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MODULES FOR WHICH EVERY ENDOMORPHISM HAS A NON-TRIVIAL INVARIANT SUBMODULE

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Abstract. All rings are commutative. Let M be a module. We introduce the property (**P**): Every endomorphism of M has a non-trivial invariant submodule. We determine the structure of all vector spaces having (**P**) over any field and all semisimple modules satisfying (**P**) over any ring. Also, we provide a structure theorem for abelian groups having this property. We conclude the paper by characterizing the class of rings for which every module satisfies (**P**) as that of the rings R for which $R/\mathfrak{m}$ is an algebraically closed field for every maximal ideal $\mathfrak{m}$ of R.

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1. Introduction

The notion studied in this article has its roots in operator theory. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable infinite dimensional complex Hilbert space $\mathcal{H}$. A closed subspace M of $\mathcal{H}$ is said to be a non-trivial invariant subspace for $T \in \mathcal{B}(\mathcal{H})$ if $M \neq 0$, $M \neq \mathcal{H}$ and $T(M) \subseteq M$. The invariant subspace problem can be stated as follows:

Does every bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ have a non-trivial invariant closed subspace?

A number of research papers have been devoted to the study of this conjecture which is still open. Following [1], the research on this problem was initiated by J. von Neumann who proved in the early thirties of the last century that every linear compact operator on a Hilbert space has a non-trivial invariant closed subspace. The proof of this result was never published. Later in 1954, Aronszajn and Smith [1] extended von Neumann's result to the Banach spaces setting. In 1966 [2], Bernstein and Robinson proved that every polynomially compact operator T on a

Hilbert space (i.e., $P(T)$ is compact for some nonzero polynomial P) has a non-trivial invariant subspace. In 1973 [5], Lomonosov showed that every bounded linear operator on a complex Banach space which commutes with a nonzero compact operator has a non-trivial invariant closed subspace. Further details about the developments on the above conjecture can be found in [4].

In this paper, we examine this problem from an algebraic point of view by extending it to a module theoretic version. Let R be a commutative ring and let M be an R-module. A submodule N of M is said to be invariant under an R-endomorphism f of M if $f(N) \subseteq N$. The module M is said to have property (**P**) if every R-endomorphism f of M has a non-trivial invariant submodule. The focus of our investigations is to explore and study modules satisfying (**P**).

In Section 2, we prove that every infinitely generated semisimple module has (**P**) (Proposition 2.12). It is shown that for a commutative field K, a K-vector space V with $dim(V) = n \geq 2$ satisfies (**P**) if and only if every monic polynomial $P(X) \in K[X]$ of degree n is reducible (Theorem 2.6). Also, we determine the structure of abelian groups having (**P**) (Theorem 2.21). Some examples are provided to show that even a semisimple module needs not have (**P**), in general.

In the main result of Section 3, we characterize the class of rings R for which every nonzero finitely generated R-module which is not simple has (**P**). It turns out that this class of rings is precisely that of rings R for which $R/\mathfrak{m}$ is an algebraically closed field for every maximal ideal $\mathfrak{m}$ of R (Theorem 3.4).

Throughout this article, all rings are commutative with identity and all modules are unital. Let R be a ring and let M be an R-module. A submodule L of M is called *non-trivial* if $L \neq 0$ and $L \neq M$. We use $Rad(M)$, $Soc(M)$, and $End_R(M)$ to denote the radical, the socle, and the endomorphism ring of M, respectively. The notation $N \subseteq M$ means that N is a subset of M and the notation $N \leq M$ means that N is a submodule of M. By $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ we denote the ring of rational numbers, the ring of integer numbers, and the set of natural numbers, respectively.

2. Modules having (P)

Proposition 2.1. *The following are equivalent for a module M:*

(i) *Every endomorphism of M has a non-trivial invariant submodule (i.e., $\forall f \in End_R(M)$, $\exists 0 \neq N \leq M$ such that $N \neq M$ and $f(N) \subseteq N$);*

(ii) *Every automorphism of M has a non-trivial invariant submodule.*

Proof. (i) $\Rightarrow$ (ii) This is immediate.

(ii) $\Rightarrow$ (i) Let f be a nonzero endomorphism of M which is not an automorphism. Then $Kerf \neq 0$ or $Imf \neq M$. Note that $Kerf \neq M$ and $Imf \neq 0$. If $Kerf \neq 0$, then $Kerf$ is a non-trivial invariant submodule under f. If $Imf \neq M$, then Imf is a non-trivial invariant submodule under f. □

Definition 2.2. A module M is said to have property (**P**) if it satisfies any of the equivalent two conditions in Proposition 2.1.

Recall that a submodule N of a module M is called *fully invariant* if $f(N) \subseteq N$ for every endomorphism f of M. It is well known that for any module M, $Soc(M)$ and $Rad(M)$ are fully invariant submodules of M. In [6], the authors studied *duo modules* (i.e., modules in which every submodule is fully invariant). It is clear that every nonzero duo module which is not simple satisfies (**P**). So for every commutative ring R which is not a field, the R-module R has (**P**).

Example 2.3. (i) It is clear that every module having a non-trivial fully invariant submodule has (**P**). In particular, every module M with non-trivial radical or non-trivial socle has (**P**).

(ii) Let M be an artinian module which is not semisimple. Then $Soc(M) \neq M$. Moreover, it is well known that $Soc(M) \neq 0$. Hence M has (**P**).

To explore modules having (**P**), it is natural to begin by investigating vector spaces over a field and semisimple modules.

Proposition 2.4. *Let K be a field. Every infinite-dimensional K-vector space has* (**P**).

Proof. Let V be a K-vector space of infinite dimension and let T be an automorphism of V. Suppose that the only invariant subspaces of V under T are 0 and V. Let $0 \neq u \in V$ and consider the nonzero subspace W of V generated by the family $\{T^k(u), k \geq 1\}$. Clearly, $T(W) \subseteq W$. Therefore $W = V$ and hence $u \in W$. So there exists $p \geq 1$ such that $u = \alpha_1 T(u) + \alpha_2 T^2(u) + \cdots + \alpha_p T^p(u)$, where $\alpha_1, \alpha_2, \ldots, \alpha_p \in K$ and $\alpha_p \neq 0$. It follows that $T^p(u)$ belongs to the nonzero subspace L of V generated by the family $\{u, T(u), \ldots, T^{p-1}(u)\}$. This implies that $T(L) \subseteq L$. Note that L is a finitely generated subspace of V. Thus $L \neq V$. This is a contradiction. Consequently, V contains a non-trivial subspace which is invariant under T. □

Next, we characterize finite-dimensional vector spaces having (**P**). We begin with the following well known remark which is included for completeness.

Remark 2.5. Let K be a field and let $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in K[X]$ be a polynomial of degree $n \geq 2$. The companion matrix of the polynomial $P(X)$ is the $n \times n$ matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \in M_n(K).$$

Let T be an endomorphism of K^n such that M is the matrix of T with respect to the standard basis. It is easy to check that the characteristic polynomial of T is $(-1)^n P(X)$.

Theorem 2.6. *Let K be a field and let $n \geq 2$ be an integer. Then the following statements are equivalent:*

(i) *The K-vector space $U = K^n$ has* (**P**)*;*

(ii) *Every monic polynomial $P(X) \in K[X]$ of degree n is reducible.*

Proof. (i) $\Rightarrow$ (ii) Assume that U has (**P**). Let $P(X)$ be a monic polynomial in $K[X]$ of degree n. By the preceding remark, there exists a nonzero endomorphism f of U such that the characteristic polynomial of f is $(-1)^n P(X)$. Then U contains a non-trivial subspace V such that $f(V) \subseteq V$. Set $h = dim(V)$. Note that $1 \leq h \leq n-1$. Moreover, there exists a subspace W of V such that $U = V \oplus W$ and $dim(W) = n - h$. Let $\mathcal{B}_1 = \{e_1, e_2, \ldots, e_h\}$ be a basis for V and let g be the restriction of f to V. We denote by A_1 the matrix of g with respect to the basis $\mathcal{B}_1$. Let $\{e_{h+1}, \ldots, e_n\}$ be a basis for W. Then $\mathcal{B}_2 = \{e_1, e_2, \ldots, e_n\}$ is a basis for U. It is easily seen that the matrix A of f with respect to the basis $\mathcal{B}$ has the form

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}.$$

Let $P_A(X)$ and $P_{A_i}(X)$ ($i \in \{1, 3\}$) be the characteristic polynomials of the matrices A and A_i ($i \in \{1, 3\}$), respectively. Then $P_A(X) = P_{A_1}(X)P_{A_3}(X)$. It follows that $(-1)^n P(X) = P_{A_1}(X)P_{A_3}(X)$. That is, $P(X) = ((-1)^n P_{A_1}(X))P_{A_3}(X)$. Note that $deg(P_{A_i}(X)) \geq 1$ for each $i \in \{1, 3\}$.

(ii) $\Rightarrow$ (i) Let T be an automorphism of U and let $P(X)$ be the characteristic polynomial of T. We denote by $A = (\alpha_{ij})_{1 \leq i,j \leq n}$ the matrix of T with respect to the standard basis. By (ii), there exists a monic irreducible polynomial $Q(X)$ which divides $P(X)$ such that $q = deg(Q(X)) \neq n$. It is well known that K has an extension field L (which is isomorphic to $K[X]/\langle Q(X)\rangle$) such that $[L:K]=q$ and $Q(X)$ has a root λ in L. Let $\{\sigma_1,\sigma_2,\ldots,\sigma_q\}$ be a basis of the K-vector space L. For all i, j in $\{1,\ldots,q\}$, there exist $\gamma_{ij}^t\in K$ $(1\le t\le q)$ such that $\sigma_i\sigma_j=\sum_{t=1}^q\gamma_{ij}^t\sigma_t$. Also, there exist $\lambda_i\in K$ $(1\le i\le q)$ such that $\lambda=\sum_{i=1}^q\lambda_i\sigma_i$. Since λ is a root of $P(X)$ in L, there exists $0\neq v=\begin{bmatrix} b_1\\ \vdots\\ b_n\end{bmatrix}\in V=L^n$ such that $Av=\lambda v$. Note that for every $i\in\{1,\ldots,n\}$, there exist $\beta_{is}\in K$ $(1\le s\le q)$ such that $b_i=\sum_{s=1}^q\beta_{is}\sigma_s$. Fix $l\in\{1,\ldots,n\}$. We have $\sum_{j=1}^n\alpha_{lj}b_j=\lambda b_l$. Hence,

$$\sum_{j=1}^n\alpha_{lj}(\sum_{s=1}^q\beta_{js}\sigma_s)=\lambda\sum_{j=1}^q\beta_{lj}\sigma_j.$$

That is,

$$\sum_{s=1}^q(\sum_{j=1}^n\alpha_{lj}\beta_{js})\sigma_s=\sum_{i=1}^q\sum_{j=1}^q\lambda_i\beta_{lj}(\sigma_i\sigma_j).$$

Therefore,

$$\sum_{s=1}^q(\sum_{j=1}^n\alpha_{lj}\beta_{js})\sigma_s=\sum_{i=1}^q\sum_{j=1}^q\lambda_i\beta_{lj}(\sum_{s=1}^q\gamma_{ij}^s\sigma_s).$$

i.e.,

$$\sum_{s=1}^q(\sum_{j=1}^n\alpha_{lj}\beta_{js})\sigma_s=\sum_{s=1}^q(\sum_{j=1}^q(\sum_{i=1}^q\lambda_i\gamma_{ij}^s)\beta_{lj})\sigma_s.$$

It follows that

$$\textstyle\sum_{j=1}^n\alpha_{lj}\beta_{js}=\sum_{j=1}^q(\sum_{i=1}^q\lambda_i\gamma_{ij}^s)\beta_{lj}\ \text{for every } s\in\{1,\ldots,q\}.$$

For every j,s in $\{1,\ldots,q\}$, set $u_j=\begin{bmatrix}\beta_{1j}\\ \vdots\\ \beta_{nj}\end{bmatrix}\in U=K^n$ and $\varepsilon_{js}=\sum_{i=1}^q\lambda_i\gamma_{ij}^s\in K$. A trivial verification shows that $Au_s=\sum_{j=1}^q\varepsilon_{js}u_j$ for all $s\in\{1,\ldots,q\}$. This implies that the K-subspace $H=\langle u_1,u_2,\ldots,u_q\rangle$ of U generated by $\{u_1,u_2,\ldots,u_q\}$ is invariant under T. This completes the proof. □

To visualize the proof of the previous theorem, we provide the following example.

Example 2.7. Let $\mathbb{R}$ denote the field of real numbers. Consider the $\mathbb{R}$-vector space $U=\mathbb{R}^4$ and the polynomial $P(X)=(X^2+1)^2\in\mathbb{R}[X]$. Then the companion matrix of $P(X)$ is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix}.$$

It is clear that $P(X)$ est divisible by $Q(X) = X^2 + 1$. Moreover, it is well known that the field $\mathbb{C}$ of complex numbers is an extension field of $\mathbb{R}$ such that $[\mathbb{C} : \mathbb{R}] = 2 = deg(Q(X))$. So, if we regard A as a matrix over $\mathbb{C}$ then A has two complex eigenvalues, namely i and $-i$, and $v = \begin{bmatrix} i \\ -1 \\ -i \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue i. With the notation of the proof of Theorem 2.6, the coefficients β_{ij} have the values $\beta_{11} = 0$, $\beta_{21} = -1$, $\beta_{31} = 0$, $\beta_{41} = 1$, $\beta_{12} = 1$, $\beta_{22} = 0$, $\beta_{32} = -1$ and $\beta_{42} = 0$ so that the vectors $v_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ generate a subspace of U that is invariant under A.

Corollary 2.8. *Let K be a finite field and let $n \geq 2$ be an integer. Then the K-vector space K^n never has* (**P**).

Proof. Since K is finite, there exists an irreducible polynomial $Q(X) \in K[X]$ with $deg(Q(X)) = n$ (see [3, Corollary 2.11]). From Theorem 2.6, we deduce that the K-vector space K^n does not have (**P**). □

Proposition 2.9. *Let K be a field and let n be an integer with $n \geq 2$. Then the following are equivalent:*

(i) *Every K-vector space of dimension t $(2 \leq t \leq n)$ has* (**P**)*;*
(ii) *Every monic polynomial $P(X) \in K[X]$ of degree t $(2 \leq t \leq n)$ has a root in the field K.*

Proof. (i) $\Rightarrow$ (ii) Let t_1 be an integer with $2 \leq t_1 \leq n$. Let $P_1(X) = X^{t_1} + a_{t_1-1}X^{t_1-1} + \cdots + a_1X + a_0 \in K[X]$ be a monic polynomial of degree t_1. Using Remark 2.5, there exists an endomorphism T_1 of K^{t_1} such that the characteristic polynomial of T_1 is $(-1)^{t_1}P_1(X)$. By Theorem 2.6, $P_1(X) = P_2(X)Q_2(X)$ where $P_2(X)$, $Q_2(X) \in K[X]$ with $1 \leq t_2 = deg(P_2(X)) < t_1$. Repeating this procedure, we show that $P_1(X) = P(X)Q(X)$ where $P(X), Q(X) \in K[X]$ and $deg(P(X)) = 1$. This shows that $P_1(X)$ has a root in K.

(ii) $\Rightarrow$ (i) Let V be a nonzero K-vector space with $2 \leq dim(V) = t \leq n$ and let T be an automorphism of V. By hypothesis, the characteristic polynomial $P_T(X)$ of T has a root λ in K. Therefore there exists a nonzero $u \in V$ such that $T(u) = \lambda u$. Let $W = Ku$ be the subspace of V generated by $\{u\}$. It is clear that $T(W) \subseteq W$. Note that $W \neq 0$ and $W \neq V$. So V has (**P**). □

Next, we determine semisimple modules satisfying (**P**). Recall that a module M is called *homogeneous semisimple* if it is generated by a single simple module; that is, M is a direct sum of simple modules which are isomorphic to each other.

Example 2.10. Consider the semisimple $\mathbb{Z}$-module $M = (\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$. Note that $Hom_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0$. Then $N = (\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \oplus 0$ is a fully invariant submodule of M by [6, Lemma 1.9]. It follows that M has (**P**). In the same manner we can see that every semisimple module which is not homogeneous has (**P**).

Proposition 2.11. *Let I be an ideal of a commutative ring R and let M be an R/I-module. Then the R-module ${}_RM$ has* (**P**) *if and only if the R/I-module ${}_{R/I}M$ has* (**P**).

Proof. Let M be a nonsimple R/I-module. Then M is an R-module and the lattices of R-submodules and R/I-submodules of M coincide. Moreover, any group endomorphism of M is an R-endomorphism of M if and only if it is an R/I-endomorphism of M. The result follows. □

Proposition 2.12. *Every infinitely generated semisimple module has* (**P**).

Proof. Let M be an infinitely generated semisimple module. By Example 2.10, there is no loss of generality in assuming that M is homogeneous semisimple. Therefore $M \cong (R/\mathfrak{m})^{(\Lambda)}$ for some maximal ideal $\mathfrak{m}$ of R and an infinite index set Λ. Then M can be viewed as an $R/\mathfrak{m}$-module. By Proposition 2.4, the $R/\mathfrak{m}$-module M has (**P**). Thus ${}_RM$ satisfies (**P**) by Proposition 2.11. □

In the following proposition, we characterize finitely generated homogeneous semisimple modules which have (**P**).

Proposition 2.13. *Let M be a homogeneous semisimple R-module such that $M \cong (R/\mathfrak{m})^n$ for some maximal ideal $\mathfrak{m}$ of R and some positive integer $n \geq 2$. Let $K = R/\mathfrak{m}$. Then the following are equivalent:*

(i) *M has $(\mathbf{P})$ as an R-module;*

(ii) *M has $(\mathbf{P})$ as a K-module;*

(iii) *Every monic polynomial $P(X) \in K[X]$ of degree n is reducible.*

Proof. This follows from Theorem 2.6 and Proposition 2.11. □

The next corollary follows easily from Proposition 2.13.

Corollary 2.14. *Let a module $M = S_1 \oplus S_2$ be a direct sum of two simple submodules S_1 and S_2 such that $S_1 \cong S_2 \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. Then the following are equivalent:*

(i) *M has $(\mathbf{P})$;*

(ii) *Every monic polynomial $P(X) \in K[X]$ of degree 2 has a root in the field $K = R/\mathfrak{m}$.*

A direct summand of a module having $(\mathbf{P})$ may not have $(\mathbf{P})$, in general, as shown below.

Example 2.15. Consider the $\mathbb{Z}$-modules $M_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $M_2 = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $M = M_1 \oplus M_2$.

(i) Let $K_2 = \mathbb{Z}/2\mathbb{Z}$ and let the polynomial $P_1(X) = X^2 - X + 1 \in K_2[X]$. It is clear that $P_1(X)$ does not have a root in K_2. Thus M_1 does not have $(\mathbf{P})$ by Corollary 2.14.

(ii) Consider the polynomial $P_2(X) = X^2 + X + 2 \in K_3[X]$, where $K_3 = \mathbb{Z}/3\mathbb{Z}$. It is easy to check that $P_2(X)$ does not have a root in K_3. From Corollary 2.14, it follows that the module M_2 does not have $(\mathbf{P})$.

(iii) From Example 2.10, we conclude that the module M has $(\mathbf{P})$. Also, note that both $M_1^{(\mathbb{N})}$ and $M_2^{(\mathbb{N})}$ have $(\mathbf{P})$ by Proposition 2.12.

The next result is a direct consequence of Corollary 2.8 and Proposition 2.13.

Corollary 2.16. *Let $\mathfrak{m}$ be a maximal ideal of a commutative ring R such that $R/\mathfrak{m}$ is a finite field (for instance, R can be the ring of integers $\mathbb{Z}$ and $\mathfrak{m} = p\mathbb{Z}$ for some prime number p). Then for any positive integer $n \geq 2$, the R-module $M = (R/\mathfrak{m})^n$ does not have $(\mathbf{P})$.*

The next proposition provides more examples of modules having $(\mathbf{P})$ over a commutative ring.

Proposition 2.17. *Let R be a commutative ring. Let M be an R-module which is not semisimple such that $Rad(M) \neq M$. Then M has a nonzero proper submodule N that is fully invariant in M. In particular, M has $(\mathbf{P})$.*

Proof. Let Ω denote the set of all maximal ideals of R. It is well known that $Rad(M) = \bigcap_{\mathfrak{m}\in\Omega} M\mathfrak{m}$. Note that $M\mathfrak{m} \neq 0$ for every $\mathfrak{m} \in \Omega$, since otherwise M will be semisimple. In addition, since $Rad(M) \neq M$, there exists a maximal ideal $\mathfrak{m}_0$ of R such that $M\mathfrak{m}_0 \neq M$. Take $N = M\mathfrak{m}_0$. It is easily seen that N is fully invariant in M. □

Using Example 2.10 and Proposition 2.17, we get the following result.

Corollary 2.18. *Let M be a nonzero finitely generated module. If M is not homogeneous semisimple, then M has* (**P**).

Next, we determine all abelian groups which have (**P**). Let $\mathbb{Q}$ denote the field of rational numbers.

Proposition 2.19. Every direct sum of copies of the $\mathbb{Z}$-module $\mathbb{Q}$ has (**P**).

Proof. Case 1: Assume that $M = \mathbb{Q}^{(I)}$ where I is an infinite index set. Notice that M has a structure of a $\mathbb{Q}$-module defined by the following operation: given $x \in M$, $r \in \mathbb{Z}$ and $0 \neq s \in \mathbb{Z}$, we put $(r/s)x = rx'$ with x' is the unique element of M which satisfies $x = sx'$. Note that x' exists and is unique because M is a divisible torsion-free $\mathbb{Z}$-module. It is easily seen that $End_{\mathbb{Z}}(M) = End_{\mathbb{Q}}(M)$. Also, it is clear that every $\mathbb{Q}$-submodule of M is a $\mathbb{Z}$-submodule of M. Applying Proposition 2.4, it follows that M has (**P**) as a $\mathbb{Q}$-module and hence also as a $\mathbb{Z}$-module.

Case 2: Assume that $M = M_1 \oplus M_2$ such that $M_i = \mathbb{Q}$ for each $i = 1, 2$. It is well known that for any $\mathbb{Z}$-endomorphism φ of $\mathbb{Q}$, there exists a nonzero $q \in \mathbb{Q}$ such that $\varphi(x) = qx$ for all $x \in \mathbb{Q}$. Now let f be a nonzero $\mathbb{Z}$-endomorphism of M. So there exist integers a_1, a_2, c_1 and c_2 and nonzero integers b_1, b_2, d_1 and d_2 such that for every $(x_1, x_2) \in \mathbb{Q}^2$, we have $f((x_1, x_2)) = ((a_1/b_1)x_1 + (c_1/d_1)x_2, (a_2/b_2)x_1 + (c_2/d_2)x_2)$. Let p be a prime integer which does not divide $b_1d_1b_2d_2$. Consider the non-trivial $\mathbb{Z}$-submodule $L = \{m/n \mid p \text{ does not divide } n\}$ of $\mathbb{Q}$. Set $N = N_1 \oplus N_2$ such that $N_i = L$ for each $i = 1, 2$. Then N is a non-trivial submodule of M that is invariant under f. This shows that M has (**P**). In the same manner we can see that every finite direct sum of copies of $\mathbb{Q}$ satisfies (**P**). □

Remark 2.20. Consider the $\mathbb{Z}$-module $M = \mathbb{Q}^2$. Let $P(X) = X^2 - 2 \in \mathbb{Q}[X]$. It is clear that $P(X)$ does not have a root in $\mathbb{Q}$. By Corollary 2.14, M considered as a $\mathbb{Q}$-module does not have (**P**). On the other hand, M viewed as a $\mathbb{Z}$-module satisfies (**P**) by Proposition 2.19.

Theorem 2.21. *The following are equivalent for a $\mathbb{Z}$-module M:*

(i) *M has $(\mathbf{P})$;*

(ii) *M satisfies any one of the following conditions:*

(a) *M is not semisimple, or*

(b) *M is a semisimple module which is infinitely generated or not homogeneous.*

Proof. (i) $\Rightarrow$ (ii) From Example 2.10, Proposition 2.12 and Corollary 2.16, it follows that a semisimple $\mathbb{Z}$-module has $(\mathbf{P})$ if and only if it is infinitely generated or not homogeneous. Now assume that M is not semisimple.

Case 1: $Rad(M) \neq M$. In this case M has $(\mathbf{P})$ by Proposition 2.17.

Case 2: $Rad(M) = M$ and $Soc(M) \neq 0$. Since M is not semisimple, we have $Soc(M) \neq M$. Hence $Soc(M)$ is a non-trivial fully invariant submodule of M. This clearly implies that M has $(\mathbf{P})$.

Case 3: $Rad(M) = M$ and $Soc(M) = 0$. In this case M is a divisible torsion-free $\mathbb{Z}$-module. Hence M is isomorphic to a direct sum of copies of $\mathbb{Q}$. Therefore M has $(\mathbf{P})$ by Proposition 2.19. □

3. Rings whose modules satisfy (P)

The aim of this section is to characterize the class of rings R over which every nonzero finitely generated R-module M which is not simple satisfies $(\mathbf{P})$. Let R be a commutative ring and consider the following properties:

$(\mathbf{P_1})$: Every nonzero finitely generated R-module M which is not simple satisfies $(\mathbf{P})$.

$(\mathbf{P_2})$: Every nonsimple R-module M with $Rad(M) \neq M$ satisfies $(\mathbf{P})$.

$(\mathbf{P_3})$: Every nonzero R-module M which is not simple satisfies $(\mathbf{P})$.

Recall that a field K is called an *algebraically closed field* if any polynomial in $K[X]$ of degree $n \geq 1$ has at least one root in K.

Proposition 3.1. *Let K be a field. Then the following are equivalent:*

(i) *K satisfies $(\mathbf{P_1})$;*

(ii) *K satisfies $(\mathbf{P_2})$;*

(iii) *K satisfies $(\mathbf{P_3})$;*

(iv) *K is algebraically closed.*

Proof. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are immediate.

(i) $\Rightarrow$ (iv) This follows from Proposition 2.9.

(iv) $\Rightarrow$ (iii) Let V be a K-vector space with $dim(V) \geq 2$. If V is of infinite dimension, then V has (**P**) by Proposition 2.4. If V is finite-dimensional, then V has (**P**) by Theorem 2.6. Therefore K satisfies ($\mathbf{P_3}$). □

Proposition 3.2. *Let R be a commutative ring. If R satisfies ($\mathbf{P_1}$) (resp., ($\mathbf{P_2}$) or ($\mathbf{P_3}$)), then R/I satisfies ($\mathbf{P_1}$) (resp., ($\mathbf{P_2}$) or ($\mathbf{P_3}$)) for every ideal I of R.*

Proof. This follows from Proposition 2.11. □

Combining Propositions 3.1 and 3.2, we obtain the following corollary.

Corollary 3.3. *Let R be a commutative ring. If R satisfies ($\mathbf{P_1}$), then the field $R/\mathfrak{m}$ is algebraically closed for every maximal ideal $\mathfrak{m}$ of R.*

We call a ring R $\mathfrak{m}$-*algebraically closed* if $R/\mathfrak{m}$ is an algebraically closed field for all maximal ideals $\mathfrak{m}$ of R.

Theorem 3.4. *The following conditions are equivalent for a commutative ring R:*

(i) *R satisfies ($\mathbf{P_1}$);*
(ii) *R satisfies ($\mathbf{P_2}$);*
(iii) *R is an $\mathfrak{m}$-algebraically closed ring.*

Proof. (ii) $\Rightarrow$ (i) This is clear.

(i) $\Rightarrow$ (iii) This follows from Corollary 3.3.

(iii) $\Rightarrow$ (ii) Using Example 2.10 and Proposition 2.17, we only need to show that every semisimple homogeneous R-module which is not simple satisfies (**P**). Let M be a nonzero semisimple homogeneous R-module such that M is not simple. Note that $M \cong (R/\mathfrak{m})^{(\Lambda)}$ for some maximal ideal $\mathfrak{m}$ of R and some index set Λ. Hence M can be considered as an $R/\mathfrak{m}$-module. Since $R/\mathfrak{m}$ is algebraically closed, it follows that the $R/\mathfrak{m}$-module ${}_{R/\mathfrak{m}}M$ satisfies (**P**) by Proposition 3.1. Therefore the R-module ${}_RM$ satisfies (**P**) by Proposition 2.11. This proves the theorem. □

Remark 3.5. It is well known that a finite field could not be algebraically closed. From Theorem 3.4, it follows that a finite ring could not satisfy ($\mathbf{P_1}$).

Next, we exhibit some examples of rings satisfying properties ($\mathbf{P_1}$) and ($\mathbf{P_2}$).

Example 3.6. (i) Let $K_1, K_2, \ldots, K_n$ be algebraically closed fields. Applying Theorem 3.4, we see that the ring $R = K_1 \times K_2 \times \cdots \times K_n$ satisfies ($\mathbf{P_2}$).

(ii) Let K be an algebraically closed field and let $R = K[X_1, \ldots, X_n]$. It is well known (see Hilbert's Nullstellensatz) that the maximal ideals of the ring R are the ideals $(X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n)$, where $a_1, a_2, \ldots, a_n \in K$. Moreover, for any $a_1, a_2, \ldots, a_n \in K$, $(X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n)$ is the kernel of the epimorphism

$$\varphi : R \to K \text{ defined by } f \mapsto f(a_1, a_2, \ldots, a_n).$$

Thus $R/(X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n)$ is isomorphic to K. This implies that $R/(X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n)$ is an algebraically closed field. Hence R satisfies $(\mathbf{P_2})$ by Theorem 3.4. Note that the ring R has infinitely many maximal ideals.

(iii) Let K be a field and let R be a subring of K. Recall that R is called a *valuation ring* of K if, for any $0 \neq x \in K$, either $x \in R$ or $x^{-1} \in R$. Note that every valuation ring of K is a local ring.

Now assume that K is an algebraically closed field and let R be a valuation ring of K. It is well known that the residue field of R is also algebraically closed. From Theorem 3.4, we see that the ring R satisfies $(\mathbf{P_2})$.

Proposition 3.7. *A finite product $R = \prod_{i=1}^{n} R_i$ $(n \geq 2)$ of rings satisfies $(\mathbf{P_2})$ if and only if so is each R_i $(1 \leq i \leq n)$.*

Proof. There is no loss of generality in assuming that $n = 2$. The necessity follows from Proposition 3.2. Conversely, let $\mathfrak{m}$ be a maximal ideal of R. Then $\mathfrak{m} = \mathfrak{m}_1 \times R_2$ or $\mathfrak{m} = R_1 \times \mathfrak{m}_2$, where $\mathfrak{m}_i$ $(i \in \{1, 2\})$ is a maximal ideal of R_i. Hence $R/\mathfrak{m} \cong R_1/\mathfrak{m}_1$ (as fields) or $R/\mathfrak{m} \cong R_2/\mathfrak{m}_2$ (as fields). Using Theorem 3.4 twice, we conclude that $R/\mathfrak{m}$ is an algebraically closed field and hence the ring R satisfies $(\mathbf{P_2})$. □

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