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Asymptotics of Eigenvalues for Regular Sturm-Liouville Problems with Spectral Parameter-Dependent Boundary Conditions and Symmetric Single Well Potential

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Abstract. In this study, we find asymptotic estimates of eigenvalues for regular Sturm-Liouville problems having the eigenvalue parameter in all boundary conditions with the symmetric single well potential that is symmetric to the midpoint of the related interval and nonincreasing on the first semi-region of the related interval.

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Keywords: Sturm-Liouville problem, symmetric single well potential, eigenvalue parameter in the boundary conditions.

1. Introduction

In this paper, we consider the boundary value problem

$$y''(t) + \{\lambda - q(t)\}\, y(t) = 0, \quad t \in [0, a], \tag{1.1}$$

$$a_1 y(0) + a_2 y'(0) = \lambda \left[a_1' y(0) + a_2' y'(0)\right], \tag{1.2}$$

$$b_1 y(a) + b_2 y'(a) = \lambda \left[b_1' y(a) + b_2' y'(a)\right], \tag{1.3}$$

where λ is a real parameter, $q(t)$ is a real-valued, continuous function and a_i, b_i, a_i', b_i' for $i = 1, 2$ are real constants. We remark that this problem (1.1)-(1.3) is different from the usual regular Sturm-Liouville problem because spectral parameter λ is seen in the boundary conditions. Such problems often arise from physical problems, for example, vibration of a string, quantum mechanics and geophysics. Fulton gives more than a hundred references about these problems in [15] and [16] (see also [22]) so his works serve as a historical guide. There are a lot of studies with Sturm-Liouville problems that the spectral parameter appears in the boundary conditions, some of them are [9–13], [17–20].

Besides, Equation (1.1) is equal to one-dimensional Schrödinger equation and especially in recent years, since quantum mechanic has gained importance, there are a lot of studies on eigenvalues of Hill's equation and Schrödinger's operator with symmetric single well potential such as anharmonic oscillator. The eigenvalues of these equations represent excitation energy and eigenfunctions are named as wavefunction in physics. A symmetric single well potential on $[0, a]$ is defined as symmetric with respect to the midpoint $a/2$ and nonincreasing on $[0, a/2]$. For example, in a symmetric hydrogen bond that occurs in many biological structures such as DNA and water, the proton free energy landscape is a symmetric single well potential. The unusual case of very strong H-bonds, features symmetric single-well proton

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free energy profiles. In this particular case the energy minimum corresponds to the proton centered in the middle of the H-bond and therefore no PT transfer barrier exits [28]. Stationary states for a particle moving in a single-well, steeper than parabolic, potential driven by Lévy noise can be bimodal. [5] and [8] have verified that the stationary states in symmetric single-well potentials can be characterized by more than two modal values. [27] discusses analytical and numerical results for nonharmonic, undamped, single-well, stochastic oscillators driven by additive noises. It focuses on average kinetic, potential, and total energies together with the corresponding distributions under random drivings, involving Gaussian white, Ornstein-Uhlenbeck, and Markovian dichotomous noises. In [6], they study the inverse nodal problem and the eigenvalue gap for the one-dimensional sloshing problem with the p-Laplacian operator. They investigate the eigenvalue gap under the restriction of symmetric single-well depth functions. The others of the eigenvalue problems with symmetric single well potential can be found in [1–3], [7], [14], [23–26].

In this study, we gain asymptotic approximations for eigenvalues λ_n of (1.1)-(1.3) with symmetric single well potential q. We note that a symmetric single well potential on $[0, a]$ means that a continuous function $q(t)$ is symmetric on $[0, a]$ and non-increasing on $\left[0, \frac{a}{2}\right]$, so we have $q(t) = q(a-t)$ mathematically. We assume without loss of generality that $q(t)$ has a mean value zero.

2. The Method

At the beginning of our method, we should remark that $q'(t)$ exists since a monotone function on an interval I is differentiable almost everywhere on I [21].

Our method is based on [4]. If we rearrange its main theorems for $a = 0$ and $b = a$, we get the following results:

Theorem 2.1. *The eigenvalues λ_n of (1.1)-(1.3) satisfy as $n \to \infty$,*

i) if $a_2' \neq 0$ and $b_2' \neq 0$,

$$\begin{aligned}
\lambda_n^{1/2} = \frac{(n+1)\pi}{a} &+ \frac{1}{(n+1)\pi}\left\{\frac{a_2'b_1' - a_1'b_2'}{a_2'b_2'} - \frac{a}{4(n+1)\pi}\int_0^a q'(x)\left(\sin\frac{2(n+1)\pi x}{a}\right)dx\right.\\
&- \frac{a^2}{(n+1)^2\pi^2}\left[\frac{-3b_1'b_2'b_2 + 3b_1\left(b_2'\right)^2 + \left(b_1'\right)^3}{3\left(b_2'\right)^3}\right]\\
&+ \frac{a^2}{(n+1)^2\pi^2}\left[\frac{-3a_1'a_2'a_2 + 3a_1\left(a_2'\right)^2 + \left(a_1'\right)^3}{3\left(a_2'\right)^3}\right] + \frac{a^2}{2(n+1)^2\pi^2}\frac{a_1'}{a_2'}\left[q(a) - q(0)\right]\\
&\left.- \frac{a^2}{2(n+1)^2\pi^2}\frac{a_1'}{a_2'}\int_0^a q'(x)\left(\cos\frac{2(n+1)\pi x}{a}\right)dx\right\} + O\left(n^{-4}\eta(n)\right) + O\left(n^{-3}\eta^2(n)\right),
\end{aligned}$$

ii) if $a_2' \neq 0$ and $b_2' = 0$,

$$\begin{aligned}
\lambda_n^{1/2} = \frac{(2n+3)\pi}{2a} &+ \frac{2}{(2n+3)\pi}\left\{\frac{a_2'b_2 - a_1'b_1'}{a_2'b_1'} - \frac{a}{2(2n+3)\pi}\int_0^a q'(x)\left(\sin\frac{(2n+3)\pi x}{a}\right)dx\right.\\
&+ \frac{4a^2}{(2n+3)^2\pi^2}\left[\frac{3b_1'b_1b_2 - b_2^3}{3\left(b_1'\right)^3}\right] + \frac{4a^2}{(2n+3)^2\pi^2}\left[\frac{-3a_1'a_2'a_2 + 3a_1\left(a_2'\right)^2 + \left(a_1'\right)^3}{3\left(a_2'\right)^3}\right]\\
&- \frac{2a^2}{(2n+3)^2\pi^2}\frac{a_1'}{a_2'}\left[q(a) + q(0)\right]\\
&\left.- \frac{2a^2}{(2n+3)^2\pi^2}\frac{a_1'}{a_2'}\int_0^a q'(x)\left(\cos\frac{(2n+3)\pi x}{a}\right)dx\right\} + O\left(n^{-4}\eta(n)\right) + O\left(n^{-3}\eta^2(n)\right),
\end{aligned}$$

where $\eta(\lambda) := \sup_{0\leq t\leq a} F(t, \lambda)$ so that

$$F(t,\lambda) := \begin{cases} \left|\int_t^a e^{2i\lambda^{1/2}x} q'(x)\,dx\right| / \int_t^a |q'(x)|\,dx, & \text{if } \int_t^a |q'(x)|\,dx \neq 0, \\ 0 & , \quad \text{if } \int_t^a |q'(x)|\,dx = 0. \end{cases}$$

Theorem 2.2. *The eigenvalues λ_n of (1.1)-(1.3) satisfy as $n \to \infty$,*

i) if $a_2' = 0$ and $b_2' \neq 0$,

$$\begin{aligned} \lambda_n^{1/2} = {} & \frac{(2n+3)\pi}{2a} + \frac{2}{(2n+3)\pi}\left\{\frac{a_1'b_1' - a_2b_2'}{a_1'b_2'} - \frac{a}{2(2n+3)\pi}\int_0^a q'(x)\left(\sin\frac{(2n+3)\pi x}{a}\right)dx\right. \\ & - \frac{4a^2}{(2n+3)^2\pi^2}\left[\frac{3b_1\left(b_2'\right)^2 - 3b_1'b_2'b_2 + \left(b_1'\right)^3}{3\left(b_2'\right)^3}\right] \\ & + \frac{4a^2}{(2n+3)^2\pi^2}\left[\frac{-3a_1'a_1a_2 + a_2^3}{3\left(a_1'\right)^3}\right] + \frac{2a^2}{(2n+3)^2\pi^2}\frac{a_2}{a_1'}\left[q(a) + q(0)\right] \\ & \left. + \frac{2a^2}{(2n+3)^2\pi^2}\frac{a_2}{a_1'}\int_0^a q'(x)\left(\cos\frac{(2n+3)\pi x}{a}\right)dx\right\} + O\left(n^{-4}\eta(n)\right) + O\left(n^{-3}\eta^2(n)\right), \end{aligned}$$

ii) if $a_2' = 0$ and $b_2' = 0$,

$$\begin{aligned} \lambda_n^{1/2} = {} & \frac{(n+2)\pi}{a} + \frac{1}{(n+2)\pi}\left\{\frac{a_1'b_2 - a_2b_1'}{a_1'b_1'} + \frac{a}{4(n+2)\pi}\int_0^a q'(x)\left(\sin\frac{2(n+2)\pi x}{a}\right)dx\right. \\ & - \frac{a^2}{(n+2)^2\pi^2}\left[\frac{3b_1'b_1b_2 - b_2^3}{3\left(b_1'\right)^3}\right] + \frac{a^2}{(n+2)^2\pi^2}\left[\frac{-3a_1'a_1a_2 + a_2^3}{3\left(a_1'\right)^3}\right] \\ & - \frac{a^2}{(n+2)^2\pi^2}\frac{a_2}{2a_1'}\left[q(a) - q(0)\right] \\ & \left. + \frac{a^2}{(n+2)^2\pi^2}\frac{a_2}{2a_1'}\int_0^a q'(x)\left(\cos\frac{2(n+2)\pi x}{a}\right)dx\right\} + O\left(n^{-4}\eta(n)\right) + O\left(n^{-3}\eta^2(n)\right). \end{aligned}$$

3. The Main Results

To obtain asymptotics approximations for eigenvalues λ_n of (1.1)-(1.3) with symmetric single well potential q, we calculate integral terms of Theorem 2.1 and Theorem 2.2. Therefore, we first compute the following integrals:

Lemma 3.1. *If $q(t)$ is a symmetric single well potential, then*

i)

$$\int_0^a q'(x)\sin\left(\frac{2(n+1)\pi x}{a}\right)dx = 2\int_0^{a/2} q'(x)\sin\left(\frac{2(n+1)\pi x}{a}\right)dx,$$

ii)

$$\int_0^a q'(x)\sin\left(\frac{(2n+3)\pi x}{a}\right)dx = 2\int_0^{a/2} q'(x)\sin\left(\frac{(2n+3)\pi x}{a}\right)dx,$$

iii)

$$\int_0^a q'(x)\sin\left(\frac{2(n+2)\pi x}{a}\right)dx = 2\int_0^{a/2} q'(x)\sin\left(\frac{2(n+2)\pi x}{a}\right)dx,$$

iv)

$$\int_0^a q'(x)\cos\left(\frac{2(n+1)\pi x}{a}\right)dx = 0,$$

v)

$$\int_0^a q'(x)\cos\left(\frac{(2n+3)\pi x}{a}\right)dx = 0,$$

vi)

$$\int_0^a q'(x)\cos\left(\frac{2(n+2)\pi x}{a}\right)dx = 0.$$

Proof. i)

$$\begin{aligned}\int_0^a q'(x)\sin\left(\frac{2(n+1)\pi x}{a}\right)dx &= \int_0^{a/2} q'(x)\sin\left(\frac{2(n+1)\pi x}{a}\right)dx + \int_{a/2}^{a} q'(x)\sin\left(\frac{2(n+1)\pi x}{a}\right)dx \\ &= \int_0^{a/2} q'(x)\sin\left(\frac{2(n+1)\pi x}{a}\right)dx - \int_{a/2}^{a} q'(a-x)\sin\left(\frac{2(n+1)\pi x}{a}\right)dx \\ &= \int_0^{a/2} q'(x)\sin\left(\frac{2(n+1)\pi x}{a}\right)dx + \int_{a/2}^{0} q'(u)\sin\left(\frac{2(n+1)\pi(a-u)}{a}\right)du \\ &= \int_0^{a/2} q'(x)\sin\left(\frac{2(n+1)\pi x}{a}\right)dx + \int_{0}^{a/2} q'(u)\sin\left(\frac{2(n+1)\pi u}{a}\right)du \\ &= 2\int_0^{a/2} q'(x)\sin\left(\frac{2(n+1)\pi x}{a}\right)dx.\end{aligned}$$

The second equality holds since $q(t)$ is symmetric and $q'(t)$ exists, so $q'(t) = -q'(a-t)$.
ii) and iii) can be proved similarly to i).
iv) Because of $q'(t) = -q'(a-t)$, we have

$$\begin{aligned}\int_0^a q'(x)\cos\left(\frac{2(n+1)\pi x}{a}\right)dx &= \int_0^{a/2} q'(x)\cos\left(\frac{2(n+1)\pi x}{a}\right)dx + \int_{a/2}^{a} q'(x)\cos\left(\frac{2(n+1)\pi x}{a}\right)dx \\ &= \int_0^{a/2} q'(x)\cos\left(\frac{2(n+1)\pi x}{a}\right)dx - \int_{a/2}^{a} q'(a-x)\cos\left(\frac{2(n+1)\pi x}{a}\right)dx \\ &= \int_0^{a/2} q'(x)\cos\left(\frac{2(n+1)\pi x}{a}\right)dx + \int_{a/2}^{0} q'(u)\cos\left(\frac{2(n+1)\pi}{a}(a-u)\right)du \\ &= \int_0^{a/2} q'(x)\cos\left(\frac{2(n+1)\pi x}{a}\right)dx + \int_{a/2}^{0} q'(u)\cos\left(\frac{2(n+1)\pi}{a}u\right)du = 0.\end{aligned}$$

v) and vi) can be proved similarly to iv). □

We obtain asymptotics approximations for symmetric single well potential as following:

Theorem 3.2. *The eigenvalues λ_n of* (1.1)-(1.3) *satisfy as $n \to \infty$,*

i) if $a_2' \neq 0$ and $b_2' \neq 0$,

$$\begin{aligned}\lambda_n^{1/2} = \frac{(n+1)\pi}{a} + \frac{1}{(n+1)\pi}\Bigg\{&\frac{a_2'b_1' - a_1'b_2'}{a_2'b_2'} - \frac{a}{2(n+1)\pi}\int_0^{a/2} q'(x)\left(\sin\frac{2(n+1)\pi x}{a}\right)dx \\ &- \frac{a^2}{(n+1)^2\pi^2}\left[\frac{-3b_1'b_2'b_2 + 3b_1\left(b_2'\right)^2 + \left(b_1'\right)^3}{3\left(b_2'\right)^3}\right] \\ &+ \frac{a^2}{(n+1)^2\pi^2}\left[\frac{-3a_1'a_2'a_2 + 3a_1\left(a_2'\right)^2 + \left(a_1'\right)^3}{3\left(a_2'\right)^3}\right]\Bigg\} + O\left(n^{-4}\eta(n)\right) + O\left(n^{-3}\eta^2(n)\right),\end{aligned}$$

ii) if $a_2' \neq 0$ and $b_2' = 0$,

$$
\begin{aligned}
\lambda_n^{1/2} = & \frac{(2n+3)\pi}{2a} + \frac{2}{(2n+3)\pi}\left\{\frac{a_2' b_2 - a_1' b_1'}{a_2' b_1'} - \frac{a}{(2n+3)\pi}\int_0^{a/2} q'(x)\left(\sin\frac{(2n+3)\pi x}{a}\right)dx\right. \\
& + \frac{4a^2}{(2n+3)^2\pi^2}\left[\frac{3b_1' b_1 b_2 - b_2^3}{3\left(b_1'\right)^3}\right] + \frac{4a^2}{(2n+3)^2\pi^2}\left[\frac{-3a_1' a_2' a_2 + 3a_1\left(a_2'\right)^2 + \left(a_1'\right)^3}{3\left(a_2'\right)^3}\right] \\
& \left. - \frac{4a^2}{(2n+3)^2\pi^2}\frac{a_1'}{a_2'}q(0)\right\} + O\left(n^{-4}\eta(n)\right) + O\left(n^{-3}\eta^2(n)\right).
\end{aligned}
$$

Proof. $q(0)$ equals to $q(a)$ because of symmetric potential. By using this and Lemma 3.1 i), ii), iv), v) in Theorem 2.1, the theorem is proved. □

Theorem 3.3. *The eigenvalues λ_n of (1.1)-(1.3) satisfy as $n \to \infty$,*

i) if $a_2' = 0$ and $b_2' \neq 0$,

$$
\begin{aligned}
\lambda_n^{1/2} = & \frac{(2n+3)\pi}{2a} + \frac{2}{(2n+3)\pi}\left\{\frac{a_1' b_1' - a_2 b_2'}{a_1' b_2'} - \frac{a}{(2n+3)\pi}\int_0^{a/2} q'(x)\left(\sin\frac{(2n+3)\pi x}{a}\right)dx\right. \\
& - \frac{4a^2}{(2n+3)^2\pi^2}\left[\frac{3b_1\left(b_2'\right)^2 - 3b_1' b_2' b_2 + \left(b_1'\right)^3}{3\left(b_2'\right)^3}\right] \\
& \left. + \frac{4a^2}{(2n+3)^2\pi^2}\left[\frac{-3a_1' a_1 a_2 + a_2^3}{3\left(a_1'\right)^3}\right] + \frac{4a^2}{(2n+3)^2\pi^2}\frac{a_2}{a_1'}q(0)\right\} + O\left(n^{-4}\eta(n)\right) + O\left(n^{-3}\eta^2(n)\right),
\end{aligned}
$$

ii) if $a_2' = 0$ and $b_2' = 0$,

$$
\begin{aligned}
\lambda_n^{1/2} = & \frac{(n+2)\pi}{a} + \frac{1}{(n+2)\pi}\left\{\frac{a_1' b_2 - a_2 b_1'}{a_1' b_1'} + \frac{a}{2(n+2)\pi}\int_0^{a/2} q'(x)\left(\sin\frac{2(n+2)\pi x}{a}\right)dx\right. \\
& \left. - \frac{a^2}{(n+2)^2\pi^2}\left[\frac{3b_1' b_1 b_2 - b_2^3}{3\left(b_1'\right)^3}\right] + \frac{a^2}{(n+2)^2\pi^2}\left[\frac{-3a_1' a_1 a_2 + a_2^3}{3\left(a_1'\right)^3}\right]\right\} + O\left(n^{-4}\eta(n)\right) + O\left(n^{-3}\eta^2(n)\right).
\end{aligned}
$$

Proof. The theorem follows from substitution of $q(0) = q(a)$ and Lemma 3.1 ii), iii), v), vi) into Theorem 2.2. □

Example 3.4. Let us consider the equation

$$
y''(t) + \{\lambda - q(t)\}\, y(t) = 0, \quad t \in [0, \pi],
$$

where $q(t) = \frac{1}{4}\left(t - \frac{\pi}{2}\right)^4 + \frac{1}{2}\left(t - \frac{\pi}{2}\right)^2$. This equation is known anharmonic oscillator in quantum mechanics and $q(t)$ is symmetric single well. We assumed that $q(t)$ has mean value zero, so we should get $q(t)$ as following:

$$
q(t) = \frac{1}{4}\left(t - \frac{\pi}{2}\right)^4 + \frac{1}{2}\left(t - \frac{\pi}{2}\right)^2 - \frac{\pi^2}{24} - \frac{\pi^4}{320}.
$$

In this situation, by evaluating integral terms in Theorem 3.2 and Theorem 3.3, we find as $n \to \infty$

for $a_2' \neq 0$ and $b_2' \neq 0$,

$$
\begin{aligned}
\lambda_n^{1/2} &= n+1+\frac{1}{(n+1)\pi}\left\{\frac{a_2'b_1'-a_1'b_2'}{a_2'b_2'}+\frac{\pi}{32\,(n+1)^4}\left[(\pi^2+4)n(n+2)+\pi^2-2\right]\right.\\
&\quad -\frac{1}{(n+1)^2}\left[\frac{-3b_1'b_2'b_2+3b_1\left(b_2'\right)^2+\left(b_1'\right)^3}{3\left(b_2'\right)^3}\right]+\left.\frac{1}{(n+1)^2}\left[\frac{-3a_1'a_2'a_2+3a_1\left(a_2'\right)^2+\left(a_1'\right)^3}{3\left(a_2'\right)^3}\right]\right\}\\
&\quad +O\left(n^{-4}\eta(n)\right)+O\left(n^{-3}\eta^2(n)\right),
\end{aligned}
$$

for $a_2' \neq 0$ and $b_2' = 0$,

$$
\begin{aligned}
\lambda_n^{1/2} &= \frac{(2n+3)}{2}+\frac{2}{(2n+3)\pi}\left\{\frac{a_2'b_2-a_1'b_1'}{a_2'b_1'}\right.\\
&\quad -\frac{1}{8(2n+3)^5}\left[8(4n^2+12n+3)(-1)^{n+1}-\pi(2n+3)\left\{4(\pi^2+4)n(n+3)+9\pi^2+12\right\}\right]\\
&\quad +\frac{4}{(2n+3)^2}\left[\frac{3b_1'b_1b_2-b_2^3}{3\left(b_1'\right)^3}\right]+\frac{4}{(2n+3)^2}\left[\frac{-3a_1'a_2'a_2+3a_1\left(a_2'\right)^2+\left(a_1'\right)^3}{3\left(a_2'\right)^3}\right]\\
&\quad -\left.\frac{4}{(2n+3)^2}\frac{a_1'}{a_2'}\left[\frac{\pi^4}{80}+\frac{\pi^2}{12}\right]\right\}+O\left(n^{-4}\eta(n)\right)+O\left(n^{-3}\eta^2(n)\right),
\end{aligned}
$$

for $a_2' = 0$ and $b_2' \neq 0$,

$$
\begin{aligned}
\lambda_n^{1/2} &= \frac{(2n+3)}{2}+\frac{2}{(2n+3)\pi}\left\{\frac{a_1'b_1'-a_2b_2'}{a_1'b_2'}\right.\\
&\quad -\frac{1}{8(2n+3)^5}\left[8(4n^2+12n+3)(-1)^{n+1}-\pi(2n+3)\left\{4(\pi^2+4)n(n+3)+9\pi^2+12\right\}\right]\\
&\quad -\frac{4}{(2n+3)^2}\left[\frac{3b_1\left(b_2'\right)^2-3b_1'b_2'b_2+\left(b_1'\right)^3}{3\left(b_2'\right)^3}\right]+\frac{4}{(2n+3)^2}\left[\frac{-3a_1'a_1a_2+a_2^3}{3\left(a_1'\right)^3}\right]\\
&\quad +\left.\frac{4}{(2n+3)^2}\frac{a_2}{a_1'}\left[\frac{\pi^4}{80}+\frac{\pi^2}{12}\right]\right\}+O\left(n^{-4}\eta(n)\right)+O\left(n^{-3}\eta^2(n)\right),
\end{aligned}
$$

for $a_2' = 0$ and $b_2' = 0$,

$$
\begin{aligned}
\lambda_n^{1/2} &= n+2+\frac{1}{(n+2)\pi}\left\{\frac{a_1'b_2-a_2b_1'}{a_1'b_1'}-\frac{\pi}{32\,(n+2)^4}\left[(\pi^2+4)n(n+4)+4\pi^2+10\right]\right.\\
&\quad -\frac{1}{(n+2)^2}\left[\frac{3b_1'b_1b_2-b_2^3}{3\left(b_1'\right)^3}\right]+\left.\frac{1}{(n+2)^2}\left[\frac{-3a_1'a_1a_2+a_2^3}{3\left(a_1'\right)^3}\right]\right\}+O\left(n^{-4}\eta(n)\right)+O\left(n^{-3}\eta^2(n)\right).
\end{aligned}
$$

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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