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$\mathcal{L}$-STABLE RINGS

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Abstract. If $\mathcal{L}(R)$ is a set of left ideals defined in any ring R, we say that R is $\mathcal{L}$-stable if it has stable range 1 relative to the set $\mathcal{L}(R)$. We explore $\mathcal{L}$-stability in general, characterize when it passes to related classes of rings, and explore which classes of rings are $\mathcal{L}$-stable for some $\mathcal{L}$. Some well known examples of $\mathcal{L}$-stable rings are presented, and we show that the Dedekind finite rings are $\mathcal{L}$-stable for a suitable $\mathcal{L}$.

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1. Introduction

A ring R is said to have stable range 1 if, for any $a \in R$ and left ideal $L \subseteq R$, $Ra + L = R$ implies $a - u \in L$ for some unit u in R. Here we insist only that this holds for all L in some prescribed set $\mathcal{L}(R)$ of left ideals of R, and say that R is $\mathcal{L}$-stable in this case. Hence Bass' rings of stable range 1 arise if $\mathcal{L}(R)$ is the set of all left ideals in R, and it is known that Kaplansky's uniquely generated rings and Ehrlich's rings with internal cancellation arise in this way for other choices of $\mathcal{L}$. In this paper, we explore $\mathcal{L}$-stability in general, derive some properties of this phenomenon, show that it captures many well known results, and characterize when $\mathcal{L}$-stability passes to related rings. This in turn yields new information about left uniquely generated and internally cancellable rings. More importantly, we show that the Dedekind finite rings also arise as the set of $\mathcal{L}$-stable rings for a suitable $\mathcal{L}$, which gives a new perspective on these rings.

Throughout this paper R always denotes an associative ring with unity $1 \neq 0$. We write $J(R)$, $U(R)$ and $M_n(R)$ respectively for the Jacobson radical of R, the unit group of R and the ring of $n \times n$ matrices over R. The notation $A \lhd R$ signifies that A is an ideal of R, and left and right annihilators of a set $X \subseteq R$ are written respectively as $\mathtt{l}(X)$ and $\mathtt{r}(X)$. We denote the ring of integers by $\mathbb{Z}$ and write $\mathbb{Z}_n$

for the ring of integers modulo n. The term "regular ring" means von Neumann regular ring. If M is an R-module then $N \leq M$, $N \subseteq^{ess} M$ and $N \subseteq^{\oplus} M$ signify, respectively, that N is a submodule, an essential submodule, and a direct summand of M. The class of local rings will be denoted by {local}, with similar notation for other classes. If each element of a set $X \subseteq R$ has a ring property $\mathfrak{p}$, we say X has $\mathfrak{p}$.

2. Left idealtors

We are interested in rings R of *stable range 1 (SR1)*, that is every element $a \in R$ is *SR1* in the sense that a satisfies the following condition

$Ra + L = R$, L *a left ideal of* R, *implies that* $a - u \in L$ *for some unit* u *of* R.[1]

The concept of a stable range originated in 1964 in Bass [2, §4]. [2] In this paper we restrict the choice of the left ideal L. To that end we make:

Definition 2.1. A *left-ideal-map* $\mathcal{L}$ is a function that associates to every ring Ra well-defined non-empty set $\mathcal{L}(R)$ of left ideals of R.

For reasons that will appear shortly, the left-ideal-maps we are interested in must have the property that every ring isomorphism is $\mathcal{L}$-fit and $\mathcal{L}$-full, where these notions are defined (for any onto ring morphism) as follows.

Definition 2.2. Let $\mathcal{L}$ be a left-ideal-map, and let $\theta : R \to S$ be an onto ring morphism.

(1) θ is called $\mathcal{L}$-*fit* if $L \in \mathcal{L}(R)$ implies $\theta(L) \in \mathcal{L}(S)$. [3]

(2) θ is called $\mathcal{L}$-*full* if $X \in \mathcal{L}(S)$ implies $X = \theta(L)$ for some $L \in \mathcal{L}(R)$.

Lemma 2.3. *Let* $\mathcal{L}$ *be any left-ideal-map. The following are equivalent.*

(1) *Every ring isomorphism is* $\mathcal{L}$*-fit.*

(2) *Every ring isomorphism is* $\mathcal{L}$*-full.*

Proof. Let $\sigma : R \to S$ be a ring isomorphism.

(1)⇒(2). If $X \in \mathcal{L}(S)$ then $\sigma^{-1}(X) \in \mathcal{L}(R)$ by (1). So $X = \sigma(L)$ where $L = \sigma^{-1}(X) \in \mathcal{L}(R)$.

(2)⇒(1). Let $L \in \mathcal{L}(R)$. By (2), $L = \sigma^{-1}(X)$ for some $X \in \mathcal{L}(S)$, so $\sigma(L) = X \in \mathcal{L}(S)$. □

[1]We need not say "left" SR1 because Vaserstein [19, Theorem 2.1] proved that this condition is left-right symmetric.

[2]The term "stable range 1" was first used by Lam [13]

[3]Note that our assumption that θ is onto guarantees that $\theta(L)$ is in fact a left ideal of S.

Definition 2.4. A left-ideal-map $\mathcal{L}$ is *natural* if the conditions in Lemma 2.3 hold. In this case we call $\mathcal{L}$ a left *idealtor.*

With this we can define the main concepts arising in this paper.

Definition 2.5. Let $\mathcal{L}$ *be a left -idealtor.* An element $a \in R$ *is* $\mathcal{L}$*-stable* if $Ra + L = R$, $L \in \mathcal{L}(R)$, implies that $a - u \in L$ for some unit u of R. Then R is an $\mathcal{L}$ *stable ring* if each element of R is $\mathcal{L}$*-stable.* A class $\mathfrak{C}$ of rings is *afforded* by an idealtor $\mathcal{L}$ if $\mathfrak{C} = \{\mathcal{L}\text{-stable}\}$*—the class of all* $\mathcal{L}$*-stable rings, and* $\mathfrak{C}$ is *affordable* if this happens for some idealtor $\mathcal{L}$.

Thus the SR1 rings are afforded by the left idealtor $\mathcal{B}$ where $\mathcal{B}(R) = \{L \mid L \leq {}_RR\}$[4] is the class of *all* left ideals of R.

As we shall see, there are other important affordable classes of rings, and the aim of this paper is to describe some of these and use this idealtor framework to study the properties of the various stability-classes and the relations between them.

We are insisting that every idealtor $\mathcal{L}$ is natural, that is $\mathcal{L}$ has the property that all ring isomorphisms are both $\mathcal{L}$-fit and $\mathcal{L}$-full. The reason for this is because otherwise $\mathcal{L}$-stability may not be preserved under ring isomorphisms. To construct an example where this happens, recall that a ring R is *Dedekind finite* (*DF*)[5] if $ba = 1$ in R implies $ab = 1$. Given a division ring D, define
$M_\omega(D) = end({}_DV)$ where ${}_DV$ is a vector space with basis $\{v_0, v_1, v_2, \ldots\}$.
It is well known that $M_\omega(D)$ is not DF (consider the "shift" operator $v_i \mapsto v_{i+1}$).

Example 2.6. Given a division ring D, let $E = M_\omega(D)$ and let $S = E \times O$ where O is the zero ring. With this define a left-ideal-map $\mathcal{X}$

$$\mathcal{X}(E) = \{E\} \qquad \text{and} \qquad \mathcal{X}(R) = \{0\} \text{ for any ring } R \neq E.$$

Then $E \cong S$ as rings, E is $\mathcal{X}$-stable, but S is not $\mathcal{X}$-stable.

Proof. First $E \cong S$ as rings via $\alpha \mapsto (\alpha, 0)$ for $\alpha \in E$. To see that E is $\mathcal{L}$-stable, assume that $E\alpha + L = E$, $\alpha \in E$, $L \in \mathcal{X}(E)$. Since $\mathcal{X}(E) = \{E\}$ we have $L = E$ so $\alpha - 1 \in L$, as required.

To see that S is not $\mathcal{X}$-stable, we show that *if* S is $\mathcal{X}$-stable then S is Dedekind finite (a contradiction as $S \cong E$). So let $ba = 1$ in S. Then $Sa + 0 = S$ and $0 \in \mathcal{X}(S)$ as $S \neq E$. If S is $\mathcal{X}$-stable this implies $a - u \in 0$ where $u \in U(R)$. Thus a is a unit so, as $ab = 1$, we get $ba = 1$. □

[4]We abuse notation as in calculus where one speaks of the function $f(x) = x^2 + 1$.

[5]Also called directly finite, or von Neumann finite.

Note that the left-ideal-map $\mathcal{X}$ in Example 2.6 is not natural because $E \mapsto R \notin \mathcal{X}(R)$. Hence $\mathcal{X}$ is not a left idealtor. However:

Proposition 2.7. *Let $\mathcal{L}$ be any left idealtor. If $\sigma : R \to S$ is a ring isomorphism, then R is $\mathcal{L}$-stable if and only if S is $\mathcal{L}$-stable.*

Proof. Let R be $\mathcal{L}$-stable. To show that S is $\mathcal{L}$-stable, let $Sb + X = S$, $X \in \mathcal{L}(S)$, $b \in S$. Apply σ^{-1} to get $R\sigma^{-1}(b) + \sigma^{-1}(X) = R$. But $\sigma^{-1}(X) \in \mathcal{L}(S)$ because σ^{-1} is $\mathcal{L}$-fit by hypothesis, so the fact that R is $\mathcal{L}$-stable shows that $\sigma^{-1}(b) - u \in \sigma^{-1}(X)$ where $u \in U(R)$. Applying σ shows that $b - \sigma(u) \in X$. Since $\sigma(u) \in U(S)$, this proves that S is $\mathcal{L}$-stable. The converse is analogous. □

If $\mathcal{L}$ is a left idealtor and $\theta : R \to S$ is any onto ring morphism, we regard θ as a map

$$\theta : \mathcal{L}(R) \to \mathcal{L}(S) \qquad \text{where} \qquad L \mapsto \theta(L)$$

Clearly:
(1) θ is $\mathcal{L}$-fit if and only if $\theta[\mathcal{L}(R)] \subseteq \mathcal{L}(S)$.
(2) θ is $\mathcal{L}$-full if and only if $\mathcal{L}(S) \subseteq \theta[\mathcal{L}(R)]$.
(3) θ is $\mathcal{L}$-fit and $\mathcal{L}$-full if and only if $\theta[\mathcal{L}(R)] = \mathcal{L}(S)$.

Proposition 2.8. *Let $\mathcal{L}$ be a left idealtor, and let $\sigma : R \to S$ be a ring isomorphism. Then*

(1) *$|\mathcal{L}(R)| = |\mathcal{L}(S)|$ via the bijection $L \mapsto \sigma(L)$ from $\mathcal{L}(R) \to \mathcal{L}(S)$.*
(2) *$\mathcal{L}(S) = \{\sigma(L) \mid L \in \mathcal{L}(R)\}$.*

Proof. Because $\mathcal{L}$ is natural, σ is $\mathcal{L}$-fit so $L \mapsto \sigma(L)$ defines a map $\mathcal{L}(R) \to \mathcal{L}(S)$. Similarly $X \mapsto \sigma^{-1}(X)$ carries $\mathcal{L}(S) \to \mathcal{L}(R)$. As these maps are mutually inverse, (1) and (2) follow. □

We close this section by listing some useful facts about when onto ring morphisms are full or fit. The routine proofs are omitted.

Lemma 2.9. *Let $\mathcal{L}$ be a left idealtor, and let φ and θ denote onto ring morphisms.*

(1) *If φ and θ are $\mathcal{L}$-fit ($\mathcal{L}$-full) so also is their composite $\varphi \circ \theta$.*
(2) *If σ, τ are ring isomorphisms then $\theta \circ \sigma$ (respectively $\tau \circ \theta$) is $\mathcal{L}$-fit ($\mathcal{L}$ -full) if and only if the same is true for θ.*
(3) *$\theta : R \to S$ is $\mathcal{L}$-fit ($\mathcal{L}$-full) if and only if the same is true of the coset map $R \to R/ker(\theta)$.*

3. Some affordable classes of rings

The motivating example for the study of $\mathcal{L}$-stability is Bass' class of rings with stable range 1, afforded by the left idealtor $\mathcal{B}(R) = \{L \mid L \leq_R R\}$. Here are two more well known examples.

For the second example, a ring R is called left *uniquely generated* (left *UG*) if

$$Ra = Rb,\ a, b \in R, \text{ implies that } a = ub \text{ for some unit } u \text{ of } R.$$

These rings were introduced in 1949 by Kaplansky [11] in his classic paper on elementary divisors. He showed that rings whose right zero-divisors are in the Jacobson radical (for example local rings) are left UG rings, and gave an example of a commutative ring that was not UG. In 1995 Canfell [4] proved the following result (we include a proof for completeness).

Example 3.1. The class of all left UG rings is afforded by the left idealtor $\mathcal{K}(R) = \{\mathtt{l}(b) \mid b \in R\}$.[6]

Proof. Let R be left UG. If $Ra + \mathtt{l}(b) = R$ then $Rab = Rb$ so, as R is left UG, $ab = ub$ with $u \in U(R)$. Hence $a - u \in \mathtt{l}(b)$, so R is $\mathcal{K}$-stable. Conversely, if R is $\mathcal{K}$-stable and $Ra = Rb$, write $a = pb$ and $b = qa$ where $p, q \in R$. Then $b = qpb$, so $1 - qp \in \mathtt{l}(b)$ and we have $Rp + \mathtt{l}(b) = R$. Since p is $\mathcal{K}$-stable we have $p - u \in \mathtt{l}(b)$ for some $u \in U(R)$, so $pb = ub$, that is, $a = ub$. □

In preparation for the third example, an element $a \in R$ is called *regular* (*unit-regular*) if $a = aba$ for some element (some unit) $b \in R$. It is easy to check that a is unit-regular if and only if a is the product of a unit and an idempotent (in either order). The following result appears in [12, Theorem 3.2]; we have a much shorter proof.

Lemma 3.2. *Every unit-regular element is SR1.*

Proof. Let $a \in R$ be unit-regular and assume $Ra + L = R$, $L \leq_R R$. Write $a = ue$ where $u \in U(R)$ and $e^2 = e$. Then $Ra = Re$ so $Re + L = R$, say $re + x = 1$, $r \in R$, $x \in L$. Define $v = 1 - (1 - e)re$ so $v \in U(R)$. Then

$$e - v = e - 1 + (1 - e)re = (1 - e)(-1 + re) = (1 - e)(-x) \in L.$$

Finally, $a - uv = ue - uv = u(e - v) \in uL \subseteq L$ where $uv \in U(R)$, as required. □

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[6] $\mathcal{K}$ is natural because, if $\sigma : R \to S$ is an isomorphism then $\sigma[\mathtt{l}_R(b)] = \mathtt{l}_S[\sigma(b)]$.

A module M is said to have *internal cancellation* if, whenever $M = K \oplus N = K' \oplus N'$ as modules where $K \cong K'$, then necessarily $N \cong N'$. In 1976 Ehrlich [7] proved:[7]

Proposition 3.3. (**Ehrlich's Theorem**) *For a ring R, ${}_RR$ has internal cancellation if and only if every regular element of R is unit-regular.*

Thus "internal cancellation" is right-left symmetric condition and, in 2005, Khurana and Lam [12] called these rings *IC rings* and gave a detailed survey of them. Earlier, in 2002, H. Chen [5] proved the following result, which is our third example. Again we supply a short proof.

Example 3.4. The IC rings are afforded by the left idealtor $\mathcal{E}(R) = \{Re \mid e^2 = e \in R\}$.

Proof. Suppose R is left $\mathcal{E}$-stable. To see that R is IC, let $a \in R$ be regular, say $a = aba$, and write $e = ba$. Then $e^2 = e$ and $Ra = Re$, so $Ra + R(1-e) = R$. As a is $\mathcal{E}$-stable we have $a - u \in R(1-e)$ for some $u \in U(R)$. Hence $ue = ae = a$, so $a(u^{-1}a) = ae = a$, as required.

Conversely, if R is IC, let $Ra + Re = R$, $e^2 = e$, say $ra + se = 1$. We must show that $a - u \in Re$ for some $u \in U(R)$. Write $\bar{e} = 1 - e$. As $ra + se = 1$ we have $ra\bar{e} = \bar{e}$, so $a\bar{e}(ra\bar{e}) = a\bar{e}^2 = a\bar{e}$. Thus $a\bar{e}$ is regular, hence unit-regular (by hypothesis), whence SR1 (by Lemma 3.2). But $ra\bar{e} = \bar{e} = 1 - e$, so $Ra\bar{e} + Re = R$. As $a\bar{e}$ is SR1 it follows that $a\bar{e} - u \in Re$ for some $u \in U(R)$. Finally $a\bar{e} = a - ae$ so $a - u = (a\bar{e} + ae) - u = (a\bar{e} - u) + ae \in Re$, as required. Finally, $\mathcal{E}$ is clearly natural. □

We now turn to a new fourth example of an important class of rings that is affordable, and the following proposition will be used frequently. Note that a ring R is DF if and only if $Ra = R$, $a \in R$, implies $aR = R$.

Proposition 3.5. *Assume a left idealtor $\mathcal{L}$ satisfies the following condition:*

For each ring R, there exists $C \in \mathcal{L}(R)$ such that $C \subseteq J(R)$. (**i**)

Then $\{\mathcal{L}$-stable$\} \subseteq \{DF\}$.[8] *The reverse inclusion may fail.*

Proof. Let R be $\mathcal{L}$-stable. By (i) choose $C \in \mathcal{L}(R)$ where $C \subseteq J(R)$. To prove R is DF, let $Ra = R$. Then certainly $Ra + C = R$ so, as R is $\mathcal{L}$-stable, let $a - u \in C$,

[7] See also [12, Page 204]

[8] As we are assuming, this asserts that the class of all $\mathcal{L}$-stable rings equals the class of DF rings.

$u \in U(R)$. But if we write $a - u = c$ then $a = u + c$ is a unit too as $c \in C \subseteq J(R)$. Hence $aR = R$, proving that R is DF.

The reverse inclusion may fail because the left idealtor $\mathcal{B}(R) = \{L \mid L \leq_R R\}$ satisfies (i) and $\mathbb{Z}$ is DF, but $2 \in \mathbb{Z}$ is not $\mathcal{B}$-stable ($2\mathbb{Z} + 5\mathbb{Z} = \mathbb{Z}$ but $2 - u \notin 5\mathbb{Z}$ for any $u \in U(\mathbb{Z})$). □

Corollary 3.6. *If R is SR1, left UG or IC, then R is Dedekind finite* (*by Proposition 3.5*).

Theorem 3.7. *The Dedekind finite rings are afforded by the left idealtor $\mathcal{D}(R) = \{L \mid L \subseteq J(R)\}$.*

Proof. We have $\{\mathcal{D}\text{-stable}\} \subseteq \{\text{DF}\}$ by Proposition 3.5. Conversely, assume R is DF and let $Ra + L = R$, $L \in \mathcal{D}(R)$. Thus $L \subseteq J(R)$, and so $Ra = R$. But then a is a unit (R is DF), so $a - u \in L$ where $u = a \in U(R)$. So R is $\mathcal{D}$-stable. It is clear that $\mathcal{D}$ is natural. □

In fact, there are several left idealtors that afford the DF rings. To see this the following Lemma is useful (and will be required several times below).

Lemma 3.8. *Let $\mathcal{M}$ and $\mathcal{L}$ be left idealtors.*

If $\mathcal{M}(R) \supseteq \mathcal{L}(R)$ for all rings R, then $\{\mathcal{M}\text{-stable}\} \subseteq \{\mathcal{L}\text{-stable}\}$.

In particular, $\{\text{SR1}\} \subseteq \{\mathcal{L}\text{-stable}\}$ for every left idealtor $\mathcal{L}$.

Proof. Let R be $\mathcal{M}$-stable, and suppose $Ra + L = R$, $L \in \mathcal{L}(R)$. By hypothesis $L \in \mathcal{M}(R)$ so, as a is $\mathcal{M}$-stable, we have $a - u \in L$ for some unit u. This proves that a is $\mathcal{L}$-stable. The last statement holds because $\mathcal{B}(R) \supseteq \mathcal{L}(R)$ for each $\mathcal{L}$, and $\mathcal{B}$ affords the SR1 rings. □

Proposition 3.9. *In addition to $\mathcal{D}$, the Dedekind finite rings are afforded by the left idealtors*:

$$\mathcal{D}_n(R) = \{L \leq\ _R R \mid L \textit{ nil}\}, \qquad \mathcal{D}_J(R) = \{J(R)\}, \quad \textit{and} \quad \mathcal{D}_0(R) = \{0\}.$$

Proof. These left idealtors are all natural. For any ring R we have $\mathcal{D}(R) \supseteq \mathcal{D}_i(R)$ for $i = 0$, J, n. With Lemma 3.8 and Proposition 3.5 this gives:

$$\{\mathcal{D}\text{-stable}\} \subseteq \{\mathcal{D}_i\text{-stable}\} \subseteq \{\text{DF}\} \quad \text{for each } i.$$

Since $\{\text{DF}\} = \{\mathcal{D}\text{-stable}\}$ by Theorem 3.7, we are done. □

The idealtors $\mathcal{D}$, $\mathcal{D}_n$, $\mathcal{D}_J$ and $\mathcal{D}_0$ all satisfy condition (**i**) in Proposition 3.5, so we ask:

Question 1. If a left idealtor $\mathcal{L}$ affords the DF rings, must (i) in Proposition 3.5 hold for $\mathcal{L}$?

Example 3.10. We have $\{\text{SR1}\} \subset \{\text{left UG}\} \subset \{\text{IC}\} \subset \{\text{DF}\}$.

Proof. We have $\{\text{SR1}\} \subseteq \{\text{left UG}\} \subseteq \{\text{IC}\}$ by Lemma 3.8 using $\mathcal{B}(R) \supseteq \mathcal{K}(R) \supseteq \mathcal{E}(R)$, and all these classes are in $\{\text{DF}\}$ by Corollary 3.6. Each inclusion is strict because, respectively:

(1) $\mathbb{Z}$ is UG, but it is not SR1 because $2\mathbb{Z} + 5\mathbb{Z} = \mathbb{Z}$ but $2 - u \notin 5\mathbb{Z}$ for every $u \in U(\mathbb{Z})$.
(2) (Kaplansky's example [11, Page 466]) $K_5 = \{(n, \lambda) \in \mathbb{Z} \times \mathbb{Z}_5[x] \mid \lambda(\bar{0}) = \bar{n}\}$,where $\bar{k} = k + 5\mathbb{Z}$ in $\mathbb{Z}_5$, is IC by [12, Examples 2.1(4)] (being commutative). But K_5 is not left UG because $(0, x)$ and $(0, \bar{2}x)$ are not unit multiples.
(3) George Bergman has an example of a regular DF ring that is not unit-regular, see [9, Example 5.10], and so is not IC by Ehrlich's theorem.

□

Note that the left idealtor $\mathcal{B}$ satisfies (**i**) but does not afford the DF rings because of Bergman's example—because $\{\text{left UG}\} \subset \{\text{DF}\}$ in Example 3.10.

The conditions SR1, IC and DF are left-right symmetric, so the following (open) question arises.

Question 2. Does $\{\text{left UG}\} \subseteq \{\text{right UG}\}$?

4. Morphisms

If $\sigma : R \to S$ is a ring isomorphism, Proposition 2.7 shows that R is $\mathcal{L}$-stable implies S is $\mathcal{L}$-stable for any left idealtor $\mathcal{L}$. The next result describes the situation for an arbitrary onto ring morphism, stating it first for elements. We say that *units lift* modulo an ideal $A \lhd R$ if $x \in U(R/A)$ implies that $x = u + A$ for some $u \in U(R)$. This holds whenever $A \subseteq J(R)$.

Lemma 4.1. *Let $\mathcal{L}$ be a left idealtor, and let $\theta : R \to S$ be an onto ring morphism. Then for any element $a \in R$ we have*

(1) *If θ is $\mathcal{L}$-fit, then $\theta(a)$ is $\mathcal{L}$-stable in S, then a is $\mathcal{L}$-stable in R provided either*
(a) *$ker(\theta) \subseteq J(R)$ or* (b) *units lift modulo $ker(\theta)$ and $ker(\theta) \subseteq L$ for all $L \in \mathcal{L}(R)$.*

(2) *If θ is $\mathcal{L}$-full, then a is $\mathcal{L}$-stable in R, then $\theta(a)$ is $\mathcal{L}$-stable in S provided either*
(a) *$ker(\theta) \subseteq J(R)$ or* (c) *$L + ker(\theta) \in \mathcal{L}(R)$ for all $L \in \mathcal{L}(R)$.*

Proof. For clarity write $\theta(r) = \bar{r}$ when $r \in R$, and write $ker(\theta) = A$.

(1). Assume $\bar{a}$ is $\mathcal{L}$-stable in $S = \bar{R}$. Let $Ra + L = R$, $L \in \mathcal{L}(R)$, say $ra + l = 1$ where $r \in R$ and $l \in L$. Then $\bar{r}\bar{a} + \bar{l} = \bar{1}$, so $\bar{R}\bar{a} + \bar{L} = \bar{R}$. Here $\bar{L} \in \mathcal{L}(S)$ because θ is $\mathcal{L}$-fit, and $\bar{a}$ is $\mathcal{L}$-stable in S by hypothesis. So we have $\bar{a} - \bar{u} \in \bar{L}$ for some $\bar{u} \in U(S)$. Hence

$$a - u \in L + A \quad \text{where } \bar{u} \in U(S). \qquad (ii)$$

(a). By (ii) let $a - u - l \in A$ where $l \in L$. Writing $c = a - u - l$ we have $a - (u + c) = l \in L$. Moreover, $u + c \in U(R)$ because $c \in A \subseteq J(R)$ by (a). Hence a is $\mathcal{L}$-stable, proving (1) in this case.
(b). Now (ii) gives $a - u \in L + A = L$ and, as $\bar{u} \in U(S)$, we may assume $u \in U(R)$ again by (b). This proves (1) in this case.

(2). Assume a is $\mathcal{L}$-stable in R. Let $S\bar{a} + X = S$, $X \in \mathcal{L}(S)$. As θ is $\mathcal{L}$-full write $X = \bar{L}$ where $L \in \mathcal{L}(R)$. Then $S\bar{a} + \bar{L} = S$, say $ra + l - 1 \in A$, $r \in R$, $l \in L$. It follows that $Ra + L + A = R$. This implies $Ra + L = R$ in both cases (a) and (c). But then, as a is $\mathcal{L}$-stable in R, we have $a - u \in L$ where $u \in U(R)$. Hence $\bar{a} - \bar{u} \in \bar{L} = X$ and $\bar{u} \in U(S)$, proving (2). □

Lemma 4.1 deals with particular elements. Here is the corresponding result for rings.

Theorem 4.2. *Let $\theta : R \to S$ be an onto ring morphism and let $\mathcal{L}$ be a left idealtor.*

(1) *If S is $\mathcal{L}$-stable, thenR is $\mathcal{L}$-stable if θ is $\mathcal{L}$-fit and either $ker(\theta) \subseteq J(R)$ or units lift modulo $ker(\theta)$, and $ker(\theta) \subseteq L$ for all $L \in \mathcal{L}(R)$.*
(2) *If R is $\mathcal{L}$-stable, then $\mathcal{L}$ -stable if θ is $\mathcal{L}$-full and either $ker(\theta) \subseteq J(R)$ or $L + ker(\theta) \in \mathcal{L}(R)$ for all $L \in \mathcal{L}(R)$.*

• For SR1 rings it is clear that every onto ring morphism $\theta : R \to S$ is $\mathcal{B}$-fit and $\mathcal{B}$-full. So if R is SR1 then S is SR1 by (2). However, the converse can fail (consider $\mathbb{Z} \to \mathbb{Z}_2$).

• For any left idealtor $\mathcal{L}$ we are assuming that every ring isomorphism is $\mathcal{L}$-fit and $\mathcal{L}$-full. Hence Proposition 2.7 ($R \cong S$ implies R is $\mathcal{L}$-stable $\Leftrightarrow$ S is $\mathcal{L}$-stable) is a consequence of Theorem 4.2.

The simplest and most useful case of Theorem 4.2 is when $ker(\theta) \subseteq J(R)$:

Corollary 4.3. *Let $\theta : R \to S$ be an onto ring morphism, $ker(\theta) \subseteq J(R)$. For any left idealtor $\mathcal{L}$*

(1) *If θ is $\mathcal{L}$-fit, then S is $\mathcal{L}$ -stable implies R is $\mathcal{L}$-stable.*
(2) *If θ is $\mathcal{L}$-full, then R is $\mathcal{L}$ -stable implies S is $\mathcal{L}$-stable.*

• The truth of (1) in Corollary 4.3 does not imply that $ker(\theta) \subseteq J(R)$. For example the ring

$$R = \mathbb{Z}_{(2,3)} = \{\tfrac{n}{d} \in \mathbb{Q} \mid 2 \nmid d \text{ and } 3 \nmid d\}$$

is SR1 by Bass' theorem because it is semilocal (in fact $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ is semismple). Here a ring R is *semilocal* if $R/J(R)$ is semisimple. However, the image $R/3R \cong \mathbb{Z}_2$ is SR1, but $3R \nsubseteq J(R) = 6R$.

Example 4.4. For a ring R write $J = J(R)$. Let $A \lhd R$ and write $\bar{R} = R/A$.

(1) R is SR1 implies $\bar{R}$ is SR1. Conversely if $A \subseteq J$.
(2) Let $A \subseteq J$. If $\bar{R}$ is left UG so also is R provided $b \in R$ implies $\overline{\mathtt{l}_R(b)} = \mathtt{l}_{\bar{R}}(\bar{c})$ for $c \in R$. Let $A \subseteq J$. If R is left UG so also is $\bar{R}$ provided $c \in R$ implies $\mathtt{l}_{\bar{R}}(\bar{c}) = \overline{\mathtt{l}_R(b)}$ for $b \in R$.
(3) Let $A \subseteq J$. If $\bar{R}$ is IC, then R is always IC. Conversely, if idempotents lift modulo A.
(4) Let $A \subseteq J$. If R/A is DF, then R is always DF. The converse fails.

Proof. For a left idealtor $\mathcal{L}$, we say that $A \lhd R$ is $\mathcal{L}$-fit/$\mathcal{L}$-full if the coset map $R \to R/A$ has the same property.

(1). Use the left idealtor $\mathcal{B}(R) = \{L \mid L \leq {}_R R\}$, so every onto ring morphism is $\mathcal{B}$-fit and $\mathcal{B}$-full, and both implications follow by Theorem 4.2. The converse fails if $A \nsubseteq J$ (consider $\mathbb{Z} \to \mathbb{Z}_2$).

(2). Use $\mathcal{K}(R) = \{\mathtt{l}_R(b) \mid b \in R\}$. The provisos assert A is $\mathcal{K}$-fit ($\mathcal{K}$-full), so Corollary 4.3 applies.

(3). Use $\mathcal{E}(R) = \{Re \mid e^2 = e \in R\}$. Then A is always $\mathcal{E}$-fit because, if $e^2 = e$, then $\overline{Re} = \bar{R}\bar{e}$ where $\bar{e}^2 = \bar{e}$. If A is lifting and $\bar{R}\bar{e} \in \mathcal{E}(\bar{R})$, $\bar{e}^2 = \bar{e}$, we may assume that $e^2 = e$. Then $\bar{R}\bar{e} = \overline{Re}$ where $Re \in \mathcal{E}(R)$. Hence A is also $\mathcal{E}$-full. So Corollary 4.3 applies in both cases.

(4). Use $\mathcal{D}(R) = \{L \mid L \subseteq J(R)\}$. Here A is always $\mathcal{D}$-fit: If $L \in \mathcal{D}(R)$ then $L \subseteq J$ so L (and hence) $\bar{L}$ is quasi-regular, whence $\bar{L} \subseteq J(\bar{R})$ that is $\bar{L} \in \mathcal{D}(\bar{R})$. Thus, the first statement follows using Corollary 4.3. The converse fails by considering the map $\mathbb{Z}_2 \times M_\omega(\mathbb{Z}_2) \to \mathbb{Z}_2$. □

The fact that *every* image of an SR1 ring is again SR1 is a special case of the following Lemma.

Lemma 4.5. *Let $\mathcal{L}$ be any left idealtor, let $\theta : R \to S$ be an onto ring morphism, and assume $\theta^{-1}(X) + ker(\theta) \in \mathcal{L}(R)$ for all $X \in \mathcal{L}(S)$. Then S is $\mathcal{L}$-stable if R is $\mathcal{L}$-stable.*

Proof. As before write $\theta(r) = \bar{r} \in \bar{R} = S$. Suppose $\bar{R}\bar{a} + X = \bar{R}$, $X \in \mathcal{L}(S)$. As θ is onto, we have $X = \theta[\theta^{-1}(X)] = \overline{\theta^{-1}(X)}$. It follows that $Ra + \theta^{-1}(X) + ker(\theta) = R$. By hypothesis, there exists $u \in U(R)$ where $a - u \in \theta^{-1}(X) + ker(\theta)$. Thus $\bar{a} - \bar{u} \in X$ and $\bar{u} \in U(S)$, as required. □

Lemma 4.6. *Full Lemma. Let $\theta : R \to S$ be an onto ring morphism. For a left idealtor $\mathcal{L}$*

(1) *If $\theta^{-1}(X) \in \mathcal{L}(R)$ for every $X \in \mathcal{L}(S)$, then θ is $\mathcal{L}$-full.*
(2) *The converse of* (1) *holds if $ker(\theta) \subseteq L$ for all $L \in \mathcal{L}(R)$.*

Proof. (1). As θ is onto, we have $\theta[\theta^{-1}(X)] = X$ for any left ideal X of S.

(2). Assume θ is $\mathcal{L}$-full. If $X \in \mathcal{L}(S)$, write $X = \theta(L)$ for some $L \in \mathcal{L}(R)$. If $r \in \theta^{-1}(X)$ then $\theta(r) \in X = \theta(L)$, say $\theta(r) = \theta(l)$ for some $l \in L$. This means that $r - l \in ker(\theta)$, and it follows that $\theta^{-1}(X) \subseteq L + ker(\theta)$. But $ker(\theta) \subseteq \theta^{-1}(X)$ always holds, and $L \subseteq \theta^{-1}(X)$ because $X \subseteq \theta(L)$, proving that $\theta^{-1}(X) = L + ker(\theta) = L \in \mathcal{L}(R)$ by hypothesis, as required. □

5. Elementary properties

We begin with a look at the set of all $\mathcal{L}$-stable elements in a ring R, where $\mathcal{L}$ is any left idealtor.

Lemma 5.1. *If $\mathcal{L}$ is any left idealtor, the following hold for each ring R.*

(1) *$u^{-1}Lu \in \mathcal{L}(R)$ for any $L \in \mathcal{L}(R)$ and any unit u of R.*
(2) *$Lu \in \mathcal{L}(R)$ for any $L \in \mathcal{L}(R)$ and any unit u of R.*

Proof. If $u \in U(R)$, consider the conjugation isomorphism $\sigma_u : R \to R$ where $\sigma_u(r) = uru^{-1}$ for all $r \in R$. Since $\mathcal{L}$ is natural, σ_u is $\mathcal{L}$-fit, which proves (1). Then (2) follows because $vL = L$ for any unit v and any left ideal L. □

Definition 5.2. If $\mathcal{L}$ s a left idealtor write $S_{\mathcal{L}}(R) = \{a \in R \mid a \text{ is } \mathcal{L}\text{-}stable\}$ for any ring R.

In the SR1 case, the next result appears in [6, Lemma 17], and answers an open question in the left UG case [17, page 2561].

Theorem 5.3. (**Product Theorem**) *For any left idealtor $\mathcal{L}$, $S_{\mathcal{L}}(R)$ is closed under multiplication.*

Proof. If a and d are $\mathcal{L}$-stable we show that da is also $\mathcal{L}$-stable. So let

$$Rda + L = R,\ L \in \mathcal{L}(R),\ \text{say } rda + b = 1,\ r \in R,\ b \in L.$$

Thus $Ra + L = R$ so (as a is $\mathcal{L}$-stable) let $a - u \in L$, for some unit u. Write $c = a - u \in L$. Then $1 = rd(c + u) + b$, so $rdu + (rdc + b) = 1$. Thus

$$rdu + g = 1, \quad \text{where } g = rdc + b \in L \quad (\text{because } c, b \in L).$$

Multiply on the left by u, and then on the right by u^{-1}, to obtain

$$urd + ugu^{-1} = 1, \quad \text{from which} \quad Rd + uLu^{=1} = R.$$

But $uLu^{-1} \in \mathcal{L}(R)$ by Lemma 5.1. So, as d is $\mathcal{L}$-stable, let $d - v \in uLu^{-1}$ where $v \in U(R)$, say $d - v = uhu^{-1}$ where $h \in L$. Thus $du - vu = uh$ so (since $u = a - c$) we obtain

$$da - vu = d(c + u) - (du - uh) = dc + uh \in L \quad \text{because } c, h \in L.$$

As vu is a unit, this shows da is $\mathcal{L}$-stable, as required. □

Example 5.4. Let R be a PID with infinitely many primes but having a finite unit group. Then $S_{\mathcal{B}}(R) = \{0\} \cup U(R)$ where $\mathcal{B}$ is the SR1 left idealtor. In particular $S_{\mathcal{B}}(\mathbb{Z}) = \{0, 1, -1\}$.

Proof. Clearly $\{0\} \cup U(R) \subseteq S_{\mathcal{B}}(R)$. Suppose $a \in S_{\mathcal{B}}(R) \setminus (\{0\} \cup U(R))$. Let p be any prime not dividing a. Then $Ra + Rp = R$ as Rp is maximal. As $a \in S_{\mathcal{B}}(R)$, $a - u \in Rp$ for some $u \in U(R)$, that is $p \mid (a - u)$ for some $u \in U(R)$. If we write $U(R) = \{u_1, u_2, \dots, u_n\}$ this means that $p \mid \Pi_{i=1}^{n}(a - u_i)$, a contradiction as there are infinitely many primes p not dividing a. □

Write $ureg(R)$ for the set of all unit-regular elements in a ring R, and let

$$Z_r(R) = \{z \in R \mid \mathtt{r}(z) \subseteq^{ess} R_R\}$$

denote the *right singular ideal* of R. Recall that a ring is called *left Kasch* if $\mathtt{r}(L) \neq 0$ for all (all maximal) left ideals $L \neq R$ of R, equivalently if every simple left R-module embeds in ${}_RR$.

Clearly a ring R is $\mathcal{L}$-stable if and only if $S_{\mathcal{L}}(R) = R$. The next result indicates how large $S_{\mathcal{L}}(R)$ is in general.

Proposition 5.5. *Let $\mathcal{L}$ be a left idealtor. The following hold for any ring R.*

(1) $J(R) \subseteq S_{\mathcal{L}}(R)$.
(2) $ureg(R) \subseteq S_{\mathcal{L}}(R)$.

(3) $Z_r(R) \subseteq S_{\mathcal{L}}(R)$ *provided* $\mathtt{r}(L) \neq 0$ *whenever* $R \neq L \in \mathcal{L}(R)$–(*say* R *is left Kasch*).

Proof. (1). Let $Ra + L = R$ where $L \in \mathcal{L}(R)$ and $a \in J(R)$. Then $L = R$ so $a - 1 \in L$.

(2). This follows using Lemma 3.2 and Lemma 3.8.

(3). Suppose $Rz + L = R$ where $z \in Z_r(R)$ and $L \in \mathcal{L}(R)$. Taking right annihilators we obtain

$$\mathtt{r}(z) \cap \mathtt{r}(L) = \mathtt{r}(R) = 0, \quad \text{so } \mathtt{r}(L) = 0 \text{ as } z \in Z_r(R).$$

By hypothesis $L = R$, so $a - u \in L$ for *any* $u \in U(R)$. □

This shows immediately that every local and every unit-regular ring is $\mathcal{L}$-stable for any left idealtor $\mathcal{L}$, and so are all SR1 (taking $\mathcal{L} = \mathcal{B}$). It is well known [9, Corollary 4.7] that matrix rings over division rings are unit-regular, so every semisimple ring is $\mathcal{L}$-stable for any left idealtor $\mathcal{L}$. We refer to Proposition 5.5 frequently below.

Proposition 5.6. *Let* $\mathcal{L}$ *be a left idealtor, let* $A \lhd R$, *and assume that the coset map* $\theta : R \to R/A$ *is* $\mathcal{L}$*-full and* $\mathcal{L}$*-fit, that* $A \subseteq J(R)$, *and that the following conditions are satisfied.*

(a) *Units lift modulo* A.

(b) $A \subseteq L$ *for all* $L \in \mathcal{L}(R)$.

(c) $L + A \in \mathcal{L}(R)$ *for all* $L \in \mathcal{L}(R)$.

Then θ *induces the following onto monoid morphism* (*still denoted* θ) :

$$\theta : S_{\mathcal{L}}(R) \to S_{\mathcal{L}}(R/A) \quad \textit{where} \quad \theta(a) = a + A.$$

Moreover $\theta(a) = \theta(b)$ *if and only if* $a - b \in J$.

Proof. If $a \in S_{\mathcal{L}}(R)$ then $a + A \in S_{\mathcal{L}}(\bar{R})$ by Lemma 4.1 using (c) and the fact that θ is $\mathcal{L}$-full. Hence the map $\theta : S_{\mathcal{L}}(R) \to S_{\mathcal{L}}(R/A)$ is well-defined; it clearly preserves multiplication and the unity (so is a monoid morphism), and the last condition is obvious. So it remains to see that θ is onto, and this again follows by Lemma 4.1 using (a), (b), and the fact that θ is $\mathcal{L}$-fit. □

The monoid $S_{\mathcal{L}}(R)$ also has the following "translation" property.

Lemma 5.7. *Let* $\mathcal{L}$ *be a left idealtor. Then* $S_{\mathcal{L}}(R) + J(R) \subseteq S_{\mathcal{L}}(R)$.

Proof. Let $R(r + c) + L = R$, where $r \in R$ is $\mathcal{L}$-stable, $c \in J(R)$ and $L \in \mathcal{L}(R)$. It follows that $Rr + J(R) + L = R$, whence $Rr + L = R$. By hypothesis, let $r - u \in L$ where $u \in U(R)$. Thus $(r + c) - (u + c) \in L$, and $u + c \in U(R)$ because $c \in J(R)$. This proves the Lemma. □

For a left idealtor $\mathcal{L}$, a ring R is $\mathcal{L}$-stable if and only if $S_{\mathcal{L}}(R) = R$. So $S_{\mathcal{L}}(R)$ is a subring of R in this case. If $a \in S_{\mathcal{L}}(R)$ then $-a = (-1)a \in S_{\mathcal{L}}(R)$ by Theorem 5.3, so $S_{\mathcal{L}}(R)$ is a subring of R if and only if it is closed under addition. This suggests the following question:

Question 3. When is $S_{\mathcal{L}}(R)$ closed under addition? By Example 5.4 *the answer is "no" for $R = \mathbb{Z}$ in the* SR1 *case.*

The class of SR1 rings is affordable and plays an important role among such classes.

Theorem 5.8. (1) *If $\mathfrak{C}$ is an affordable class of rings then* $\{\text{SR1}\} \subseteq \mathfrak{C}$.
(2) *If $\mathfrak{C}$ is affordable and $\mathfrak{C} \subseteq \{\text{SR1}\}$ then $\mathfrak{C} = \{\text{SR1}\}$.*

Proof. (1). The left idealtor $\mathcal{B}(R) = \{L \mid L \leq_R R\}$ affords the SR1 rings. Suppose $\mathfrak{C}$ of is afforded by a left idealtor $\mathcal{L}$. Then $\mathcal{B}(R) \supseteq \mathcal{L}(R)$ for each ring R because $\mathcal{L}(R)$ *consists* of left ideals. Hence $\{\mathcal{B}\text{-stable}\} \subseteq \{\mathcal{L}\text{-stable}\}$ by Lemma 3.8. In other words $\{\text{SR1}\} \subseteq \mathfrak{C}$.

(2). Let $\mathfrak{C} \subseteq \{\text{SR1}\}$ where $\mathfrak{C}$ is affordable. By (1), $\{\text{SR1}\} \subseteq \mathfrak{C}$ too. □

Theorem 5.8 makes it easy to identify when a class $\mathfrak{C}$ of rings is *not* affordable:

Simply find an SR1 ring that is not in $\mathfrak{C}$.

For example, $M_2(\mathbb{R})$ is SR1 [19, Corollary 2.9] but it is neither commutative nor semilocal. So {commutative} and {semilocal} are *not* affordable classes of rings.

A ring R is an *exchange* ring if, for all $a \in R$, there exists $e^2 = e \in Ra$ with $1 - e \in R(1 - a)$, equivalently if $R = A + B$ where A and B are left ideals, then there exists $e^2 = e \in A$ with $1 - e \in B$. This condition is left-right symmetric ([20] or [16]). A ring is exchange if and only every left (right) ideal L is *lifting*, that is if idempotents lift modulo L [15, Corollary 1.3]. Hence the semilocal ring

$$R = \mathbb{Z}_{(2,3)} = \{\tfrac{n}{d} \in \mathbb{Q} \mid 2 \nmid d \text{ and } 3 \nmid d\}$$

is SR1, but it is not exchange as $J(\mathbb{Z}_{(2,3)})$ is not lifting. Thus {exchange} is not affordable.

A ring R is called *semiregular*, *semiperfect*, *local*, if $J(R)$ is lifting and $R/J(R)$ is regular, semisimple and a division ring, respectively. And R is *prime* (*semiprime*) if, for ideals A and B, $AB = 0$ implies $A = 0$ or $B = 0$ ($A^k = 0$, $k \geq 1$ implies $A = 0$). Finally R is left *P-injective (*left *mininjective)* [18] if every R-linear map $L \to {}_RR$, $L \leq {}_RR$, extends to R where L is any principal (respectively simple) left ideal.

Remark 5.9. If $\mathfrak{C} \subseteq \mathfrak{D}$ are classes of rings and $\mathfrak{D}$ is affordable then $\mathfrak{C}$ need not be affordable. For an example, {Dedekind finite} is affordable (Theorem 3.7) but the subclass of commutative rings is not.

Example 5.10. None of the following classes of rings are affordable:

(1) {commutative}, {semilocal}.
(2) Any class $\mathfrak{C}$ of rings in which $J(R)$ is lifting for each $R \in \mathfrak{C}$.
(3) {exchange}, {semiperfect}, {semiregular}, {local}, {unit-regular}.
(4) {prime}, {semiprime}, {left mininjective}, {left P-injective}, {left Kasch}.

Proof. (1) is proved above; (2) follows by the above argument that the exchange rings are not affordable; and (2)⇒(3) because each of these classes consists of exchange rings [15, Proposition 1.6 and Theorem 2.1], so $J(R)$ (indeed every left ideal) is lifting. Finally, the ring $R = \begin{bmatrix} D & D \\ & D \end{bmatrix}$, D a division ring, is SR1 by Example 4.4(1) but enjoys none of the properties in (4)—see [18]. □

A celebrated theorem of Bass [2] asserts that every semilocal ring is SR1. If R is semilocal then $R/J(R)$ is semisimple and so is a finite product of matrix rings over division rings. Such a product is unit-regular [9, Corollary 4.7], which leads to:

Definition 5.11. Call a ring R *casilocal* if $R/J(R)$ is unit-regular.[9]

Hence every semilocal ring is casilocal, but the converse fails as unit-regular rings exist that are not semisimple [9, Example 5.15]. However we have:

Theorem 5.12. (**Extended Bass Theorem**) *Every casilocal ring is* SR1. *The converse fails.*

Proof. Let R be casilocal. Then $R/J(R)$ is unit-regular, and hence is SR1 by Lemma 3.2 (or Proposition 5.5). Then R is SR1 by Example 4.4(1). The converse fails by an example of Estes and Ohm [8, Theorem 4.4] of a commutative SR1 domain that is not a division ring (and so not unit-regular). □

We know that {casilocal} ⊂ {SR1}; we also have {casilocal} ⊂ {IC}. To prove this we need the following result of H. Chen (private comunication); see also [17, Lemma 24].

Lemma 5.13. *Let $a \in R$ where R is a ring. If $a \in R$ is regular in R and $a + J(R)$ is unit-regular in $R/J(R)$, then a is unit-regular in R.*

[9] "casi" is spanish for "almost".

Proof. If $a + J(R)$ is unit-regular then it is SR1 (by Lemma 3.2 or Proposition 5.5), and so is $\mathcal{B}$-stable. As every ring morphism is $\mathcal{B}$-fit, Lemma 4.1 shows that a is SR1 too.

But we are also assuming that a is regular, say $axa = a$, $x \in R$, where we may assume $xax = x$. Then $1 - xa \in \mathtt{l}(x) = \mathtt{l}(xa)$, so $R = Ra + \mathtt{l}(xa)$. As a is left SR1 we obtain $a - u \in \mathtt{l}(xa) = \mathtt{l}(x)$ for some unit u in R. Hence $ax = ux$, so $a = axa = uxa$. Thus $u^{-1}a = xa$, so $au^{-1}a = axa = a$. This proves that a is unit-regular in R. □

Theorem 5.14. *Every casilocal ring is IC. The converse is false.*

Proof. Every casilocal ring is SR1, and hence IC easily. The converse is false because $\mathbb{Z}$ is IC (being commutative) but not casilocal. □

There is another class of rings that plays a role here. Camillo and Yu [3, page 4743] call a ring R *semi-unit-regular* (*SUR*) if $R/J(R)$ is unit-regular and $J(R)$ is lifting. In other words, R is SUR if and only if R is casilocal and $J(R)$ is lifting.

Example 5.15. Every SUR ring is casilocal; the converse fails. Hence {SUR} is not affordable.

Proof. Clearly, SUR rings are casilocal. The converse fails because the ring

$$R = \mathbb{Z}_{(2,3)} = \{\tfrac{n}{d} \in \mathbb{Q} \mid 2 \nmid d \text{ and } 3 \nmid d\}$$

is casilocal (in fact it is semilocal as $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$), but R is not SUR because $J(R)$ is not lifting. Finally, {SUR} is not affordable by Theorem 5.8 because R is SR1 but not SUR. □

The SUR rings are all SR1 because unit-regular rings are SR1, and they are exchange (being semiregular from the definitions [15, Proposition 1.6]).

Example 5.16. We have the following inclusions for classes of rings:

$$\left.\begin{array}{r}\{\text{semilocal}\} \subset \\ \{\text{SUR}\} \subset\end{array}\right\} \{\text{casilocal}\} \quad \subset \quad \{\text{IC}\} \quad \subset \quad \{\text{DF}\}$$

Proof. We have observed that semilocal implies casilocal, and the other strict inclusions follow by Example 5.15, Theorem 5.14, and Example 3.10. Finally, $\mathbb{Z}_{(2,3)}$ is semilocal but not SUR; while any unit-regular ring that is not semisimple [9, Example 5.15] is SUR but not semilocal. □

For a left idealtor $\mathcal{L}$, an element a in a ring R is $\mathcal{L}$-stable if $Ra+L = R$, $L \in \mathcal{L}(R)$, implies that $a-u \in L$ for some unit u. We now investigate the situation where u is only required to be *left invertible*, that is $Ru = R$. Our starting point is Vaserstein's proof [19, Theorem 2.6] that all left SR1 rings are DF. His argument motivates:

Lemma 5.17. *Fix a left idealtor $\mathcal{L}$ and a ring R. If $a \in R$, the following conditions are equivalent*:

(1) *If $axa = a$, $x \in R$, then $R(1 - xa) \in \mathcal{L}(R)$.*
(2) *If $f^2 = f \in \mathtt{r}(a)$ and $1 - f \in Ra$, then $Rf \in \mathcal{L}(R)$.*

Proof. (1)⇒(2). If f is as in (2), write $1 - f = xa$, $x \in R$. Then $axa = a$, so (1) applies.

(2)⇒(1). If $axa = a$ write $f = 1 - xa$. Then the hypotheses in (2) are satisfied. □

Definition 5.18. For a ring R *and a left idealtor* $\mathcal{L}$, an element $a \in R$ will be called $\mathcal{L}$*-Vaserstein* if the conditions in Lemma 5.17 are satisfied.

If a ring R is SR1, left UG or IC then R is $\mathcal{L}$-Vaserstein using, respectively, the left idealtors:

$\mathcal{B}(R) = \{L \mid L \le R\}$, $\mathcal{K}(R) = \{\mathtt{l}(b) \mid b \in R\}$ and $\mathcal{E}(R) = \{Re \mid e^2 = e \in R)$. (*iii*)

For convenience, if $\mathcal{L}$ is a left idealtor call $a \in R$ "*left*" $\mathcal{L}$-stable if

$Ra + L = R$, $L \in \mathcal{L}(R)$, *implies* $a - x \in R$ *for some* $x \in R$ *with* $Rx = R$.

Lemma 5.19. *Let $a \in R$ be $\mathcal{L}$-Vaserstein, and let $\mathcal{L}$ be any left idealtor. If a is "left" $\mathcal{L}$-stable, then $ab = 1$, $b \in R$ implies $ba = 1$.*

Proof. If $ab = 1$, write $f = 1 - ba$. Then $f = f^2$, $1 - f = ba \in Ra$ and $af = a - aba = 0$. As a is $\mathcal{L}$-Vaserstein, $Rf \in \mathcal{L}(R)$. But $ba + f = 1$ so $Ra + Rf = R$. As a is "left" $\mathcal{L}$-stable, let

$$a - x \in Rf \quad \text{where} \quad Rx = R. \qquad \square$$

Now observe that $fb = b - bab = 0$, so $(a - x)b \in Rf\,b = 0$. Thus $xb = ab = 1$, so x is right invertible too, and hence is a unit. But then b is also a unit (because $xb = 1$), whence a is a unit (because $ab = 1$). It follows that $ba = 1$, as required. Note: In fact $a = b^{-1} = x$. □

In an $\mathcal{L}$-Vaserstein ring, we can weaken the $\mathcal{L}$-stability requirement using Lemma 5.19 and give a condition that the ring is DF.

Theorem 5.20. *Let $\mathcal{L}$ be a left idealtor, and let R be an $\mathcal{L}$-Vaserstein ring. Then*:

(1) *If R is "left" $\mathcal{L}$-stable then R is Dedekind finite.*

(2) *R is $\mathcal{L}$-stable $\Leftrightarrow$ R is "left" $\mathcal{L}$-stable.*

Proof. Each $a \in R$ is $\mathcal{L}$-Vaserstein by hypothesis, so (1) holds by Lemma 5.19. But then $Rx = R$ implies x is a unit, and (2) follows. □

Corollary 5.21. *The conclusions of Theorem 5.20 hold for SR1, left UG and IC rings* (*using* (**iii**)).

6. Congruences and coverings

We know that $\mathcal{B}(R) = \{L \mid L \leq {}_R R\}$ affords the SR1 rings; it is not the only left idealtor to do so.

Example 6.1. The SR1 rings are afforded by the left idealtor $\mathcal{B}_1(R) = \{Rb \mid b \in R\}$.

Proof. Clearly $\mathcal{B}_1$ is natural. As $\mathcal{B}_1(R) \subseteq \mathcal{B}(R)$ for all R, we have $\{\mathcal{B}_1\text{-stable}\} \supseteq \{\mathcal{B}\text{-stable}\}$ by Lemma 3.8. For the other inclusion, let R be $\mathcal{B}_1$-stable. To see that R is $\mathcal{B}$-stable let $Ra + L = R$, $L \in \mathcal{B}(R)$, say $ra + b = 1$, $r \in R$, $b \in L$. Hence $Ra + Rb = R$ so, as R is $\mathcal{B}_1$-stable, $a - u \in Rb \subseteq L$ for some $u \in U(R)$, as required. □

The use of left idealtors to prove theorems about classes of rings depends upon the following.

Definition 6.2. Two left idealtors $\mathcal{M}$ and $\mathcal{N}$ will be called *congruent* (written $\mathcal{M} \equiv \mathcal{N}$) if

$\mathcal{M}$ and $\mathcal{N}$ afford the same class of rings, that is if $\{\mathcal{M}\text{-}stable\} = \{\mathcal{N}\text{-}stable\}$.

Clearly $\equiv$ is an equivalence relation on the class of left idealtors. But it is not equality. For example, $\mathcal{B} \equiv \mathcal{B}_1$ in Example 6.1, but $\mathcal{B} \neq \mathcal{B}_1$ as functions. Also $\mathcal{D} \equiv \mathcal{D}_n \equiv \mathcal{D}_J \equiv \mathcal{D}_0$ by Proposition 3.9, but no two are equal.

Congruence is closely related to the following "ordering" relation on the class of left idealtors.

Definition 6.3. If $\mathcal{M}$ and $\mathcal{L}$ are left idealtors, we say that $\mathcal{M}$ *covers* $\mathcal{L}$ and write $\mathcal{M} \geq^c \mathcal{L}$ if :

For each ring R: $b \in L \in \mathcal{L}(R)$ implies that $b \in M \subseteq L$ for some $M \in \mathcal{M}(R)$.

Note. If $\mathcal{M} \geq^c \mathcal{L}$, then each $L \in \mathcal{L}(R)$ contains some $M \in \mathcal{M}(R)$—take $b = 0$ in Definition 6.3.

Proposition 6.4. *Let $\mathcal{M}$ and $\mathcal{L}$ be any left idealtors, and let R denote a ring. Then*

(1) $\mathcal{M}(R) \supseteq \mathcal{L}(R)$ *for all rings* $R \quad \Rightarrow \quad \mathcal{M} \geq^c \mathcal{L} \quad \Rightarrow \quad \{\mathcal{M}\text{-stable}\} \subseteq \{\mathcal{L}\text{-stable}\}$.

(2) *If* $\mathcal{M} \geq^c \mathcal{L}$ *and* $\mathcal{L} \geq^c \mathcal{N}$ *then* $\mathcal{M} \geq^c \mathcal{N}$.

(3) *If* $\mathcal{M} \geq^c \mathcal{L}$ *and* $\mathcal{L} \geq^c \mathcal{M}$ *then* $\mathcal{M} \equiv \mathcal{L}$. *The converse fails.*

Proof. (1). The second implication restates Lemma 3.8. For the first implication, assume that $\mathcal{M}(R) \supseteq \mathcal{L}(R)$ for all rings R, and let $b \in L \in \mathcal{L}(R)$. Then $b \in M \subseteq L$ where $M = L \in \mathcal{M}(R)$, as required.

(2). This is a routine consequence of Definition 6.3.

(3). The first assertion is clear by the second implication in (1). To see that the converse is false, let R be a ring with $J(R) \neq 0$ and consider the left idealtors

$$\mathcal{D}_J(R) = \{J(R)\} \quad \text{and} \quad \mathcal{D}_0(R) = \{0\}.$$

Then $\mathcal{D}_J \equiv \mathcal{D}_0$ because they both afford the Dedekind finite rings (Proposition 3.9). To see that neither $\mathcal{D}_0$ nor $\mathcal{D}_J$ covers the other, fix a ring R with $0 \neq b \in J(R)$. We show that $\mathcal{D}_0 \geq^c \mathcal{D}_J$ and $\mathcal{D}_J \geq^c \mathcal{D}_0$ are both impossible by showing that each of them lead to a contradiction:

• If $\mathcal{D}_0 \geq^c \mathcal{D}_J$ then $b \in J(R) \in \mathcal{D}_J(R)$ so $b \in M \subseteq J(R)$ where $M \in \mathcal{D}_0(R)$. Thus $b = 0$.

• If $\mathcal{D}_J \geq^c \mathcal{D}_0$ then $0 \in \langle 0 \rangle \in \mathcal{D}_0(R)$ so $0 \in M \subseteq \langle 0 \rangle$ where $M \in \mathcal{D}_J(R)$. Hence $J(R) = M = 0$. □

Question 4. Can Definition 6.3 be refined so that the converse holds in (3) of Proposition 6.4?

Having congruent idealtors is useful because it means having more than one way to describe a class of rings. We now turn to two important congruences, and illustrate how they can be used.

Definition 6.5. If $\mathcal{L}$ is any left idealtor, define the left idealtors $\mathcal{L}^1$ and $\mathcal{L}^J$ as follows:

$$\mathcal{L}^1(R) = \mathcal{L}(R) \cup \{R\} \qquad \text{and} \qquad \mathcal{L}^J(R) = \{L + C \mid L \in \mathcal{L}(R), C \subseteq J(R).$$

We leave to the reader the task of verifying that $\mathcal{L}^1$ and $\mathcal{L}^J$ are natural.

Example 6.6. $\mathcal{L} \equiv \mathcal{L}^1$ for any left idealtor $\mathcal{L}$. Hence if $\mathcal{L}$ affords a class of rings we can always assume that $R \in \mathcal{L}(R)$ for each ring R by replacing $\mathcal{L}$ by $\mathcal{L}^1$.

Proof. Since $\mathcal{L}^1(R) \supseteq \mathcal{L}(R)$ for each ring R, we have $\{\mathcal{L}^1\text{-stable}\} \subseteq \{\mathcal{L}\text{-stable}\}$ by Lemma 3.8. We show this is equality. If R is $\mathcal{L}$-stable let $Ra + X = R$ where $a \in R$ and $X \in \mathcal{L}^1(R)$. If $X \in \mathcal{L}(R)$ then $a - u \in X$, $u \in U(R)$, as R is $\mathcal{L}$-stable; if $X = R$ then $a - 1 \in X$. Either way R is $\mathcal{L}^1$-stable. □

Example 6.7. $\mathcal{L} \equiv \mathcal{L}^J$ for any left idealtor $\mathcal{L}$.

Proof. Since $\mathcal{L}^J(R) \supseteq \mathcal{L}(R)$ for each ring R, we have $\{\mathcal{L}^J\text{-stable}\} \subseteq \{\mathcal{L}\text{-stable}\}$ by Lemma 3.8. For the other inclusion, let R be $\mathcal{L}$-stable. If $Ra + X = R$, $a \in R$, $X \in \mathcal{L}^J(R)$, write $X = L + C$, $L \in \mathcal{L}(R)$, $C \subseteq J(R)$. Then $Ra + (L + C) = R$, so $Ra + L = R$ because $C \subseteq J(R)$. But R is $\mathcal{L}$-stable so $a - u \in L$ for some unit u. As $L \subseteq X$ this shows that a is $\mathcal{L}^J$-stable. □

Example 6.7 provides a new proof of an old result, namely that if R is semiperfect then

$$R \text{ is SR1} \quad \Leftrightarrow \quad R \text{ is left UG} \quad \Leftrightarrow \quad R \text{ is IC.} \qquad (iv)$$

We know $\{\text{SR1}\} \subseteq \{\text{left UG}\} \subseteq \{\text{IC}\}$. The IC rings are afforded by $\mathcal{E}(R) = \{Re \mid e^2 = e \in R\}$, and also (Example 6.7) by $\mathcal{E}^J(R) = \{Re + C \mid e^2 = e \in R, C \subseteq J(R)\}$. But in a semiperfect ring this is the set of *all* left ideals L (because R/L has a projective cover), and so $\mathcal{E}^J$ affords the SR rings. Hence $\{\text{IC}\} \subseteq \{\text{SR1}\}$.

It is also well known that the "semiperfect" requirement for (iv) can be weakened to "exchange" [12, Theorem 6.5]. In fact there is a version of this for elements. To describe it, recall that a ring R is exchange if and only if the following equivalent conditions are satisfied:

(1) For all $a \in R$ there exists $e^2 = e \in Ra$ with $1 - e \in R(1 - a)$,
(2) If $R = A + L$, $A \leq_R R$, $L \leq_R R$, then $e^2 = e \in A$ exists with $1 - e \in L$.

Definition 6.8. Call an element $a \in R$ *left exchange* if

$$Ra + L = R,\ L \leq {}_R R \quad \text{implies} \quad e^2 = e \in Ra \text{ exists with } 1 - e \in L.$$

It is routine to verify that a ring R is exchange if and only if every element is left exchange.

An element a is called left *SR1*, left *UG*[10], or left *IC* if it is stable for the left idealtors

$$\mathcal{B}(R) = \{L \mid L \leq_R R\}, \quad \mathcal{K}(R) = \{\mathrm{l}(b) \mid b \in R\}, \quad \text{and } \mathcal{E}(R) = \{Re \mid e^2 = eR\},$$

respectively. With this we can give an elementary version of [12, Theorem 6.5].

[10]It is more natural to call an element $a \in R$ left UG if $Ra = Rb$ implies $b = ua$ for some unit $u \in R$. However, this does *not* imply that a is $\mathcal{L}$-stable for $\mathcal{L}(R) = \{\mathrm{l}(b) \mid b \in R\}$ by [17, Theorem 6].

Proposition 6.9. *Let a be any element in a ring R.*

(1) *Always*: *a is left SR1 $\Rightarrow$ a is left UG $\Rightarrow$ a is left IC.*

(2) *If a is left exchange*: *a is left SR1 $\Leftrightarrow$ a is left UG $\Leftrightarrow$ a is left IC.*

Proof. (1). This follows routinely from the definitions.

(2). Assume that a is both left IC and left exchange, and let $R = Ra + L$ where $L \leq_R R$. As a is left exchange, choose $e^2 = e \in Ra$ with $1 - e \in L$. Observe:

$$R = Re + R(1-e) \subseteq Ra + R(1-e), \quad \text{so} \quad Ra + R(1-e) = R.$$

As $1-e$ is an idempotent, the fact that a is left IC means there exists a unit u with the property that $a - u \in R(1-e) \subseteq L$. This proves that a is left SR1. □

Definition 6.10. Given a left idealtor $\mathcal{L}$, *define*

$$\mathcal{L}^c(R) = \{M \leq {}_R R \mid M \cong L \text{ for some } L \in \mathcal{L}(R)\}.$$

Then $\mathcal{L}^c$ is natural and (*in view of Lemma* 6.11 *below*) *we call $\mathcal{L}^c$ the closure of the left idealtor $\mathcal{L}$, and say that $\mathcal{L}$ is closed if $\mathcal{L}^c = \mathcal{L}$.*

Lemma 6.11. *If $\mathcal{L}$ is a left idealtor, then*: (1) *$\mathcal{L}(R) \subseteq \mathcal{L}^c(R)$ for each ring R.* (2) *$\mathcal{L}^c = \mathcal{L}^{cc}$.*

Proof. (1) is a routine verification. As to (2), apply (1) to $\mathcal{L}^c$ to get $\mathcal{L}^c(R) \subseteq \mathcal{L}^{cc}(R)$ for all R. Let $X \in \mathcal{L}^{cc}(R)$, say $X \cong M \in \mathcal{L}^c(R)$. Then, in turn, let $M \cong L \in \mathcal{L}(R)$, so $X \cong M \cong L \in \mathcal{L}(R)$. Thus $X \in \mathcal{L}^c(R)$, so $\mathcal{L}^{cc}(R) \subseteq \mathcal{L}^c(R)$. Hence $\mathcal{L}^c(R) = \mathcal{L}^{cc}(R)$ for all rings R, that is $\mathcal{L}^c = \mathcal{L}^{cc}$. □

The SR1 left idealtors $\mathcal{B}(R) = \{L \mid L \leq {}_R R\}$ and $\mathcal{B}_1(R) = \{Rb \mid b \in R\}$ are both closed. However neither $\mathcal{K}(R) = \{\mathtt{l}_R(b) \mid b \in R\}$ nor $\mathcal{E}(R) = \{Re \mid e^2 = e \in R\}$ is closed (affording the left UG and IC rings, respectively). For example, let $R = \mathbb{Z}$ and $M = 2\mathbb{Z}$. Then:

$M \cong \mathbb{Z} = \mathtt{l}(0) \in \mathcal{K}(\mathbb{Z})$, but $M \neq \mathtt{l}(k)$ for all $k \in \mathbb{Z}$, so $\mathcal{K}$ is not closed;

$M \cong \mathbb{Z} = R1 \in \mathcal{E}(\mathbb{Z})$, but $M \neq Re$ for all $e^2 = e \in \mathbb{Z}$, so $\mathcal{E}$ is not closed.

In Proposition 3.9, four left idealtors affording the DF rings are given:

$\mathcal{D}(R) = \{L \leq {}_R R \mid L \subseteq J(R)\}$, $\mathcal{D}_n(R) = \{L \leq {}_R R \mid L \text{ is nil}\}$, $\mathcal{D}_J(R) = \{J(R)\}$, $\mathcal{D}_0(R) = \{0\}$.

Clearly $\mathcal{D}_0$ is closed. But none of $\mathcal{D}$, $\mathcal{D}_J$, or $\mathcal{D}_n$ is closed. Indeed, consider $R = \begin{bmatrix} D & D \\ & D \end{bmatrix}$ where D is a division ring, with left ideals $K = \begin{bmatrix} D & 0 \\ & 0 \end{bmatrix}$ and $J =$

$J(R) = \begin{bmatrix} 0 & D \\ & 0 \end{bmatrix}$. Then $K \cong J$, and J is in each of $\mathcal{D}(R)$, $\mathcal{D}_J(R)$ and $\mathcal{D}_n(R)$, but K is in none of them.

Question 5. Does there exist a closed left idealtor affording the left UG rings? The IC rings?

A ring R is called a left *C2 ring* if every left ideal isomorphic to a summand of ${}_RR$ is itself a summand [18, Section 7.2]. Consider the (non-closed) left idealtor $\mathcal{E}(R) = \{Re \mid e^2 = e \in R\}$ for the IC rings.

Example 6.12. For a ring R : $\mathcal{E}^c(R) = \mathcal{E}(R)$ if and only if R is a left C2 ring.

Proof. $\mathcal{E}^c(R) = \{L \leq {}_RR \mid L \cong Re$ for some $e^2 = e \in R\}$. Assume $\mathcal{E}^c(R) = \mathcal{E}(R)$. If L is a left ideal of R and $L \cong Re$, $e^2 = e$, then $L \in \mathcal{E}^c(R) = \mathcal{E}(R)$, so $L = Rf$ for some $f^2 = f$. This shows that R is left C2. The converse is proved the same way. □

Incidentally, the left C2 rings are not affordable by Theorem 5.8. In fact, if D is a division ring, the ring $R = \begin{bmatrix} D & D \\ & D \end{bmatrix}$ is SR1, but it is not left C2 because $J(R) = \begin{bmatrix} 0 & D \\ & 0 \end{bmatrix} \cong Re$ where $e^2 = e = \begin{bmatrix} 1 & 0 \\ & 0 \end{bmatrix}$ and $J(R)$ is not a direct summand of ${}_RR$.

An element a in a ring R is called *left PP* if the following equivalent statements hold:

(1) $\mathtt{l}(a)$ is a direct summand of ${}_RR$ (2) Ra is projective.

All regular rings and all domains are left PP. It is known [1, Theorem 11] that every commutative PP ring is UG, and one asks if every left PP-ring is left UG. The answer is "no": If D is a division ring the ring $M_\omega(D)$ is regular (and so left PP) but not left UG because it is not Dedekind finite. So we ask:

Question 6. Is every Dedekind finite, left PP ring left UG?

Note that the answer is "yes" if R is IC (and left PP).

7. Corners and products

We begin by describing when $\mathcal{L}$-stablility of a ring R passes to a corner eRe, $e^2 = e \in R$. An idempotent $e^2 = e \in R$ is called *quasi-normal* if the following equivalent conditions are satisfied:

(1) $eR(1-e)Re = 0$.
(2) The map $R \to eRe$ given by $r \mapsto ere$ is a ring morphism.

Theorem 7.1. *Let $\mathcal{L}$ be a left idealtor and let $e^2 = e \in R$. If R is $\mathcal{L}$-stable, then eRe is $\mathcal{L}$-stable provided both conditions* (a) *and* (b) *below are satisfied*:

(a) *If $X \in \mathcal{L}(eRe)$ then $RX \in \mathcal{L}(R)$.*
(b) *Either* (b_1) *or* (b_2) *holds*:
  (b_1) *Every $\mathcal{L}$-stable ring is Dedekind finite.*
  (b_2) *e is quasi-normal.*

Proof. Let R be $\mathcal{L}$-stable, write $S = eRe$, and let $Sa + X = S$ where $a \in S$ and $X \in \mathcal{L}(S)$. We want $a - w \in X$ for some unit w of S. Write $sa + x = e$, $s \in S$, $x \in X$. Then

$$(s + 1 - e)(a + 1 - e) + x = (sa + 1 - e) + x = 1.$$

Hence $R(a + 1 - e) + RX = R$. Using condition (a), let $(a + 1 - e) - v \in RX$ for some $v \in U(R)$. Write $b = (a + 1 - e) - v$, so $b \in RX$ and hence $be = b$. Since $a + 1 - e - b = v$, we obtain

$$(a + 1 - e - b)u = 1 \quad \text{where } u = v^{-1} \in U(R). \tag{v}$$

Multiply both sides by e to get $(a - eb)ue = e$. In addition we have $eb \in e(RX) = eR(eX) \subseteq X$ because X is a left ideal of S. It follows that

$$(a - eb)eue = e, \quad eb \in X \tag{vi}$$

If we write $w = a - eb$, then $w \in S$ has a right inverse in S.

Case (b_1). Here it follows that w is a unit in S because S is Dedekind finite whenever R is. As $a - w = eb \in X$, this shows that a is L-stable in S, as required.

Case (b_2). Now we show that $eue \in U(S)$ (whence $a - eb \in U(S)$ by (vi)). As in (v) we have $u(a + 1 - e - b) = 1$, whence $eu(a - be) = e$. Now condition (b_2) shows that $erse = erese$ for all $r, s \in R$, so we obtain $(eu)e(a - ebe) = e$. This with (**vi**) shows that eue is a unit in S, and we are done as before. ☐

Corollary 7.2. *Let $\mathcal{L}$ be any left idealtor. If R is $\mathcal{L}$-stable then eRe is $\mathcal{L}$-stable if $e^2 = e \in R$ is central and $\mathcal{L}(eRe) \subseteq \mathcal{L}(R)$.*

Proof. Clearly (b_2) holds. For (a): If $X \in \mathcal{L}(eRe)$ then $RX = R(eX) = eReX = X$. It follows by hypothesis that $RX \in \mathcal{L}(eRe) \subseteq \mathcal{L}(R)$. □

Corollary 7.3. *Each of the ring properties SR1, left UG, IC and DF passes to corners.*

Proof. First consider SR1, IC and DF. Then (b_1) holds by Corollary 3.6. To verify (a) use, respectively, the left idealtors

$$\mathcal{B}(R) = \{L \mid L \leq_R R\}, \qquad \mathcal{E}(R) = \{Re \mid e^2 = e \in R\}, \qquad \mathcal{D}_0(R) = \{0\}.$$

Then (a) is clear for $\mathcal{B}$ and $\mathcal{D}_0$, and it holds for $\mathcal{E}$ because $RSf = Rf$ whenever $f^2 = f \in S = eRe$. The fact that left UG passes to corners comes from [17, Theorems 29 and 30] where it is shown that if the Morita context ring $C = \begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is left UG, then R is left UG. □

Question 7. Is there a left UG ring R in which Condition (a) *fails for* $\mathcal{K}(R) = \{\mathtt{l}(b) \mid b \in R\}$?

Question 8. Let $\mathcal{L}$ be any left idealtor. When is $\mathcal{L}$-stability a Morita invariant? That is:

(1) *For a ring R, when does: R is $\mathcal{L}$-stable imply eRe is $\mathcal{L}$-stable where $e^2 = e$ and $ReR = R$?*

(2) *For a ring R, when does: R is $\mathcal{L}$-stable imply $M_n(R)$ is $\mathcal{L}$-stable for $n \geq 2$?*

8. Direct products

We now investigate when $\mathcal{L}$-stability passes from a direct product to its factors, and back.

Theorem 8.1. *Let $R = \Pi_{i \in I} R_i$ denote a direct product of rings R_i with canonical projections $\pi_k : R \to R_k$ for each $k \in I$. Let $\mathcal{L}$ denote a left idealtor. Then*

(1) *R is $\mathcal{L}$-stable $\Rightarrow$ each R_i is $\mathcal{L}$-stable provided $L_i \in \mathcal{L}(R_i)$ for each i implies $\Pi_{i \in I} L_i \in \mathcal{L}(R)$.*

(2) *Each R_i is $\mathcal{L}$-stable $\Rightarrow$ R is $\mathcal{L}$-stable provided $L \in \mathcal{L}(R)$ implies $L = \Pi_{i \in I} L_i$ for $L_i \in \mathcal{L}(R_i)$.*

Proof. (1). Assume that R is left $\mathcal{L}$-stable. Suppose $R_i a_i + L_i = R_i$ with $L_i \in \mathcal{L}(R_i)$ and $a_i \in R_i$, say $r_i a_i + x_i = 1_{R_i}$ where $x_i \in L_i$. Then, $\langle r_i \rangle \langle a_i \rangle + \langle x_i \rangle = 1_R$, and $\langle x_i \rangle \in \Pi_{i \in I} L_i \in \mathcal{L}(R)$ by the proviso. By hypothesis $\langle a_i \rangle - \langle u_i \rangle \in \langle x_i \rangle$ where $\langle u_i \rangle$ is a unit in R. Thus $a_i - u_i = x_i \in L_i$ for each i, and each u_i is a unit in R_i. This proves (1).

(2). Now assume that each R_i is $\mathcal{L}$-stable. Suppose $R\langle a_i\rangle + L = R$ where $L \in \mathcal{L}(R)$. By the proviso, $L = \Pi_{i\in I} L_i$ where $L_i \in \mathcal{L}(R_i)$ for each i. Hence $\langle r_i\rangle\langle a_i\rangle + \langle x_i\rangle = \langle 1_{R_i}\rangle$ where $r_i \in R_i$ and $x_i \in L_i$ for each i. It follows that $R_i a_i + L_i = R_i$ so, by hypothesis, $a_i - u_i \in L_i$ for some unit u_i in R_i. Finally $\langle a_i\rangle - \langle u_i\rangle \in \Pi_{i\in I} L_i = L$ where $\langle u_i\rangle$ is a unit in R. This proves (2). □

Corollary 8.2. *Let $R = \Pi_{i\in I} R_i$ be a direct product of rings. Then R is SR1, left UG, IC or DF if and only if the same holds for each R_i.*

Proof. The proof of Corollary 7.3 adapts. □

Question 9. Let $R = \Pi_{i\in I} R_i$ with canonical projections $\pi_k : R \to R_k$. The provisos in (1) *and* (2) *of Theorem 8.1 imply, respectively, that each π_k is $\mathcal{L}$-full* ($\mathcal{L}$*-fit*)*. When do the converses hold*?

We conclude this Section with a result about a finite direct product R, viewed internally: $R = S_1 \oplus \cdots \oplus S_n$ where $S_i \lhd R$ for each i. Then $S_i = e_i R e_i$ where $e_i^2 = e_i$ is central for each i, the e_i are orthogonal, and $1 = e_1 + \cdots + e_n$.

Theorem 8.3. *Let $\mathcal{L}$ be any left idealtor and let $R = S_1 \oplus \cdots \oplus S_n$ where $S_i \lhd R$ for each i. Then*

(1) *R is $\mathcal{L}$-stable $\Rightarrow$ every S_i is $\mathcal{L}$-stable provided $\mathcal{L}(S_i) \subseteq \mathcal{L}(R)$ for each i.*
(2) *Every S_i is $\mathcal{L}$-stable $\Rightarrow$ R is $\mathcal{L}$-stable provided $\{S_i \cap L \mid L \in \mathcal{L}(R)\} \subseteq \mathcal{L}(S_i)$ for each i.*

Proof. Write $S_i = e_i R e_i$ where $e_i^2 = e_i$ is central, $e_1 + \cdots + e_n = 1$, and $\{e_1, \ldots, e_n\}$ is orthogonal.

(1). This follows from Theorem 7.1: Condition (b_2) is satisfied because e_i is central; and condition (a) holds because if $X \in \mathcal{L}(S_i)$ then $RX = R(e_i X) = S_i X = X \in \mathcal{L}(R)$ by the proviso.

(2). Let $Ra + L = R$, $a \in R$, $L \in \mathcal{L}(R)$. Multiplying by e_i gives $S_i a e_i + L e_i = S_i$. Observe that $L e_i = S_i \cap L \in \mathcal{L}(S_i)$ by the proviso. Since S_i is $\mathcal{L}$-stable, there exists $u_i \in U(S_i)$ such that $a e_i - u_i \in L e_i$. Write $u = \Sigma_{i=1}^n u_i$ so u is a unit in R (with inverse $\Sigma_{i=1}^n v_i$ where $u_i v_i = e_i = v_i u_i$ for each i). Finally, we obtain $a - u = \Sigma_{i=1}^n (a e_i - u_i) \in \Sigma_{i=1}^n L e_i = \Sigma_{i=1}^n e_i L \subseteq L$, as required. □

Corollary 8.4. *Let $R = S_1 \oplus \cdots \oplus S_n$ where $S_i \lhd R$ for each i. Then R enjoys each of the ring properties SR1, left UG, IC and DF if and only if the same is true of each S_i.*

Proof. As in Theorem 8.3, write $S_i = e_iRe_i$ where $e_i^2 = e_i$ is central in R. Each property passes to every S_i by Corollary 7.3 because $S_i = e_iRe_i$ is a corner of R.

So it remains to check proviso (2) of Theorem 8.3 in each case. It is clear that it holds for SR1 and DF using the left idealtors $\mathcal{B}(R) = \{L \mid L \leq_R R\}$ and $\mathcal{D}_0(R) = \{0\}$. For left UG, using $\mathcal{K}(R) = \{\mathtt{l}(b) \mid b \in R\}$, the proviso in (2) also holds because $S_i \cap \mathtt{l}_R(b) = \mathtt{l}_{S_i}(b)$. Finally for IC, using $\mathcal{E}(R) = \{Re \mid e^2 = e \in R\}$, the proviso in (2) holds because $Re_i \cap Rf = Re_if$ for any idempotent $f \in R$ (e_i is central in R). □

9. Ideal extensions and matrix rings

If S is a (unital) subring of a ring R, then R is said to be an *extension* of S. In this section we examine a particular extension type and characterize when a stability condition passes from S to R, and conversely. The ring extensions we are interested in are described as follows:

Definition 9.1. A ring R is called an *ideal extension* of a (unital) *subring* S *if* $R = S \oplus A$ *where* $A \lhd R$ *and* $A \subseteq J(R)$.[11]

For example the formal power series ring $R = S[[x]]$ is an ideal extension of S. If S is any ring and A is a general ring (no unity) with $J(A) = A$, then the abelian group $S \oplus A$ becomes an ideal extension if we define multiplication by $(s, a)(t, b) = (st, sb + at + ab)$.

Theorem 9.2. *Let* $R = S \oplus A$ *be an ideal extension, and let* $\mathcal{L}$ *be a left idealtor. Define* $\theta : R \to S$ *by* $\theta(s + a) = s$ *for all* $s \in S$ *and* $a \in A$. *Then*

(1) *If* R *is* $\mathcal{L}$*-stable then* S *is* $\mathcal{L}$*-stable provided* θ *is* $\mathcal{L}$*-full.*

(2) *If* S *is* $\mathcal{L}$*-stable then* R *is* $\mathcal{L}$*-stable provided* θ *is* $\mathcal{L}$*-fit.*

Proof. For clarity write $\bar{r} = \theta(r)$ and $\bar{L} = \theta(L)$ for any $r \in R$ and any left ideal $L \subseteq R$. Note that θ is an onto ring morphism with kernel A, and that $\bar{s} = s$ for all $s \in S$. Clearly $U(S) \subseteq U(R)$; in fact $U(R) = U(S) \oplus A$ because $A \subseteq J(R)$.

(1). If R is $\mathcal{L}$-stable, let $Sb + X = S$, $b \in S$, $X \in \mathcal{L}(S)$, say $1 = sb + x$, $s \in S$, $x \in X$. As θ is $\mathcal{L}$-full, $X = \bar{L}$ where $L \in \mathcal{L}(R)$. Write $x = \bar{l}$, $l \in L$. Then $\bar{x} = x$ because $x \in S$, so

$$\overline{1 - sb - l} = \overline{x - l} = \bar{x} - \bar{l} = x - \bar{l} = 0.$$

[11]If the requirement that $A \subseteq J(R)$ is dropped then R is called a *Dorroh* extension of R. For example the polynomial ring $S[x]$ is a Dorroh extension of S.

Hence $1 - sb - l \in A$, so $Rb + L + A = R$. As $A \subseteq J(R)$ we obtain $Rb + L = R$. Since $L \in \mathcal{L}(R)$ and R is $\mathcal{L}$-stable, let $b - u \in L$ where $u \in U(R)$. But $\bar{b} = b$ so it follows that

$$b - \bar{u} = \bar{b} - \bar{u} = \overline{b - u} \in \bar{L} = X.$$

Since $\bar{u} \in U(S)$, this proves (1).

(2). Assume that S is $\mathcal{L}$-stable and let $r \in R$; we must show r is $\mathcal{L}$-stable in R. Write $r = s + a$, $s \in S$, $a \in A$. Since $A \subseteq J(R)$, it suffices (by Lemma 5.7) to show that s is $\mathcal{L}$-stable in R.

To that end, let $Rs + L = R$, $L \in \mathcal{L}(R)$, say $ps + l = 1$, $p \in R$, $l \in L$. Then $1 = \bar{1} = \bar{p}\bar{s} + \bar{l}$, so $S = Ss + \bar{L}$. Moreover $\bar{L} \in \mathcal{L}(S)$ because θ is $\mathcal{L}$-fit, so $s - u \in \bar{L}$ for some $u \in U(S) \subseteq U(R)$. If $s - u = \bar{x}$ where $x \in L$, then $s - u - x \in ker(\theta) = A$, say $s - u - x = a \in A$. Finally $s - (u + a) = x \in L$, and we are done because $u + a$ is a unit of R. □

Corollary 9.3. *Let $R = S \oplus A$ be an ideal extension. Then*

(1) *R has SR1, IC or DF if and only if S has the same property.*
(2) *If R is left UG, then S is left UG. The converse holds if for each $b \in R$, $\theta[\mathtt{l}_R(b)] = \mathtt{l}_S(s)$ for some $s \in S$.*

Proof. Let θ be as in Theorem 9.2. Note that $ker(\theta) = A \subseteq J(R)$.

• SR1. If $\mathcal{B}(R) = \{L \mid L \leq_R R\}$ then θ is both $\mathcal{B}$-fit and $\mathcal{B}$-full, so Theorem 9.2 applies.

• IC. Use $\mathcal{E}(R) = \{Re \mid e^2 = e \in R\}$. Then θ is $\mathcal{E}$-full because $\theta(Re) = Se$ for all $e^2 = e \in S$, and θ is $\mathcal{E}$-fit because $\theta(Rf) = S\,\theta(f)$ for all $f^2 = f \in R$. Hence we are done by Theorem 9.2.

• DF. Using $\mathcal{D}_0(R) = \{0\}$, again θ is both $\mathcal{D}_0$-fit and $\mathcal{D}_0$-full, so Theorem 9.2 applies.

• Left UG. Use $\mathcal{K}(R) = \{\mathtt{l}(b) \mid b \in R\}$. Then θ is $\mathcal{K}$-full because of the following Claim:

Claim. For any $c \in S$, $\mathtt{l}_S(c) = \theta[\mathtt{l}_R(c)]$.

Proof. For convenience, write $\theta(r) = \bar{r}$ for all $r \in R$, and recall that $\bar{s} = s$ for all $s \in S$.

$\mathtt{l}_S(c) \subseteq \theta[\mathtt{l}_R(c)]$. If $s \in \mathtt{l}_S(c)$ then $s = \theta(s) \in \theta[\mathtt{l}_R(c)]$.

$\mathtt{l}_S(c) \supseteq \theta[\mathtt{l}_R(c)]$. If $b \in \mathtt{l}_R(c)$ then $bc = 0$ so $\theta(b)c = \bar{b}c = \bar{b}\bar{c} = \overline{bc} = \bar{0} = 0$; that is $\theta(b) \in \mathtt{l}_S(c)$.

Hence R is left UG implies S is left UG by Theorem 9.2. By the same theorem, the converse holds if θ is $\mathcal{K}$-fit (for each $b \in R$, $\theta[\mathtt{1}_R(b)] = \mathtt{1}_S(s)$ for some $s \in S$). □

Let $R = S[[x]]$ denote the ring of *formal power series* over a ring S. As usual, we identify S with the subring of constant series, and write $\langle x \rangle$ for the ideal of series with zero constant term. It is well known that $U(R) = U(S)$, and that $J(R) = J(S) \oplus \langle x \rangle$. Hence $R = S \oplus \langle x \rangle$ is an ideal extension.

Corollary 9.4. *The conclusions of Corollary 9.3 hold for power series rings.*

10. Context-null matrix rings

Consider the Morita context ring $R = \begin{bmatrix} R_1 & V \\ W & R_2 \end{bmatrix}$ where R_1 and R_2 are rings with bimodules $V = {}_{R_1}V_{R_2}$ and $W = {}_{R_2}W_{R_1}$. If $VW = 0$ and $WV = 0$ then R is called the *context-null extension* of R_1 and R_2 by the bimodules V and W,[12] and the multiplication takes the form

$$\begin{bmatrix} a & v \\ w & b \end{bmatrix} \begin{bmatrix} a' & v' \\ w' & b' \end{bmatrix} = \begin{bmatrix} aa' & av' + vb' \\ wa' + bw' & bb' \end{bmatrix}.$$

Note that the diagonals multiply "directly" as in a direct product.

With this in mind, write $S = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$ for the "diagonal" subring and write $A = \begin{bmatrix} 0 & V \\ W & 0 \end{bmatrix}$. Then the context-null extension R takes the form $R = S \oplus A$ and so is an ideal extension ($A \subseteq J(R)$ because $A^2 = 0$). Hence Theorem 9.2 can be applied. Rather than state the details here, we are going to generalize this to the $n \times n$ case.

Let $R_1, \ldots, R_n$ be rings and, whenever $i \neq j$, let V_{ij} be an R_i-R_j-bimodule. Assume that there exist multiplications $V_{ij}V_{ji} \subseteq R_i$ for each i, j, and $V_{ij}V_{jk} \subseteq V_{ik}$ when $i \neq k$, such that

$$R = M_n[R_i, V_{ij}] = \begin{bmatrix} R_1 & V_{12} & \cdots & V_{1n} \\ V_{21} & R_2 & & V_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_{n1} & V_{n2} & \cdots & R_n \end{bmatrix}$$

is an associative ring using matrix operations, called a *generalized* $n \times n$ *matrix ring* over the rings R_i. The prototype example is $R = end({}_RM)$ where $M = M_1 \oplus \cdots \oplus M_n$, $R_i = end({}_RM_i)$ for each i, and $V_{ij} = hom_R(M_i, M_j)$ when $i \neq j$.

[12]These extensions arise as follows: Every element of a ring R is regular or quasi-regular if and only if R is either regular, local, or a context-null extension of division rings R and S [14, Theorem 2]. Grover and Khurana [10] discuss a generalization of these rings.

Definition 10.1. A generalized matrix ring $R = M_n[R_i, V_{ij}]$ over the rings $R_1, \ldots,$ R_n is called a *context-null extension* of the rings R_i, denoted by $R = CN_n[R_i, V_{ij}]$, if $V_{pj}V_{jq} = 0$ whenever $j \neq p$ or $j \neq q$.

Thus the case $n = 2$ is described above. For $n = 4$ the multiplication in $CN_4[R_i, V_{ij}]$ becomes

$$\begin{bmatrix} a & v_{12} & v_{13} & v_{14} \\ v_{21} & b & v_{23} & v_{24} \\ v_{31} & v_{32} & c & v_{34} \\ v_{41} & v_{42} & v_{43} & d \end{bmatrix} \begin{bmatrix} p & u_{12} & u_{13} & u_{14} \\ u_{21} & q & u_{23} & u_{24} \\ u_{31} & u_{32} & r & u_{34} \\ u_{41} & u_{42} & u_{43} & s \end{bmatrix} =$$
$$\begin{bmatrix} ap & au_{12} + v_{12}q & au_{13} + v_{13}r & au_{14} + v_{14}s \\ v_{21}p + bu_{21} & bq & bu_{23} + v_{23}r & bu_{24} + v_{24}s \\ v_{31}p + cu_{31} & v_{32}q + u_{32}c & cr & cu_{34} + v_{34}s \\ v_{41}p + du_{41} & v_{42}q + du_{42} & v_{43}r + du_{43} & ds \end{bmatrix}$$

where the diagonals multiply "directly" as in the 2×2 case above. Furthermore, by deleting pairs of columns and the corresponding rows, each of the 2×2 rings $CN_2[R_i, V_{ij}]$ arises as a corner of $CN_4[R_i, V_{ij}]$.

In the general $n \times n$ case, write $R = CN_n[R_i, V_{ij}]$. If $R = \begin{bmatrix} R_1 & V_{12} & \cdots & V_{1n} \\ V_{21} & R_2 & & V_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_{n1} & V_{n2} & \cdots & R_n \end{bmatrix}$, let $S = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix}$ and $A = \begin{bmatrix} 0 & V_{12} & \cdots & V_{1n} \\ V_{21} & 0 & & V_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_{n1} & V_{n2} & \cdots & 0 \end{bmatrix}$. Then S is a subring of R, $A \lhd R$, and $A \subseteq J(R)$ because $A^2 = 0$. That is, $R = S \oplus A$ is an ideal extension. Hence we obtain.

Example 10.2. The ring $CN_n[R_i, V_{ij}]$ has SR1, IC or DF if and only if each factor ring R_i has the same property.

Proof. Since $R = S \oplus A$ is an ideal extension and $S \cong R_1 \times \cdots \times R_n$ as rings, the result follows using Proposition 2.7, Corollary 8.2 and Corollary 9.3. $\square$

Theorem 10.3. *Let $R_1, \ldots, R_n$ be rings and let $R = CN_n[R_i, V_{ij}]$ be a generalized context-null extension. Then* (*with the notation above*) *we have*:

$$R = S \oplus A \text{ is an ideal extension and } A \subseteq J(R) \text{ because } A^2 = 0.$$

Define $\theta : R \to S$ by $\theta(s + a) = s$ where $s \in S$ and $a \in A$. If $\mathcal{L}$ is a left idealtor then

(1) *R is $\mathcal{L}$-stable $\Rightarrow$ Each R_i is $\mathcal{L}$-stable provided*

(a) *$X \in \mathcal{L}(S)$ implies $X = \theta(L)$ for some $L \in \mathcal{L}(R)$.*

(b) *$L_i \in \mathcal{L}(R_i)$ for each i implies $\Pi_{i=1}^n L_i \in \mathcal{L}(S)$,*

(2) *Each* R_i *is* $\mathcal{L}$*-stable* $\Rightarrow$ R *is* $\mathcal{L}$*-stable provided*:

(c) $L \in \mathcal{L}(S)$ *implies* $L = \Pi_{i=1}^n L_i$ *for* $L_i \in \mathcal{L}(R_i)$.

(d) $L \in \mathcal{L}(R)$ *implies* $\theta(L) \in \mathcal{L}(S)$.

Proof. We have $R \overset{\theta}{\to} S \overset{\sigma}{\to} \Pi_{i=1}^n R_i$ where $\sigma[diag(r_1, \dots, r_n)] = (r_1, \dots, r_n)$ where $r_i \in R_i$ for each i. Since σ is an isomorphism we have (by Lemma 2.9) that θ is $\mathcal{L}$-fit/$\mathcal{L}$-full if and only if $\sigma \circ \theta$ is $\mathcal{L}$-fit/$\mathcal{L}$-full. Hence, for determining whether θ is $\mathcal{L}$-fit/$\mathcal{L}$-full we may assume that $S = \Pi_{i=1}^n R_i$, and apply Theorem 8.1.

(1). Assume R is $\mathcal{L}$-stable. Then S is $\mathcal{L}$-stable by Theorem 9.2 using (a). Now, with (b), each R_i is $\mathcal{L}$-stable by Theorem 8.1.

(2). Assume each R_i is $\mathcal{L}$-stable. Then $S = \Pi_{i=1}^n R_i$ is $\mathcal{L}$-stable by (c) and Theorem 8.1. Hence, because of (d), $S \oplus A$ is $\mathcal{L}$-stable by Theorem 9.2. □

Remark 10.4. Another version of Theorem 10.3 is valid if Theorem 8.3 is used in place of Theorem 8.1. We leave the details to the reader.

11. Triangular matrix rings

If $V_{ij} = 0$ whenever $i > j$ then the generalized matrix ring $M_n[R_i, V_{ij}]$ becomes upper triangular, and is called an $n \times n$ *generalized upper triangular* matrix ring over the rings R_i, and denoted by $T_n[R_i, V_{ij}]$. The case $n = 2$ is the usual split-null extension $\begin{bmatrix} R_1 & V_{12} \\ & R_2 \end{bmatrix}$. The following theorem is the analogue of Theorem 10.3 for general context-null extensions. The routine proof is omitted.

Theorem 11.1. *Let* $R_1, \dots, R_n$ *be rings and let* $R = T_n[R_i, V_{ij}]$ *be a generalized upper triangular matrix ring over the* R_i*. Let* $S \subseteq R$ *be the subring of diagonal matrices, and let* $A \lhd R$ *denote the ideal of matrices with zero diagonal. Then all the conclusions of Theorem 10.3 are valid.*

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