

RM ALGEBRAS AND COMMUTATIVE MOONS

Andrzej Walendziak

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ABSTRACT. Some generalizations of BCI algebras (the RM, BH, CI, BCH, BH**, BCH**, and *aRM** algebras) satisfying the identity $(x \rightarrow 1) \rightarrow y = (y \rightarrow 1) \rightarrow x$ are considered. The connections of these algebras and various generalizations of commutative groups (such as, for example, involutive commutative moons and commutative (weakly) groups) are described. In particular, it is proved that an RM algebra verifying this identity is equivalent to an involutive commutative moon.

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1. Introduction

In 1966, K. Iséki [6] introduced BCI algebras as algebraic models of BCI-logic. In 1983, Q. P. Hu and X. Li [2] defined BCH algebras, which are a generalization of BCI algebras. Later on, in 1998, Y. B. Jun et al. [7] introduced the notion of BH algebras. It is known that BCI and BCH algebras are contained in the class of BH algebras. Next, in 2009, the new class of algebras called CI algebras was introduced by B. L. Meng [11]. These algebras are a generalization of BCH algebras (hence also a generalization of BCI algebras). BH and CI algebras are also called aRM and RME algebras, respectively (see [3], [4], [14]). All of the algebras mentioned above are contained in the class of RM algebras (an RM algebra is an algebra $(A; \rightarrow, 1)$ of type $(2, 0)$ satisfying the identities: $x \rightarrow x = 1$ and $1 \rightarrow x = x$). Deductive systems and congruences in RM algebras were studied by A. Walendziak [12]. The implicative and commutative properties for some subclasses of the class of RM algebras were investigated in [13,14]. In 1985, T. Lei and C. Xi [9] defined p-semisimple BCI algebras and proved that p-semisimple BCI algebras are equivalent with commutative groups. The p-semisimple BCI algebras have been extensively investigated in many papers (for example [1], [8], [10], [16]).

In this paper, we consider some generalizations of BCI algebras (the RM, BH, CI, BCH, BH**, BCH**, and *aRM** algebras) satisfying the identity $(x \rightarrow 1) \rightarrow y = (y \rightarrow 1) \rightarrow x$, which plays a very important role in the theory of p-semisimple

BCI algebras. We describe the connections of considered algebras and various generalizations of commutative groups (such as, for example, involutive commutative moons and commutative goops, which were introduced by A. Iorgulescu [5]). In particular, we prove that RM algebras verifying $(x \rightarrow 1) \rightarrow y = (y \rightarrow 1) \rightarrow x$ are equivalent with involutive commutative moons.

2. Preliminaries

2.1 Generalizations of BCI algebras

Let $\mathcal{A} = (A; \rightarrow, 1)$ be an algebra of type $(2, 0)$. We define the binary relation \leq by: for all $x, y \in A$,

$$x \leq y \iff x \rightarrow y = 1.$$

We consider the following list of properties (cf. [3]) that can be satisfied by \mathcal{A} :

$$(An) \text{ (Antisymmetry) } x \rightarrow y = 1 = y \rightarrow x \implies x = y,$$

$$(An') \text{ (Antisymmetry) } x \leq y \text{ and } y \leq x \implies x = y,$$

$$(B) (y \rightarrow z) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1,$$

$$(B') y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z),$$

$$(BB) (y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1,$$

$$(BB') y \rightarrow z \leq (z \rightarrow x) \rightarrow (y \rightarrow x),$$

$$(D1) x \rightarrow ((x \rightarrow 1) \rightarrow 1) = 1,$$

$$(D1') x \leq (x \rightarrow 1) \rightarrow 1,$$

$$(Ex) x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z),$$

$$(M) 1 \rightarrow x = x,$$

$$(Re) \text{ (Reflexivity) } x \rightarrow x = 1,$$

$$(Re') \text{ (Reflexivity) } x \leq x,$$

$$(*) x \rightarrow y = 1 \implies (z \rightarrow x) \rightarrow (z \rightarrow y) = 1,$$

$$(*)' x \leq y \implies z \rightarrow x \leq z \rightarrow y,$$

$$(**) x \rightarrow y = 1 \implies (y \rightarrow z) \rightarrow (x \rightarrow z) = 1,$$

$$(**)' x \leq y \implies y \rightarrow z \leq x \rightarrow z,$$

$$(Tr) \text{ (Transitivity) } x \rightarrow y = 1 = y \rightarrow z \implies x \rightarrow z = 1,$$

$$(Tr') \text{ (Transitivity) } x \leq y \text{ and } y \leq z \implies x \leq z.$$

Lemma 2.1. (See Proposition 2.1 and Theorem 2.7 of [3]) *Let $\mathcal{A} = (A; \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then the following hold:*

- (i) (Re) + (Ex) + (An) imply (M),
- (ii) (M) + (B) imply (*), (**),
- (iii) (M) + (**) imply (Tr),
- (iv) (Re) + (Ex) + (*) imply (BB),
- (v) (M) + (BB) imply (B).

Definition 2.2. (See [3]) An algebra $\mathcal{A} = (A; \rightarrow, 1)$ is called:

1. *RM algebra* if it verifies the axioms: (Re), (M).
2. *aRM algebra* (or *BH algebra*) if it verifies the axioms: (Re), (M), (An), i.e., it is an RM algebra with (An).
3. *RME algebra* (or *CI algebra*) if it verifies the axioms: (Re), (M), (Ex), i.e., it is an RM algebra with (Ex).
4. *BCH algebra* if it verifies the axioms: (Re), (Ex), (An), hence (M) by Lemma 2.1 (i), i.e., it is an RME algebra with (An).
5. *aRM** algebra* if it verifies the axioms: (Re), (M), (An), (**), i.e., it is an aRM algebra with (**).
6. **aRM** algebra* if it verifies the axioms: (Re), (M), (An), (*), (**), i.e., it is an aRM algebra with (*), (**).
7. *BCH** algebra* if it is a BCH algebra verifying (**).
8. *BCI algebra* if it is a BCH algebra verifying (B).

Denote by **RM**, **aRM = BH**, **RME = CI**, **BCH**, **aRM** = BH****, ***aRM****, **BCH****, and **BCI** the classes of RM, aRM, RME, BCH, aRM**, *aRM**, BCH**, and BCI algebras, respectively. From the definitions and Lemma 2.1(ii) it follows that

$$\mathbf{BCI} \subset \mathbf{BCH}^{**} \subset \mathbf{BCH} \subset \mathbf{RME} \subset \mathbf{RM},$$

$$\mathbf{BCI} \subset \mathbf{*aRM}^{**} \subset \mathbf{aRM}^{**} \subset \mathbf{aRM} \subset \mathbf{RM}, \text{ and } \mathbf{BCH} \subset \mathbf{aRM}.$$

The interrelationships between the classes of algebras mentioned before are visualized in Figure 1. (An arrow indicates proper inclusion, that is, if **X** and **Y** are classes of algebras, then $\mathbf{X} \rightarrow \mathbf{Y}$ means $\mathbf{X} \subset \mathbf{Y}$.)

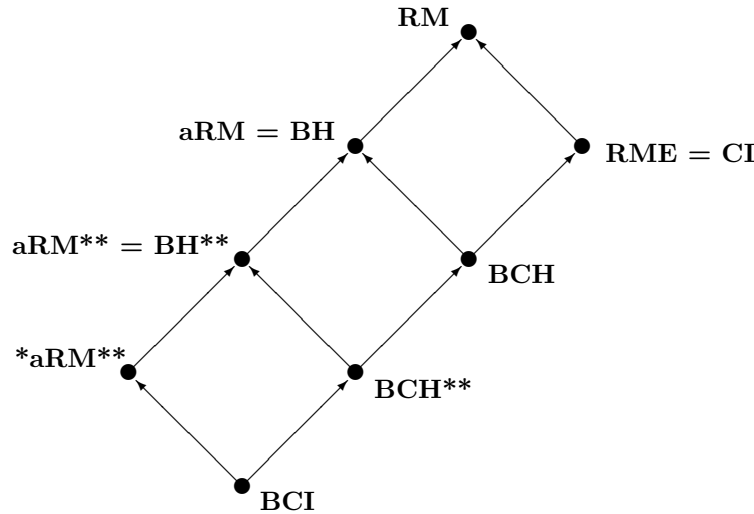


Fig. 1: The hierarchy between **BCI** and **RM**

Note that, in RM algebras, \leq is a reflexive relation; in aRM algebras, it is reflexive and antisymmetric. If \mathcal{A} is an aRM** algebra, then \leq is also transitive by Lemma 2.1(iii). Therefore \leq is an order relation in aRM** algebras (hence also in BCH**, *aRM**, and BCI algebras).

2.2 Generalizations of commutative groups

A. Iorgulescu [5] introduced and studied new generalizations of groups such as moons, goops and many others.

Definition 2.3. A *moon* is an algebra $\mathcal{G} = (G; \cdot, {}^{-1}, 1)$ of type $(2, 1, 0)$ satisfying

$$(U) \quad x \cdot 1 = x = 1 \cdot x,$$

$$(Iv) \quad x \cdot x^{-1} = 1 = x^{-1} \cdot x.$$

A moon is *involution* if it satisfies

$$(DN) \quad (x^{-1})^{-1} = x.$$

A moon is *associative* if it satisfies

$$(Ass) \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

A moon is *commutative* if it satisfies

$$(Com) \quad x \cdot y = y \cdot x.$$

Note that the associative moon is just the group.

Definition 2.4. ([5]) A *goop* is an algebra $(G; \cdot, {}^{-1}, 1)$ of type $(2, 1, 0)$ satisfying (U) and the following conditions:

$$(GP1) \quad y \cdot x^{-1} = 1 \iff x^{-1} \cdot y = 1,$$

$$(GP2) \quad y \cdot x^{-1} = 1 \iff x = y.$$

Proposition 2.5. *An algebra $\mathcal{G} = (G; \cdot, {}^{-1}, 1)$ of type $(2, 1, 0)$ is a goop if and only if it is an involutive moon satisfying*

$$(GP) \quad y \cdot x^{-1} = 1 \implies x = y.$$

Proof. Let \mathcal{G} be a goop. Observe that it satisfies (Iv). Indeed, let $x \in G$. We have

$$x = x \stackrel{(GP2)}{\iff} x \cdot x^{-1} = 1 \stackrel{(GP1)}{\iff} x^{-1} \cdot x = 1.$$

Therefore, (Iv) holds. By (Iv), $(x^{-1})^{-1} \cdot x^{-1} = 1$. From (GP2) we conclude that $(x^{-1})^{-1} = x$. Thus \mathcal{G} is an involutive moon. By (GP2), it satisfies (GP).

Conversely, let \mathcal{G} satisfy (U), (Iv), (DN) and (GP). Let $x, y \in G$. To prove (GP1), we first assume that $y \cdot x^{-1} = 1$. By (GP), $x = y$. Then $x^{-1} \cdot y = x^{-1} \cdot x \stackrel{(Iv)}{=} 1$. Now suppose that $x^{-1} \cdot y = 1$. Therefore, $1 = x^{-1} \cdot y \stackrel{(DN)}{=} x^{-1} \cdot (y^{-1})^{-1}$. Applying (GP), we see that $x^{-1} = y^{-1}$. Hence $y \cdot x^{-1} = y \cdot y^{-1} = 1$. Consequently, (GP1) holds. Using (GP) and (Iv), we have (GP2). Thus, \mathcal{G} is a goop. \square

Definition 2.6. ([15]) We say that an algebra $(G; \cdot, ^{-1}, 1)$ of type $(2, 1, 0)$ is a *weakly goop* if it is an involutive moon satisfying

$$(wGP) \quad y \cdot x^{-1} = 1 = x \cdot y^{-1} \implies x = y.$$

Example 2.7. The algebra $\mathcal{G} = (\{a, b, 1\}; \cdot, ^{-1}, 1)$, with

$$\begin{array}{c|ccc} \cdot & a & b & 1 \\ \hline a & a & 1 & a \\ b & 1 & 1 & b \\ 1 & a & b & 1 \end{array}$$

and $a^{-1} = b$, $b^{-1} = a$, $1^{-1} = 1$, is a commutative weakly goop. Since $b \cdot a^{-1} = b \cdot b = 1$ and $a \neq b$, \mathcal{G} does not satisfy (GP2). Therefore, it is not a goop.

Let **inv. com. moon**, **com. weakly goop**, **com. goop** and **com. group** denote the class of all involutive commutative moons, commutative weakly goops, commutative goops and commutative groups, respectively. From the definitions we obtain

$$\mathbf{inv. \ com. \ moon} \subset \mathbf{com. \ weakly \ goop} \subset \mathbf{com. \ goop} \subset \mathbf{com. \ group}$$

3. The conditions (p-s), (p-s1), (D1=) and (PS)

Let $\mathcal{A} = (A; \rightarrow, 1)$ be an algebra of type $(2, 0)$. Consider the following conditions that can be satisfied by \mathcal{A} :

- (p-s) $x \leq y \implies x = y$,
- (p-s1) $x \leq 1 \implies x = 1$,
- (D1=) $x = (x \rightarrow 1) \rightarrow 1$,
- (PS) $(x \rightarrow 1) \rightarrow y = (y \rightarrow 1) \rightarrow x$.

Note that, in [5], the concept of negation, $^{-1}$, is defined by $x^{-1} = x \rightarrow 1$, and hence

$$(D1=) \iff (x^{-1})^{-1} = x \quad \text{and} \quad (PS) \iff x^{-1} \rightarrow y = y^{-1} \rightarrow x.$$

Thus (D1=) is in fact the double negation property (DN) and (PS) is the property (pDNeg2), in the commutative case, from the book [5].

Remark that RM algebras satisfying (D1=) were studied in [15].

First we present connections between the conditions in the above list.

Lemma 3.1. *Let \mathcal{A} be an algebra of type $(2, 0)$. Then*

- (i) (M) + (PS) *imply* (D1=),
- (ii) (M) + (D1=) *imply* (p-s1),
- (iii) (An) + (Re) + (**) + (p-s1) *imply* (p-s),

- (iv) (D1) + (p-s) *imply* (D1=),
- (v) (Re) + (Ex) + (D1=) *imply* (p-s).
- (vi) (p-s) *implies* (An), (*), (**).

Proof. (i) Let $x \in A$. By (PS) and (M), $(x \rightarrow 1) \rightarrow 1 = (1 \rightarrow 1) \rightarrow x = x$, that is, (D1=) holds.

(ii) Let $x \in A$ and $x \leq 1$. Hence $x \rightarrow 1 = 1$. From (D1=) we have $x = (x \rightarrow 1) \rightarrow 1 = 1 \rightarrow 1 = 1$. Consequently, \mathcal{A} satisfies (p-s1).

(iii) Let $x, y \in A$ and $x \leq y$. By (**) and (Re), $y \rightarrow x \leq x \rightarrow x = 1$. From (p-s1) we obtain $y \rightarrow x = 1$, that is, $y \leq x$. Applying (An), we conclude that $x = y$. Then (p-s) holds.

(iv) This is immediate.

(v) Let $x, y \in A$ and suppose that $x \leq y$. Then $x \rightarrow y = 1$. Applying (Ex) and (Re), we get

$$y \rightarrow 1 = y \rightarrow (x \rightarrow y) = x \rightarrow (y \rightarrow y) = x \rightarrow 1.$$

Therefore, $x = (x \rightarrow 1) \rightarrow 1 = (y \rightarrow 1) \rightarrow 1 = y$ by (D1=). Thus (p-s) is satisfied.

(vi) Obvious. \square

Definition 3.2. An RM algebra $\mathcal{A} = (A; \rightarrow, 1)$ is *p-semisimple* ([5]) if (p-s) holds in \mathcal{A} .

Remark 3.3. From Lemma 3.1 (vi) it follows that p-semisimple RM algebras coincide with p-semisimple aRM algebras, with p-semisimple aRM** algebras, and with p-semisimple *aRM** algebras. Moreover, p-semisimple RME algebras coincide with p-semisimple BCH algebras, with p-semisimple BCH** algebras, and with p-semisimple BCI algebras.

Example 3.4. Consider the set $A = \{a, b, 1\}$ with the following table of \rightarrow :

| | | | |
|---------------|-----|-----|-----|
| \rightarrow | a | b | 1 |
| a | 1 | b | a |
| b | a | 1 | a |
| 1 | a | b | 1 |

The properties (Re), (M), (An), (*), (**) and (p-s) are satisfied. The algebra $\mathcal{A} = (A; \rightarrow, 1)$ does not satisfy (Ex) for $x = a$, $y = b$, $z = 1$. Therefore, \mathcal{A} is a p-semisimple *aRM** algebra, which is not an RME algebra (hence, it is also not a BCI algebra).

From Lemma 3.1(i) and (ii) we have

Lemma 3.5. *If an algebra \mathcal{A} verifies (M), then*

$$(PS) \implies (D1=) \implies (p-s1).$$

Proposition 3.6. *Let \mathcal{A} be an aRM^{**} algebra satisfying (D1). Then*

$$(p-s1) \iff (p-s) \iff (D1=).$$

Proof. $(p-s1) \implies (p-s)$ By Lemma 3.1 (iii).

$(p-s) \implies (D1=)$ Follows from Lemma 3.1 (iv).

$(D1=) \implies (p-s1)$ Obviously, by Lemma 3.1 (ii). \square

Proposition 3.7. ([16]) *Let \mathcal{A} be a BCI algebra. Then*

$$(p-s) \iff (p-s1) \iff (D1=) \iff (PS).$$

We introduced now the following definition.

Definition 3.8. Let $\mathcal{A} = (A; \rightarrow, 1)$ be an algebra of type $(2, 0)$.

9. \mathcal{A} is an $RM(PS)$ algebra if it is an RM algebra verifying (PS).
10. \mathcal{A} is an $aRM(PS)$ algebra (resp. $aRM^{**}(PS)$, $*aRM^{**}(PS)$ algebra) if it is an aRM algebra (resp. aRM ** , $*aRM^{**}$ algebra) verifying (PS).
11. \mathcal{A} is an $RME(PS)$ algebra (resp. $BCH(PS)$, $BCH^{**}(PS)$, and $BCI(PS)$ algebra) if it is an RME algebra (resp. BCH, BCH ** , and BCI algebra) verifying (PS).

Remark 3.9. From Lemma 3.1 (i)–(iii) we see that every $*aRM^{**}(PS)$ algebra is p-semisimple. The converse is not true, that is, there is a p-semisimple $*aRM^{**}$ algebra that does not satisfy (PS); see Example 3.4.

Proposition 3.10. *Let $\mathcal{A} = (A; \rightarrow, 1)$ be an algebra of type $(2, 0)$. The following are equivalent:*

- (i) \mathcal{A} is a $BCH(PS)$ algebra,
- (ii) \mathcal{A} is an $RME(PS)$ algebra,
- (iii) \mathcal{A} is a $BCH^{**}(PS)$ algebra,
- (iv) \mathcal{A} is a $BCI(PS)$ algebra,
- (v) \mathcal{A} is a p-semisimple BCI algebra.

Proof. (i) \implies (ii) This is obvious.

(ii) \implies (iii) Let \mathcal{A} verify (Re), (M), (Ex) and (PS). From Lemma 3.1 (i) and (v) we deduce that \mathcal{A} satisfies (p-s). Hence, by Lemma 3.1 (vi), (An) and $(^{**})$ hold in \mathcal{A} . Thus, \mathcal{A} is a $BCH^{**}(PS)$ algebra.

(iii) \implies (iv) By Lemma 3.1 (i)–(iii) we conclude that \mathcal{A} satisfies (p-s). From (p-s) it follows that \mathcal{A} also satisfies $(^*)$. Applying Lemma 2.1 (iv) and (v) we see that (B) holds in \mathcal{A} . Consequently, \mathcal{A} is a BCI algebra.

(iv) \implies (v) By Remark 3.9.

(v) \implies (i) This follows from Proposition 3.7. \square

Proposition 3.11. *Let $\mathcal{A} = (A; \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then \mathcal{A} is an $aRM^{**}(\mathcal{PS})$ algebra if and only if it is an $*aRM^{**}(\mathcal{PS})$ algebra.*

Proof. Let \mathcal{A} be an $aRM^{**}(\mathcal{PS})$ algebra. Then \mathcal{A} satisfies (An), (Re), (M), (**), (PS). Applying Lemma 3.1 (i)–(iii) and (vi) we deduce that (*) holds in \mathcal{A} . Thus \mathcal{A} is an $*aRM^{**}(\mathcal{PS})$ algebra. The converse is obvious. \square

Let us denote by $\mathbf{RM}(\mathcal{PS})$, $\mathbf{aRM}(\mathcal{PS})$, $\mathbf{aRM}^{**}(\mathcal{PS})$, $\mathbf{*aRM}^{**}(\mathcal{PS})$, $\mathbf{RME}(\mathcal{PS})$, $\mathbf{BCH}(\mathcal{PS})$, $\mathbf{BCH}^{**}(\mathcal{PS})$, and $\mathbf{BCI}(\mathcal{PS})$ the classes of $\mathbf{RM}(\mathcal{PS})$, $\mathbf{aRM}(\mathcal{PS})$, $\mathbf{aRM}^{**}(\mathcal{PS})$, $\mathbf{*aRM}^{**}(\mathcal{PS})$, $\mathbf{RME}(\mathcal{PS})$, $\mathbf{BCH}(\mathcal{PS})$, $\mathbf{BCH}^{**}(\mathcal{PS})$, and $\mathbf{BCI}(\mathcal{PS})$ algebras, respectively.

From Propositions 3.10 and 3.11 it follows that

$$\mathbf{RME}(\mathcal{PS}) = \mathbf{BCH}(\mathcal{PS}) = \mathbf{BCH}^{**}(\mathcal{PS}) = \mathbf{BCI}(\mathcal{PS}) = \mathbf{p-s-BCI} \quad \text{and} \\ \mathbf{aRM}^{**}(\mathcal{PS}) = \mathbf{*aRM}^{**}(\mathcal{PS}),$$

where $\mathbf{p-s-BCI}$ denotes the class of all p-semisimple BCI algebras.

Furthermore, there are inclusions:

$$\mathbf{p-s-BCI} \subset \mathbf{*aRM}^{**}(\mathcal{PS}) \subset \mathbf{aRM}(\mathcal{PS}) \subset \mathbf{RM}(\mathcal{PS}).$$

These inclusions are proper, see examples below.

Example 3.12. Consider the set $A = \{a, b, 1\}$ and the operation \rightarrow given by the following table:

| | | | |
|---------------|-----|-----|-----|
| \rightarrow | a | b | 1 |
| a | 1 | b | b |
| b | b | 1 | a |
| 1 | a | b | 1 |

It is easy to see that the properties (An), (Re), (M), (*), (**), and (PS) are satisfied; (Ex) is not satisfied for $(x, y, z) = (a, b, a)$. Hence, $\mathcal{A} = (A; \rightarrow, 1)$ is an $*aRM^{**}(\mathcal{PS})$ algebra, which is not a BCI algebra.

Example 3.13. Let $A = \{a, b, 1\}$ and \rightarrow be defined as follows:

| | | | |
|---------------|-----|-----|-----|
| \rightarrow | a | b | 1 |
| a | 1 | 1 | b |
| b | a | 1 | a |
| 1 | a | b | 1 |

The algebra $\mathcal{A} = (A; \rightarrow, 1)$ satisfies properties (An), (Re), (M), (PS). It does not satisfy (*) for $(x, y, z) = (b, a, b)$, (**) for $(x, y, z) = (a, a, b)$. Then \mathcal{A} is an $aRM(\mathcal{PS})$ algebra, which is not an $*aRM^{**}$ algebra.

Example 3.14. Consider the set $A = \{a, b, c, 1\}$ and the operation \rightarrow given by the following table:

| | | | | |
|---------------|-----|-----|-----|-----|
| \rightarrow | a | b | c | 1 |
| a | 1 | 1 | b | a |
| b | 1 | 1 | c | b |
| c | b | c | 1 | c |
| 1 | a | b | c | 1 |

Then the properties (Re), (M), (PS) are satisfied. (An) is not satisfied for $(x, y) = (a, b)$. Therefore, $\mathcal{A} = (A; \rightarrow, 1)$ is an $\text{RM}_{(\text{PS})}$ algebra, which is not an aRM algebra.

4. $\text{RM}_{(\text{PS})}$ algebras and involutive commutative moons

In this section, we establish the connections between:

- $\text{RM}_{(\text{PS})}$ algebras and involutive commutative moons,
- a $\text{RM}_{(\text{PS})}$ algebras and commutative weakly goops,
- *a $\text{RM}^{**}_{(\text{PS})}$ algebras and commutative goops.

Note that connections between goops and implicative-goops are presented in Chapter 12 of [5].

Let $\mathcal{A} = (A; \rightarrow, 1)$ be an algebra of type $(2, 0)$. Define $\Phi(\mathcal{A}) = (A; \cdot, ^{-1}, 1)$ by: for all $x, y \in A$, $x \cdot y = (y \rightarrow 1) \rightarrow x$ and $x^{-1} = x \rightarrow 1$.

Let $\mathcal{G} = (G; \cdot, ^{-1}, 1)$ be an algebra of type $(2, 1, 0)$. Define $\Psi(\mathcal{G}) = (G; \rightarrow, 1)$ by: for all $x, y \in G$, $x \rightarrow y = y \cdot x^{-1}$.

Lemma 4.1. *Let $\mathcal{A} = (A; \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then*

- (i) (M) and (D1=) imply (U) in $\Phi(\mathcal{A})$,
- (ii) (Re) and (D1=) imply (Iv) in $\Phi(\mathcal{A})$,
- (iii) (D1=) implies (DN) in $\Phi(\mathcal{A})$,
- (iv) (An) implies (wGP) in $\Phi(\mathcal{A})$,
- (v) (Re), (D1=), and (p-s) imply (GP) in $\Phi(\mathcal{A})$,
- (vi) (M) and (PS) imply (Com) in $\Phi(\mathcal{A})$.

Proof. (i) Let $x \in A$. By (M), $x \cdot 1 = (1 \rightarrow 1) \rightarrow x = x$. Applying (D1=), we get $1 \cdot x = (x \rightarrow 1) \rightarrow 1 = x$. Therefore, $\Phi(\mathcal{A})$ satisfies (U).

(ii) From (D1=) and (Re) we obtain

$$x \cdot x^{-1} = ((x \rightarrow 1) \rightarrow 1) \rightarrow x = x \rightarrow x = 1 \quad \text{and}$$

$$x^{-1} \cdot x = (x \rightarrow 1) \rightarrow (x \rightarrow 1) = 1.$$

Then (Iv) holds in $\Phi(\mathcal{A})$.

(iii) and (iv) are obvious.

(v) To prove (GP), let $x, y \in A$ and $y \cdot x^{-1} = 1$. Then $x \rightarrow y = ((x \rightarrow 1) \rightarrow 1) \rightarrow y = y \cdot x^{-1} = 1$. Hence $x \leq y$. By (p-s), $x = y$. Therefore (GP) holds in $\Phi(\mathcal{A})$.

(vi) Let (M) and (PS) are satisfied and let $x, y \in A$. From Lemma 3.1(i) we obtain (D1=). Consequently, (DN) holds in $\Phi(\mathcal{A})$. We have

$$x \cdot y \stackrel{\text{(DN)}}{=} x \cdot (y^{-1})^{-1} = y^{-1} \rightarrow x = (y \rightarrow 1) \rightarrow x \stackrel{\text{(PS)}}{=} (x \rightarrow 1) \rightarrow y = y \cdot x,$$

that is, (Com) is satisfied in $\Phi(\mathcal{A})$. \square

Lemma 4.2. *Let $\mathcal{G} = (G; \cdot, ^{-1}, 1)$ be an algebra of type $(2, 1, 0)$. Then*

- (i) (Iv) implies (Re) in $\Psi(\mathcal{G})$,
- (ii) (U) and (Iv) imply (M) in $\Psi(\mathcal{G})$,
- (iii) (Com) and (DN) imply (PS) in $\Psi(\mathcal{G})$,
- (iv) (wGP) implies (An) in $\Psi(\mathcal{G})$.

Proof. (i) Let $x \in G$. By (Iv), $x \rightarrow x = x \cdot x^{-1} = 1$, that is, (Re) holds in $\Psi(\mathcal{G})$.

(ii) Applying (U) and (Iv), we get

$$1 \rightarrow x = x \cdot 1^{-1} = x \cdot (1 \cdot 1^{-1}) = x \cdot 1 = x,$$

i.e., $\Psi(\mathcal{G})$ satisfies (M).

(iii) Let $x, y \in G$. We have

$$(x \rightarrow 1) \rightarrow y = y \cdot (x^{-1})^{-1} \stackrel{\text{(DN)}}{=} y \cdot x \stackrel{\text{(Com)}}{=} x \cdot y = (y \rightarrow 1) \rightarrow x,$$

that is, (PS) holds in $\Psi(\mathcal{G})$.

(iv) is immediate. \square

Theorem 4.3. (i) *Let $\mathcal{A} = (A; \rightarrow, 1)$ is an $RM(\text{ps})$ algebra. Then $\Phi(\mathcal{A}) = (A; \cdot, ^{-1}, 1)$ is an involutive commutative moon.*

(ii) *Let $\mathcal{G} = (G; \cdot, ^{-1}, 1)$ is an involutive commutative moon. Then $\Psi(\mathcal{G}) = (G; \rightarrow, 1)$ is an $RM(\text{ps})$ algebra.*

(iii) *Given \mathcal{A} and \mathcal{G} as above we have $\Psi\Phi(\mathcal{A}) = \mathcal{A}$ and $\Phi\Psi(\mathcal{G}) = \mathcal{G}$.*

Proof. (i) Let \mathcal{A} be an $RM(\text{ps})$ algebra. Then \mathcal{A} satisfies (Re), (M), (PS) and, by Lemma 3.1(i), also satisfies (D1=). From Lemma 4.1 it follows that (U), (Iv), (DN) and (Com) hold in $\Phi(\mathcal{A})$. Therefore, $\Phi(\mathcal{A})$ is an involutive commutative moon.

(ii) Let \mathcal{G} verify (U), (Iv), (DN) and (Com). By Lemma 4.2(i)–(iii), $\Psi(\mathcal{G})$ verifies (Re), (M) and (PS). Hence $\Psi(\mathcal{G})$ is an $RM(\text{ps})$ algebra.

(iii) Suppose $\mathcal{A} = (A; \rightarrow, 1)$ is an $RM(\text{ps})$ algebra and $x \in A$. Then, in $\Phi(\mathcal{A}) = (A; \cdot, ^{-1}, 1)$, we obtain $x^{-1} = x \rightarrow 1$ and

$$y \cdot x^{-1} = y \cdot (x \rightarrow 1) = ((x \rightarrow 1) \rightarrow 1) \rightarrow y = x \rightarrow y.$$

Thus $\Psi\Phi(\mathcal{A}) = \mathcal{A}$.

Next suppose $\mathcal{G} = (G; \cdot, ^{-1}, 1)$ is an involutive moon and $x, y \in G$. Then, in $\Psi(\mathcal{G}) = (G; \rightarrow, 1)$, we get $x \rightarrow 1 = 1 \cdot x^{-1} = x^{-1}$ and

$$(y \rightarrow 1) \rightarrow x = y^{-1} \rightarrow x = x \cdot (y^{-1})^{-1} = x \cdot y.$$

Thus $\Phi\Psi(\mathcal{G}) = \mathcal{G}$. □

Hence, by above Theorem 4.3, we have the equivalence

$$\mathbf{RM}(\mathbf{ps}) \iff \mathbf{inv. \ com. \ moon.}$$

Theorem 4.4. (i) Let $\mathcal{A} = (A; \rightarrow, 1)$ be an $\mathbf{aRM}(\mathbf{ps})$ algebra ($\mathbf{*aRM}^{**}(\mathbf{ps})$ algebra). Then $\Phi(\mathcal{A}) = (A; \cdot, ^{-1}, 1)$ is a commutative weakly goop (commutative goop, respectively).

(ii) Let $\mathcal{G} = (G; \cdot, ^{-1}, 1)$ be a commutative weakly goop (commutative goop). Then $\Psi(\mathcal{G}) = (G; \rightarrow, 1)$ is an $\mathbf{aRM}(\mathbf{ps})$ algebra ($\mathbf{*aRM}^{**}(\mathbf{ps})$ algebra, respectively).

(iii) Given \mathcal{A} and \mathcal{G} as above we have $\Psi\Phi(\mathcal{A}) = \mathcal{A}$ and $\Phi\Psi(\mathcal{G}) = \mathcal{G}$.

Proof. (i) Let \mathcal{A} be an $\mathbf{aRM}(\mathbf{ps})$ algebra. By Theorem 4.3, $\Phi(\mathcal{A})$ is an involutive commutative moon. Since (An) implies (wGP) (by Lemma 4.1(iv)), $\Phi(\mathcal{A})$ is a weakly goop. Therefore, it is a commutative weakly goop. If \mathcal{A} is an $\mathbf{*aRM}^{**}(\mathbf{ps})$ algebra then, by Lemma 3.1(i)–(iii), \mathcal{A} satisfies (p-s). From Lemma 4.1(v) we see that (GP) holds in $\Phi(\mathcal{A})$. Thus $\Phi(\mathcal{A})$ is a goop by Proposition 2.5. Finally, $\Phi(\mathcal{A})$ is a commutative goop.

(ii) Let \mathcal{G} be a commutative weakly goop. Since \mathcal{G} is an involutive commutative moon, $\Psi(\mathcal{G})$ is an $\mathbf{RM}(\mathbf{ps})$ algebra by Theorem 4.3. From Lemma 4.2 (iv) we deduce that (An) holds in $\Psi(\mathcal{G})$. Thus $\Psi(\mathcal{G})$ is an $\mathbf{aRM}(\mathbf{ps})$ algebra.

Now let \mathcal{G} be a commutative goop. From (GP) it follows that (p-s) holds in $\Psi(\mathcal{G})$. Hence, $\Psi(\mathcal{G})$ satisfies (*) and (**). Thus $\Psi(\mathcal{G})$ is an $\mathbf{aRM}^{**}(\mathbf{ps})$ algebra.

(iii) See the proof of Theorem 4.3 (iii). □

Hence, by above Theorem 4.4, we have the equivalences:

$$\mathbf{aRM}(\mathbf{ps}) \iff \mathbf{com. \ weakly \ goop} \quad \text{and} \quad \mathbf{*aRM}^{**}(\mathbf{ps}) \iff \mathbf{com. \ goop.}$$

Theorem 4.5. ([9]) If $\mathcal{A} = (A; \rightarrow, 1)$ is a p -semisimple BCI algebra (or, equivalently, $\mathbf{BCH}^{**}(\mathbf{ps})$ algebra), then $\Phi(\mathcal{A}) = (A; \cdot, ^{-1}, 1)$ is a commutative group. Conversely, if $\mathcal{G} = (G; \cdot, ^{-1}, 1)$ is a commutative group, then $\Psi(\mathcal{G}) = (G; \rightarrow, 1)$ is a p -semisimple BCI algebra.

Hence, by Theorem 4.5, we have the equivalence

$$\mathbf{p-s-BCI} = \mathbf{BCH}^{**}(\mathbf{ps}) = \mathbf{BCH}(\mathbf{ps}) = \mathbf{RME}(\mathbf{ps}) \iff \mathbf{com. \ group.}$$

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Andrzej Walendziak

Siedlce University of Natural Sciences and Humanities

Faculty of Exact and Natural Sciences

3 Maja 54, PL-08110 Siedlce, Poland

e-mail: walent@interia.pl