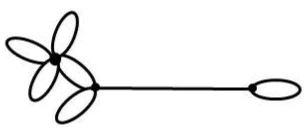

Figure 1. The graph $B_{2[3],1}$

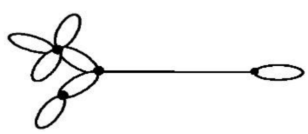

Figure 2. The graph $B_{2[3,1],1}$

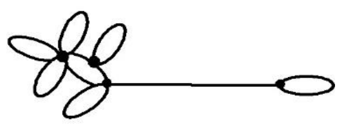

Figure 3. The graph $T_{2[3;1],1}$

3.1. Triangular graphs of order 1

First, we study the existence of a graph with only one vertex having degree equal to a triangular number. Naturally, this is the simplest case. Recall that the triangular numbers follow the rule odd, odd, even, even, odd, odd, even, even . . .. That is, if the index of a triangular number is 1 or 2 in modulo 4, then it is odd, otherwise it is even. Therefore only the triangular numbers $T_3, T_4, T_7, T_8, T_{11}, T_{12}, . . .$ are even. Therefore, to have the required graph of order one, the degree of the unique vertex must be one of these triangular numbers. Hence we showed

Theorem 3.1. A graph *G* of order 1 is a triangular graph iff the unique vertex has degree T_{4k+3} or T_{4k} for some natural number k.

3.2. Triangular graphs of order 2

Next, we study the case of two successive triangular numbers as vertex degrees of a triangular graph. That is, we want to determine which sets of two consecutive triangular numbers are realizable as graphs. According to the parity rule mentioned just above, the degree sequence of such a graph could have two consecutive odd triangular numbers or two consecutive even triangular numbers. That is the vertex degrees of such a graph must be either T_{4k+1} and T_{4k+2} or T_{4k+3} and T_{4k+4} for some natural number *k*. Hence such a graph must have the degree sequence

$$D = \left\{T_{4k+1}^{(1)}, T_{4k+2}^{(1)}\right\}$$

or

$$D = \left\{T_{4k+3}^{(1)}, T_{4k+4}^{(1)}\right\}.$$

Therefore the omega invariant of any realization of such a degree sequence would be either

$$\Omega(G) = (T_{4k+1} - 2).1 + (T_{4k+2} - 2).1 = T_{4k+1} + T_{4k+2} - 4 \quad (6)$$

or

$$\Omega(G) = (T_{4k+3} - 2).1 + (T_{4k+4} - 2).1 = T_{4k+3} + T_{4k+4} - 4, \quad (7)$$

respectively. In this case, the number r of the faces of a connected realization is

$$r = \frac{\Omega(G)}{2} + 1 = \frac{T_{4k+1}+T_{4k+2}-4}{2} + 1 = \frac{T_{4k+1}+T_{4k+2}-2}{2} \quad (8)$$

in the former case and

$$r = \frac{\Omega(G)}{2} + 1 = \frac{T_{4k+3}+T_{4k+4}-4}{2} + 1 = \frac{T_{4k+3}+T_{4k+4}-2}{2} \quad (9)$$

in the latter case. Since both vertices are of odd degrees, there is only one connected graph.
In general, we have proved the following fact:

Theorem 3.2. A graph *G* of order 2 is a triangular graph iff both vertices have odd degrees T_{4k+1} and T_{4k+2} or even degrees T_{4k+3} and T_{4k+4} for some natural number *k*.

To illustrate the former case, if we take T_1=1 and T_2=3, then we would have the graph in Figure 4:

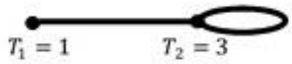

Figure 4. The graph $B_{0,1}$.

As easily seen, this connected graph has only one face which coincides with the first formula for *r* and it is $B_{0,1}$.
As the second possible example of the former case, we take $T_5 = 15$ and $T_6 = 21$. In this case, we have a unique connected graph $B_{7,10}$ with 17 faces as Figure 5.

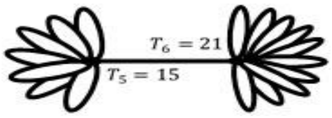

Figure 5. The graph $B_{7,10}$.

If we take the triangular numbers T_9 and T_{10}, we get the graph $B_{22,27}$ with 49 faces. We can see it in Figure 6.

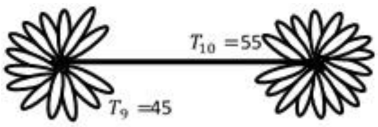

Figure 6. The graph $B_{22,27}$.

Now we consider the latter case. Recall that the maximum number of components is

$$c_{max} = \sum_{d_i \ even} a_i + \frac{1}{2} \sum_{d_i \ odd} a_i = (T_{4k+3} + T_{4k+4}). \qquad (10)$$

Therefore there are two possible graph realizations one of which is a connected graph and the other one is a disconnected graph. As an example, for $T_3 = 6$ and $T_4 = 10$, the former graph is $L_{T_3/2} \cup L_{T_4/2}$ and the latter one is $L_{5[2]}$ or $L_{3[4]}$, see Figure 7 and Figure 8.

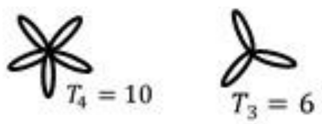

Figure 7. The graph $L_{T_3/2} \cup L_{T_4/2}$.

This graph is disconnected and named as $L_5 \cup L_{3.}$. Because of two components, we have $r = \frac{10+6-4}{2} + 2 = 8$ there are eight faces. The connected realization is $L_{5[2]}$ given in Figure 8:

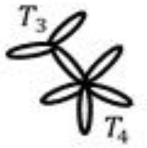

Figure 8. The graph $L_{5[2]}$.

As $r = \frac{10+6-4}{2} + 1 = 7$, there are seven faces.

Secondly, for $T_7 = 28$ and $T_8 = 36$, there is one disconnected graph realization given in Figure 9:

Figure 9. The graph $L_{T_7/2} \cup L_{T_8/2}$.

As $r = \frac{28+36-4}{2} + 2 = 32$, there are 32 faces. In this case, there is just one connected graph $L_{18[13]}$ or $L_{14[17]}$ as in Figure 10:

Figure 10. The graph $L_{18[13]}$.

Here $r = \frac{28+36-4}{2} + 1 = 31$ implies that there are 31 faces.

3.3. Triangular graphs of order 3

Next, we study the triangular graphs of order 3. In such a graph, we must have an even sum of vertex degrees. So according to the parity rule of the triangular numbers, we have exactly two possibilities:

A) Let $D = \left\{T_{4k+1}^{(1)}, T_{4k+2}^{(1)}, T_{4k+3}^{(1)}\right\}$. Then

$$\begin{aligned}\Omega(D) &= (T_{4k+1} - 2).1 + (T_{4k+2} - 2).1 + (T_{4k+3} - 2).1 \\ &= [T_{4k+1} + T_{4k+2} + T_{4k+3}] - 6 \end{aligned} \qquad (11)$$

Because of the fact that two of the vertex degrees are odd and the third one is even, we find the maximum number of components of any realization of this degree sequence as

$$c_{max} = \sum_{d_i\ even} a_i + \frac{1}{2} \sum_{d_i\ odd} a_i = 1 + \frac{1}{2}(1+1) = 2\,. \qquad (12)$$

That is, any realization of D is either connected or can have two components. In this situation, there are te following cases to consider:

In the first case, there are two possible connected realizations. To illustrate this case, let us take$T_1 = 1, T_2 = 3, T_3 = 6$. In Figure 11 and Figure 12, the two connected graphs are shown:

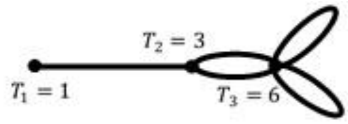

Figure 11. The graph $T_{0,1[2]}$.

and

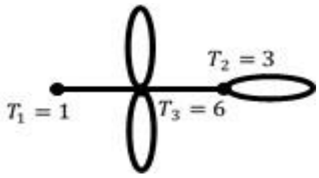

Figure 12. The graph $T_{0,2,1}$.

This graphs can respectively be denoted by $T_{0,1[2]}$ and $T_{0,2,1}$. Here, because of connectedness, we obtain $r = \frac{T_1+T_2+T_3-6}{2} + 1 = 3$, so there are three faces.

There is also one disconnected realization in this case which is given in Figure. 13:

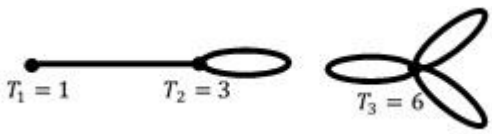

Figure 13. The graph $B_{0,1} \cup L_3$.

This graph can be denoted by $B_{0,1} \cup L_3$ and it has $r = \frac{T_1+T_2+T_3-6}{2} + 2 = 4$ faces.
For the second possibility, take $T_5 = 15, T_6 = 21, T_7 = 28$. In this case, we find that there are four realizable graphs for these vertex degrees. Three of them are connected and the last one is disconnected. The connected graphs are shown in Figure 14, Figure 15 and Figure 16:

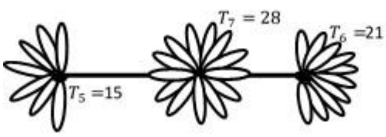

Figure 14.

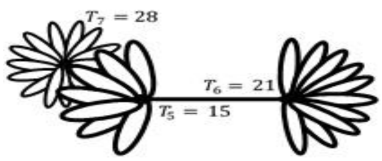

Figure 15.

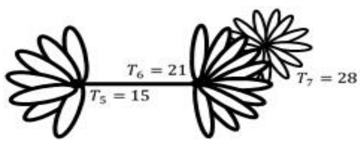

Figure 16.

These graphs can respectively be denoted by $T_{\frac{T_5-1}{2},\frac{T_7-2}{2},\frac{T_6-1}{2}}$, $T_{\frac{T_5-1}{2}\left[\frac{T_7-2}{2}\right],\frac{T_6-1}{2}}$, $T_{\frac{T_5-1}{2},\frac{T_6-1}{2}\left[,\frac{T_7-2}{2}\right]}$ and for these connected graph realizations, the number of faces is $r = \frac{T_5+T_6+T_7-6}{2} + 1 = 30$.
The unique disconnected realization is in Figure17:

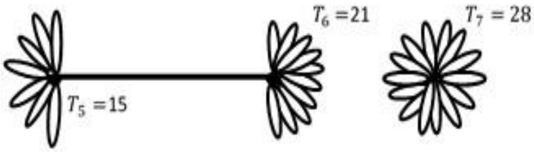

Figure 17. The graph $B_{7,10} \cup L_{14}$.

This graph can be denoted by $B_{7,10} \cup L_{14}$ and has $r = \frac{\Omega(G)}{2} + 2 = \frac{T_5+T_6+T_7-6}{2} + 2$ = 31 faces.

B) If we take the degree sequence consisting of three succesive triangular numbers as

$D = \left\{T_{4k}^{(1)}, T_{4k+1}^{(1)}, T_{4k+2}^{(1)}\right\}$, then

$$\Omega(D) = (T_{4k} - 2).1 + (T_{4k+1} - 2).1 + (T_{4k+2} - 2).1. \quad (13)$$

In this case, a careful examination shows that there are three connected graph realizations and one disconnected realization.

To illustrate this case, let us take three succesive triangular numbers as $T_4 = 10, T_5 = 15, T_6 = 21$. The connected realizations are $T_{7,4,10}, B_{7,[4],10}$ \ and $B_{7,10[4]}$ and the disconnected realization is $B_{7,10} \cup L_5$. These graphs are illustrated in Figure 18 - Figure 21.

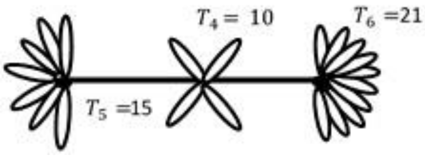

Figure 18. The graph $T_{7,4,10}$.

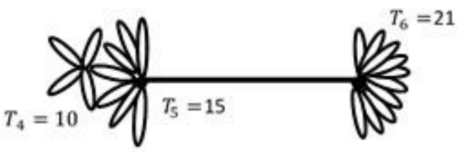

Figure 19. The graph $B_{7[4],10}$.

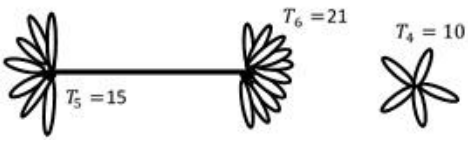

Figure 20. The graph $B_{7,10[4]}$.

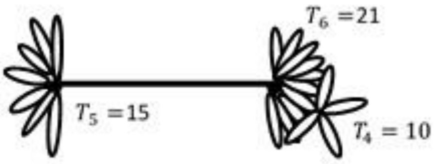

Figure 21. The graph $B_{7,10} \cup L_5$.

Hence we have proved the following fact:

Theorem 3.2. A graph G of order 3 is a triangular graph iff the degrees of the three vertices are either T_{4k}, T_{4k+1} and T_{4k+2} or T_{4k+10}, T_{4k+2}, and T_{4k+3} for some natural number k.

3.4. Triangular graphs of order 4

Similarly to the above cases, we can see that any four consecutive triangular numbers can form a realizable degree sequence giving a triangular graph of order 4. Not to give all four possible cases here, we call these four triangular numbers as $a.b.c$ and d without any order between them. We also let a and b be odd and c and d be even. With the similar arguments as above, we obtain the following result:

Theorem 3. 4. $\{a, b, c, d\}$ be a set of four consecutive triangular numbers. Then $D = \{a, b, c, d\}$ is always realizable as a triangular graph of order 4.

All possible graph realizations are as follows:

$Q_{\frac{a-1}{2},\frac{c-2}{2},\frac{d-2}{2},\frac{b-1}{2}}, Q_{\frac{a-1}{2},\frac{d-2}{2},\frac{c-2}{2},\frac{b-1}{2}}, T_{\frac{a-1}{2},\frac{c-2}{2}\left[\frac{d-2}{2}\right],\frac{b-1}{2}}, T_{\frac{a-1}{2},\frac{d-2}{2}\left[\frac{c-2}{2}\right],\frac{b-1}{2}}$,

$T_{\frac{a-1}{2}\left[\frac{d-2}{2}\right],\frac{c-2}{2},\frac{b-1}{2}}, T_{\frac{a-1}{2}\left[\frac{c-2}{2}\right],\frac{d-2}{2},\frac{b-1}{2}}, T_{\frac{a-1}{2},\frac{c-2}{2},\frac{b-1}{2}\left[\frac{d-2}{2}\right]}, T_{\frac{a-1}{2},\frac{d-2}{2},\frac{b-1}{2}\left[\frac{c-2}{2}\right]}$,

$B_{\frac{a-1}{2}\left[\frac{c-2}{2}\left[\frac{d-2}{2}\right]\right],\frac{b-1}{2}}, B_{\frac{a-1}{2}\left[\frac{d-2}{2}\left[\frac{c-2}{2}\right]\right],\frac{b-1}{2}}, B_{\frac{a-1}{2},\frac{b-1}{2}\left[\frac{c-2}{2}\left[\frac{d-2}{2}\right]\right]}, B_{\frac{a-1}{2},\frac{b-1}{2}\left[\frac{d-2}{2}\left[\frac{c-2}{2}\right]\right]}$,

$B_{\frac{a-1}{2}\left[\frac{c-2}{2}\right],\frac{b-1}{2}\left[\frac{d-2}{2}\right]}, B_{\frac{a-1}{2}\left[\frac{d-2}{2}\right],\frac{b-1}{2}\left[\frac{c-2}{2}\right]}, B_{\frac{a-1}{2}\left[\frac{c-2}{2}\right],\frac{b-1}{2} \cup L_{\frac{d}{2}}}, B_{\frac{a-1}{2}\left[\frac{d-2}{2}\right],\frac{b-1}{2} \cup L_{\frac{c}{2}}}$,

$B_{\frac{a-1}{2},\frac{b-1}{2},,\left[\frac{c-2}{2}\right] \cup L_{\frac{d}{2}}}, B_{\frac{a-1}{2},\frac{b-1}{2}\left[\frac{d-2}{2}\right] \cup L_{\frac{c}{2}}}, T_{\frac{a-1}{2},\frac{c-2}{2},\frac{b-1}{2} \cup L_{\frac{d}{2}}}, T_{\frac{a-1}{2},\frac{d-2}{2},\frac{b-1}{2} \cup L_{\frac{c}{2}}}$,

$B_{\frac{a-1}{2},\frac{b-1}{2}} \cup L_{\frac{c}{2}\left[\frac{d-2}{2}\right]}$ and $B_{\frac{a-1}{2},\frac{b-1}{2}} \cup L_{\frac{c}{2}} \cup L_{\frac{d}{2}}$.

Note that out of 22 realizations, 14 are connected and 8 of them are disconnected. Out of the disconnected ones, 7 have 2 components and one has 3 components.

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