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ON (m, n)-CLOSED IDEALS IN AMALGAMATED ALGEBRA

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Abstract. Let R be a commutative ring with $1 \neq 0$ and let m and n be integers with $1 \leq n < m$. A proper ideal I of R is called an (m, n)-closed ideal of R if whenever $a^m \in I$ for some $a \in R$ implies $a^n \in I$. Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. This paper investigates the concept of (m, n)-closed ideals in the amalgamation of A with B along J with respect f denoted by $A \bowtie^f J$. Namely, Section 2 investigates this notion to some extensions of ideals of A to $A \bowtie^f J$. Section 3 features the main result, which examines when each proper ideal of $A \bowtie^f J$ is an (m, n)-closed ideal. This allows us to give necessary and sufficient conditions for the amalgamation to inherit the radical ideal property with applications on the transfer of von Neumann regular, π-regular and semisimple properties.

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1. Introduction

We assume throughout the whole paper that all rings are commutative with $1 \neq 0$. The notion of (m, n)-closed ideal was introduced and defined by Anderson and Badawi in [2], as follow: Let R be a ring and m and n be two positive integers with $1 \leq n < m$. A proper ideal I of R is called an (m, n)-*closed ideal* of R if whenever $a^m \in I$ for some $a \in R$ implies $a^n \in I$. Also, an ideal I of R is a semi-n-absorbing ideal of R if and only if I is an $(n+1, n)$-closed ideal of R. Recall that an ideal I of R is a radical ideal if and only if I is a $(2, 1)$-closed ideal. Among other things, they gave the basic properties of semi-n-absorbing ideals and (m, n)-closed ideals and they also determined when every proper ideal of R is (m, n)-closed for integers $1 \leq n < m$. Further, they gave several examples illustrating their results. Recall that a proper ideal I of R is called an n-*absorbing ideal* of R as in [1] if $a_1, a_2, \ldots, a_{n+1} \in R$ and $a_1 a_2 \cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I. Notice that an n-absorbing ideal is an (m, n)-closed for every positive integer m. In [4], the authors studied the notions of n-absorbing ideals, strongly n-absorbing ideals and (m,n)-closed ideals in the trivial ring extension. Let A be a commutative ring and E be an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a,e)(a',e') = (aa', ae' + a'e)$. Trivial ring extensions have been studied extensively. Considerable work, part of is summarized in Glaz's book [12] and Huckaba's book [13], has been concerned with trivial ring extension these extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [3,10,12,13]. We recall that if I is a proper ideal of A, then $I \propto E$ is an ideal of $A \propto E$. And if F is a submodule of E such that $IE \subset F$, then $I \propto F$ is an ideal of $A \propto E$. The ideals of $A \propto E$ are not all of the form $I \propto E$ or $I \propto F$, but if A is an integral domain, and E an A-module divisible, the ideals of $A \propto E$ are of the form $I \propto E$ or $0 \propto F$, where I is an ideal of A and F a submodule of E. Let (A,B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called *the amalgamation of A and B along J with respect to f*, introduced and studied by D'Anna, Finocchiaro and Fontana in [9,10]. In particular, they have studied amalgamations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [7,8]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations) (cf. [16, page 2]). Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$ and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([9, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [5]) are strictly related to it ([9, Example 2.7 and Remark 2.8]). In [9], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [10], they pursued the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension. For more details on amalgamation rings, we refer the reader to [11], [14], [15].

In this paper, we study the notion of (m, n)-closed ideals in the amalgamation of A with B along J with respect f denoted by $A \bowtie^f J$. For any ring R, we denote by $Nil(R)$ (resp., $dim(R)$), the set of nilpotent elements of R (resp., the Krull dimension of R).

2. On some (m, n)-closed ideals of amalgamation $A \bowtie^f J$

To avoid unnecessary repetition, let us fix notation for the rest of the paper. Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B. All along this paper, $A \bowtie^f J$ will denote the amalgamation of A and B along J with respect to f. Let I be an ideal of A and K be an ideal of $f(A) + J$. Notice that $I \bowtie^f J := \{(i, f(i) + j)/i \in I, j \in J\}$ and $\overline{K}^f := \{(a, f(a) + j)/a \in A, j \in J, f(a) + j \in K\}$ are ideals of $A \bowtie^f J$. Our first result gives a characterization about when the ideals $I \bowtie^f J$ and $\overline{K}^f$ are (m, n)-closed ideals of $A \bowtie^f J$, for all positive integers m and n, with $1 \leq n < m$.

Proposition 2.1. *Under the above notations, the following statements hold:*

(1) *$I \bowtie^f J$ is an (m, n)-closed ideal of $A \bowtie^f J$ if and only if I is an (m, n)-closed ideal of A.*

(2) *$\overline{K}^f$ is an (m, n)-closed ideal of $A \bowtie^f J$ if and only if K is an (m, n)-closed ideal of $f(A) + J$.*

Proof. (1) Assume that $I \bowtie^f J$ is an (m, n)-closed ideal of $A \bowtie^f J$ for m and n two positive integers with $1 \leq n < m$. Let $a^m \in I$, with $a \in A$. Clearly $(a, f(a))^m \in I \bowtie^f J$. Since $I \bowtie^f J$ is an (m, n)-closed ideal of $A \bowtie^f J$, we have $(a, f(a))^n \in I \bowtie^f J$ and so $a^n \in I$. Hence, I is an (m, n)-closed ideal of A. Conversely, assume that I is an (m, n)-closed ideal of A. Let $x^m = (a, f(a) + j)^m \in I \bowtie^f J$ with $x = (a, f(a) + j) \in A \bowtie^f J$. Clearly, $a^m \in I$. Since I is an (m, n)-closed ideal of A, we have $a^n \in I$. One can easily check that $x^n = (a^n, (f(a) + j)^n) \in I \bowtie^f J$, as desired.

(2) Suppose that $\overline{K}^f$ is an (m, n)-closed ideal of $A \bowtie^f J$. We claim that K is an (m, n)-closed ideal of $f(A) + J$. Indeed, let $(f(a) + j)^m \in K$ with $(f(a)+j) \in f(A)+J$. Then $(a^m, (f(a)+j)^m) \in \overline{K}^f$. Since $\overline{K}^f$ is an (m, n)-closed ideal, $(a, f(a) + j)^n \in \overline{K}^f$. Therefore, $(f(a) + j)^n \in K$. Hence, K is an (m, n)-closed ideal of $f(A) + J$. Conversely, assume that K is an (m, n)-closed ideal of $f(A)+J$. Let $(a, f(a)+j)^m \in \overline{K}^f$ with $(a, f(a)+j) \in A \bowtie^f J$.

Obviously, $f(a)+j \in f(A)+J$ and $(f(a)+j)^m \in K$ which is an (m,n)-closed ideal. So, $(a^n,(f(a)+j)^n) \in \overline{K}^f$. Hence, $\overline{K}^f$ is an (m,n)-closed ideal of $A \bowtie^f J$, as desired. □

The following corollary is an immediate consequence of Proposition 2.1.

Corollary 2.2. *Under the above notations, the following statements hold:*

(1) *$I \bowtie^f J$ is a radical ideal of $A \bowtie^f J$ if and only if I is a radical ideal of A.*

(2) *$I \bowtie^f J$ is a semi-n-absorbing ideal of $A \bowtie^f J$ if and only if I is a semi-n-absorbing ideal of A.*

(3) *$\overline{K}^f$ is a semi-n-absorbing ideal of $A \bowtie^f J$ if and only if K is a semi-n-absorbing ideal of $f(A)+J$.*

(4) *$\overline{K}^f$ is a radical ideal of $A \bowtie^f J$ if and only if K is a radical ideal of $f(A)+J$.*

Let I be a *proper* ideal of A. The (amalgamated) duplication of A along I is a special amalgamation given by

$$A \bowtie I := A \bowtie^{id_A} I = \big\{(a,a+i) \mid a \in A, i \in I\big\}.$$

The next corollary is an immediate consequence of Proposition 2.1 and Corollary 3.5 on the transfer of (m,n)-closed ideal property to duplications.

Corollary 2.3. *Let A be a ring and I be an ideal of A. Consider K an ideal of A. Then the following statements hold:*

(1) *$K \bowtie I$ is an (m,n)-closed ideal of $A \bowtie I$ if and only if I is an (m,n)-closed ideal of A.*

(2) *$K \bowtie I$ is a semi-n-absorbing ideal of $A \bowtie I$ if and only if I is a semi-n-absorbing ideal of A.*

(3) *$K \bowtie I$ is a radical ideal of $A \bowtie I$ if and only if I is a radical ideal of A.*

Let I (resp., K) be an ideal of A (resp., $f(A)+J$). Observe that

$$\overline{I \times K}^f := \{(a, f(a)+j) \mid j \in J, a \in I, f(a)+j \in K\}$$

is an ideal of $A \bowtie^f J$. The following proposition establishes a partial result about when the ideal $\overline{I \times K}^f$ is an (m,n)-closed ideal of $A \bowtie^f J$.

Proposition 2.4. *Let m_1, n_1, m_2 and n_2 be positive integers such that $m_1 < n_1$ and $m_2 < n_2$. Under the above notations: If I is an (m_1,n_1)-closed ideal of A and K is an (m_2,n_2)-closed ideal of $f(A)+J$, then $\overline{I \times K}^f$ is an (m,n)-closed ideal of $A \bowtie^f J$ for all positive $m \leq min(m_1,m_2)$ and $n \geq max(n_1,n_2)$.*

Proof. Assume that I is an (m_1, n_1)-closed ideal of A and K is an (m_2, n_2)-closed ideal of $f(A)+J$. Notice that $\overline{I \times K}^f = (I \times K) \cap (A \bowtie^f J)$. From [2, Theorem 2.12], $I \times K$ is an (m, n)-closed ideal of $A \times (f(A) + J)$ for all positive $m \leq min(m_1, m_2)$ and $n \geq max(n_1, n_2)$. Since $A \bowtie^f J \subset A \times (f(A) + J)$, by [2, Corollary 2.11(1)], it follows that $\overline{I \times K}^f$ is an (m, n)-closed ideal of $A \bowtie^f J$ for all positive $m \leq min(m_1, m_2)$ and $n \geq max(n_1, n_2)$, as desired. □

As a direct consequence of Proposition 2.4, we obtain the following corollary:

Corollary 2.5. (1) *If I is a semi-n-absorbing ideal of A and K is a semi-n-absorbing ideal of $f(A) + J$, then $\overline{I \times K}^f$ is a semi-n-absorbing ideal of $A \bowtie^f J$.*

(2) *If I is a radical ideal of A and K is a radical ideal of $f(A)+J$, then $\overline{I \times K}^f$ is a radical ideal of $A \bowtie^f J$.*

Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Consider an ideal I (resp., H) of A (resp., $f(A) + J$) such that $f(I)J \subseteq H \subseteq J$. Observe that $I \bowtie^f H := \{(i, f(i) + h)/i \in I, h \in H\}$ is an ideal of $A \bowtie^f J$.

Remark 2.6. Under the above notations. Let m and n be two integers with $1 \leq n < m$. If $I \bowtie^f H$ is an (m, n)-closed ideal of $A \bowtie^f J$, then by using similar argument as statement (1) of Proposition 2.1, it follows that I is an (m, n)-closed ideal of A. The converse is not true, in general, as shown by the next example which exhibits an ideal I that is a $(2, 1)$-closed ideal of A such that $f(I)J \subseteq H \subset J$ but $I \bowtie^f H$ is not a $(2, 1)$-closed ideal of $A \bowtie^f J$.

Example 2.7. Let $A := \mathbb{Z}$ be the ring of integers, $B := \mathbb{Q}[[X]]$ be the ring of formal power series over $\mathbb{Q}$ in an indeterminate X, $f : \mathbb{Z} \hookrightarrow \mathbb{Q}[[X]]$ be the natural embedding and $J := X\mathbb{Q}[[X]]$. Let $H = \{XP(X); P \in f(A) + J = \mathbb{Z} + X\mathbb{Q}[[X]]\}$. Obviously, H is an ideal of $f(A) + J = \mathbb{Z} + X\mathbb{Q}[[X]]$. Note that $I := 0$ is a prime ideal of A and so is an $(m, 1)$-closed ideal of A for all positive integer m. Since $f(I)J = 0 \subset H \subset J$, $0 \bowtie^f H$ is an ideal of $\mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$. We claim that $0 \bowtie^f H$ is not a $(2, 1)$-closed ideal of $\mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$. Indeed, $0 \bowtie^f H = \{(0, XP(X)), P(X) \in \mathbb{Z} + X\mathbb{Q}[[X]]\}$, we have $(0, \sqrt{2}X)^2 = (0, 2X^2) \in 0 \bowtie^f H$ but $(0, \sqrt{2}X) \notin 0 \bowtie^f H$. Hence, $0 \bowtie^f H$ is not a $(2, 1)$-closed ideal of $\mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$.

Now, we examine about when the ideal $I \bowtie^f H$ is an (m, n)-closed ideal of $A \bowtie^f J$.

Proposition 2.8. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Let H be an ideal of $f(A)+J$ such that $f(I)J \subseteq H \subseteq J$. Then the following statements hold:*

(1) *If $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$, then I is an (m,n)-closed ideal of A, for all positive integers m and n, with $1 \leq n < m$.*

(2) *Assume that I is an $(m,1)$-closed ideal of A and $x^n \in H$ for every $x \in J$ with $n \geq 1$ a positive integer. Then $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$.*

Proof. (1) From Remark 2.6.

(2) Assume that I is an $(m,1)$-closed ideal of A. We claim that $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$. Let $x^m = (a, f(a)+j)^m \in I \bowtie^f H$ for some $x = (a, f(a)+j) \in A \bowtie^f J$. Then $a^m \in I$. Since I is an $(m,1)$-closed ideal of A, we have $a \in I$ and so $a^i \in I$ for every positive integer $1 \leq i \leq m$. Let $h = \sum_{i=1}^{n-1} \begin{pmatrix} n \\ i \end{pmatrix} f(a^i)j^{n-i} + j^n$. Observe that for every positive integer $i \leq n-1$, we have $f(a^i)j^{n-i} \in H$ (as $(a, f(a)+j)^m \in I \bowtie^f H$). Since $j^n \in H$ for every $j \in J$, by using the Binomial theorem, it follows that $x^n = (a^n, f(a^n)+h) \in I \bowtie^f H$. Hence, $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$, as desired. □

The following corollary is a consequence of Proposition 2.8.

Corollary 2.9. *Let $f : A \to B$ be a homomorphism of rings and J be an ideal of B. Assume that I is a $(m,1)$-closed ideal of A and $x^n \in (f(I)B)J$ for every $x \in J$, with $n \geq 1$ a positive integer. Then the extension ideal of I to $A \bowtie^f J$, denoted by $I^e := I \bowtie^f (f(I)B)J$ is an (m,n)-closed ideal of $A \bowtie^f J$, for all positive integers m and n, with $1 \leq n < m$.*

Proof. Applying Proposition 2.8 with $H := (f(I)B)J$, it follows that I^e is an (m,n)-closed ideal of $A \bowtie^f J$, as desired. □

Next, we show how one may use Proposition 2.8 to construct original example of (m,n)-closed ideals of the form $I \bowtie^f H$ of amalgamation $A \bowtie^f J$.

Example 2.10. Let A be a ring, E be an A-module and $B := A \propto E$ be the trivial ring extension of A by E. Let I be an ideal of A and F be a submodule of E such that $IE \subset F$, and $J := I \propto E$ be an ideal of B. Consider the ring homomorphism $f : A \hookrightarrow B$ defined by $f(a) = (a,0)$. Notice that $H := I \propto F$ is an ideal of $f(A)+J = A \propto 0 + I \propto E = (A+I) \propto E = A \propto E$ and

$f(I)J = (I \propto 0)(I \propto E) \subset I \propto F \subset J$. So, $I \bowtie^f H$ is an ideal of $A \bowtie^f J$. Let n be a positive integer and let $(i, e) \in J$. Clearly, $(i, e)^n = (i^n, ni^{n-1}e) \in H$. By Proposition 2.8, we conclude that I is an $(m, 1)$-closed ideal of A if and only if $I \bowtie^f H$ is an (m, n)-closed ideal of $A \bowtie^f J$ for every positive integer $1 \leq n < m$. (For instance, if I is a prime ideal of A, then $I \bowtie^f H$ is an $(m, 1)$-closed ideal of $A \bowtie^f J$ for every positive integer $m \geq 1$. Therefore, $I \bowtie^f H$ is a radical ideal of $A \bowtie^f J$.)

Now, we give a characterization of (m, n)-closed ideals of the form $I \bowtie^f H$ in the case (A, M) is local and J an ideal of B such that $f(M)J = 0$.

Theorem 2.11. *Let A be a local ring with maximal ideal M, $f : A \to B$ be a ring homomorphism and J be an ideal of B such that $f(M)J = 0$. Let I be a proper ideal of A. Consider an ideal H of $f(A) + J$ such that $H \subset J$ and $x^n \in H$ for every $x \in J$. Then the following statements are equivalent:*

(1) *$I \bowtie^f H$ is an (m, n)-closed ideal of $A \bowtie^f J$ for every positive integer $1 \leq n < m$.*
(2) *I is an (m, n)-closed ideal of A for every positive integer $1 \leq n < m$.*

Proof. Notice that $I \bowtie^f H$ is an ideal of $A \bowtie^f J$ since $f(I)J = 0 \subset H$.
$(1) \Rightarrow (2)$ Assume that $I \bowtie^f H$ is an (m, n)-closed ideal of $A \bowtie^f J$ for every positive integer $1 \leq n < m$. Then by Remark 2.6, it follows that I is an (m, n)-closed ideal of A for every positive integer $1 \leq n < m$.
$(2) \Rightarrow (1)$ Let $x = (a, f(a) + j) \in A \bowtie^f J$ such that $x^m \in I \bowtie^f H$. Then a^m is an element of I which is an (m, n)-closed ideal of A. Therefore, $a^n \in I$. Let $1 \leq l \leq n - 1$. Two cases are then possible:
Case 1: $a^l \in M$. Then $f(a^l)j^{n-l} = 0$ and so by using the Binomial theorem, it follows that $(a, f(a) + j)^n = (a^n, f(a^n) + j^n) \in I \bowtie^f H$.
Case 2: $a^l \notin M$. Then a^l is invertible in A and so $a^n \in I$ is invertible, which is a contradiction since I is a proper ideal of A. So, $x^n = (a^n, (f(a) + j)^n) \in I \bowtie^f H$. Hence, in all cases, $I \bowtie^f H$ is an (m, n)-closed ideal of $A \bowtie^f J$. $\square$

Before giving an explicit example of Theorem 2.11, we establish the following lemma which will be useful.

Lemma 2.12. *Let (A, M) be local ring such that $M^n = 0$, where n is a positive integer. Then every ideal of A is an n-absorbing ideal of A. In particular, every proper ideal of A is an (m, n)-ideal of A, for every positive integer $1 \leq n < m$.*

Proof. Let I be a proper ideal of A. Let $a_1,\dots,a_{n+1}\in A$ such that $\prod_{i=1}^{n+1} a_i \in I$. Two cases are then possible:
Case 1: There exists $j\in\{1,\dots,n+1\}$ such that $a_j\notin M$. Then a_j is invertible and so it follows that $\prod_{i=1,i\neq j}^{n+1} a_i \in I$, as desired.
Case 2: For every $j\in\{1,\dots,n+1\}$, $a_j\in M$. So, for every $j\in\{1,\dots,n+1\}$, a_j is not invertible in A. Therefore, $\prod_{i=1}^{n} a_i = 0\in I$. Thus in all cases, I is an n-absorbing ideal of A. In particular, I is an (m,n)-closed ideal of A. □

Next, we show how one may use Theorem 2.11 and Lemma 2.12 to construct original examples of (m,n)-closed ideals of the form $I\bowtie^f H$.

Example 2.13. Let (A,M) be a local ring with a maximal ideal M such that $M^n=0$, and E be an $\frac{A}{M}$-vector space. Consider the ring homomorphism $f: A\hookrightarrow B := A\propto E$ defined by $f(a)=(a,0)$, for every $a\in A$. Let $J := M\propto E$ be an ideal of B and $H = I\propto E$ be an ideal of $f(A)+J$, where $I\subsetneq M$ is an ideal A with m and n two positive integers such that $2\le n\le m$. One can easily check that $J^n=0$ and for every ideal I of A, we get $f(I)J=(I\propto E)(M\propto E)\subseteq H$, and $J^n\subseteq H$. Hence, by Theorem 2.11, the ideal $I\bowtie^f H$ is an (m,n)-closed ideal of $A\bowtie^f J$ since I is an n-absorbing ideal of A (by Lemma 2.12).

We denote by $Char(R)$, the characteristic of a ring R. We close this section by giving a characterization of (m,n)-closed ideals of the form $I\bowtie^f H$ under the condition "$Char(f(A)+J)=n$".

Proposition 2.14. *Let $f: A\to B$ be a ring homomorphism and J be an ideal of B. Assume that $char(f(A)+J)=n$. Let H be an ideal of $f(A)+J$ such that $f(I)J\subset H\subset J$ and for every $j\in J, j^n\in H$. Then $I\bowtie^f H$ is an (m,n)-closed ideal of $A\bowtie^f J$ if and only if I is an (m,n)-closed ideal of A, for all positive integer $m\ge n$.*

Proof. Assume that I is an (m,n)-closed ideal of A, for all positive integer $m\ge n$. Let $(a,f(a)+j)^m\in I\bowtie^f H$ for every $(a,f(a)+j)\in A\bowtie^f J$. From assumption, it follows that $a^n\in I$. Using the fact that $\mathrm{char}(f(A)+J)=n$ and $j^n\in H$, then $(a,f(a)+j)^n=(a^n,f(a^n)+j^n)\in I\bowtie^f H$. Hence, $I\bowtie^f H$ is an (m,n)-closed ideal of $A\bowtie^f J$. The converse is trivial via Remark 2.6. □

3. When every proper ideal of amalgamation $A\bowtie^f J$ is an (m,n)-closed ideal?

The following result gives a characterization about when every proper ideal of the amalgamation $A\bowtie^f J$ is an (m,n)-closed ideal, for some integers $1\le n<m$.

Theorem 3.1. *Assume that $f^{-1}(J)$ is a radical ideal of A. Then every proper ideal of $A \bowtie^f J$ is an (m,n)-closed ideal of $A \bowtie^f J$ if and only if the following statements hold:*

(i) *Every proper ideal of A is an (m,n)-closed ideal of A.*
(ii) *Every proper ideal of $f(A)+J$ is an (m,n)-closed ideal of $f(A)+J$.*

The proof of this theorem requires the following lemmas.

Lemma 3.2. [2, Theorem 2.14] *Let R be a commutative ring and m and n integers with $1 \leq n < m$. Then the following statements are equivalent.*

(1) *Every proper ideal of R is an (m,n)-closed ideal of R.*
(2) *$dim(R) = 0$ and $w^n = 0$ for every $w \in Nil(R)$.*

Lemma 3.3. [6, Lemma 2.10] *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then:*

$$Nil(A \bowtie^f J) := \{(a, f(a)+j)/a \in Nil(A), j \in Nil(B) \cap J\}.$$

Lemma 3.4. [10, Proposition 4.1] *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then: $dim(A \bowtie^f J) = Max(dim(A), dim(f(A)+J))$.*

Proof of Theorem 3.1: Assume that every proper ideal of $A \bowtie^f J$ is an (m,n)-closed ideal of $A \bowtie^f J$.
(i) By Lemmas 3.2 and 3.4, it follows that $dim(A \bowtie^f J) = Max(dim(A), dim(f(A)+J)) = 0$. So, $dim(A) = 0$. Next, let $a \in Nil(A)$. Then $(a, f(a)) \in Nil(A \bowtie^f J)$. Using the fact that every proper ideal of $A \bowtie^f J$ is an (m,n)-closed ideal of $A \bowtie^f J$, then by Lemma 3.2, it follows that $(a, f(a))^n = (0,0)$. Therefore, $a^n = 0$. Hence, every proper ideal of A is an (m,n)-closed ideal of A.
(ii) With similar argument as (i) above, it follows that $dim(f(A)+J) = 0$. Let $f(a)+j \in Nil(f(A)+J)$. Clearly, $f(a)^k \in J$ for some integer $k \geq 1$. So, $a^k \in f^{-1}(J)$ which is radical. Therefore, $a \in f^{-1}(J)$. Consequently, $f(a)+j \in J$. By Lemma 3.3, $(0, f(a)+j) \in Nil(A \bowtie^f J)$. Hence, $(0, f(a)+j)^n = (0,0)$ and so $(f(a)+j)^n = 0$. From Lemma 3.2, it follows that every proper ideal of $f(A)+J$ is an (m,n)-closed ideal of $f(A)+J$. Conversely, assume that every proper ideal of A (resp., $f(A)+J$) is an (m,n)-closed ideal of A (resp., $f(A)+J$). We claim that every proper ideal of $A \bowtie^f J$ is an (m,n)-closed ideal of $A \bowtie^f J$. Indeed, by Lemma 3.4, $dim(A \bowtie^f J) = Max(dim(A), dim(f(A)+J)) = 0$ since $dim(A) = dim(f(A)+J) = 0$. It remains to show that for all $(a, f(a)+j) \in Nil(A \bowtie^f J)$, $(a, f(a)+j)^n = 0$. Let $(a, f(a)+j) \in Nil(A \bowtie^f J)$. Then $a \in Nil(A)$ and $f(a)+j \in Nil(f(A)+J)$. By Lemma 3.2, it follows that $a^n = 0$ and $(f(a)+j)^n = 0$.

Therefore, $(a, f(a)+j)^n = 0$. Hence, every proper ideal of $A \bowtie^f J$ is an (m,n)-closed ideal of $A \bowtie^f J$, as desired. □

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.5. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then the following statements hold:*

(1) *Every proper ideal of $A \bowtie^f J$ is a radical ideal of $A \bowtie^f J$ if and only if the following statements hold:*
   (i) *Every proper ideal of A is a radical ideal of A.*
   (ii) *Every proper ideal of $f(A)+J$ is a radical ideal of $f(A)+J$.*

(2) *Assume that $f^{-1}(J)$ is a radical ideal of A. Then every proper ideal of $A \bowtie^f J$ is a semi-n-absorbing ideal of $A \bowtie^f J$ if and only if the following statements hold:*
   (i) *Every proper ideal of A is a semi-n-absorbing ideal of A.*
   (ii) *Every proper ideal of $f(A)+J$ is a semi-n-absorbing of $f(A)+J$.*

Remark 3.6. Observe that the assumption "$f^{-1}(J)$ is a radical ideal of A" is omitted in Corollary 3.5(1). Indeed, if every proper ideal of $A \bowtie^f J$ is a radical ideal of $A \bowtie^f J$, then $f^{-1}(J) \times \{0\}$ is a radical ideal of $A \bowtie^f J$ and so $f^{-1}(J) \simeq f^{-1}(J) \times \{0\}$ is also a radical ideal of A. Conversely if the statements (i) and (ii) hold, then $f^{-1}(J)$ is radical ideal of A.

Theorem 3.1 and Corollary 3.5 cover the special case of duplications, as recorded below.

Corollary 3.7. *Let A be a ring and I be an ideal of A. Then the following statements hold:*

(1) *Assume that I is a radical ideal of A. Then every proper ideal of $A \bowtie I$ is an (m,n)-closed ideal of $A \bowtie I$ if and only every proper ideal of A is an (m,n)-closed ideal of A.*

(2) *Every proper ideal of $A \bowtie I$ is a radical ideal of $A \bowtie I$ if and only every proper ideal of A is a radical ideal of A.*

(3) *Assume that I is a radical ideal of A. Then every proper ideal of $A \bowtie I$ is a semi-n-absorbing ideal of $A \bowtie I$ if and only every proper ideal of A is a semi-n-absorbing ideal of A.*

Theorem 3.1 recovers a known result for trivial ring extensions which is [4, Theorem 6.12].

Corollary 3.8. *Let A be an integral domain with quotient field K, E be a K-vector space, and $B := A \propto E$ be the trivial ring extension of A by E. Then the following statements are equivalent:*

(1) *Every proper ideal of A is an (m,n)-closed ideal of A for some integers $1 \leq n < m$.*
(2) *Every proper ideal of B is an (m,n)-closed ideal of B for some integers $1 \leq n < m$.*

Proof. Consider the injective ring homomorphism $f : A \hookrightarrow B$ defined by $f(a) = (a, 0)$, for every $a \in A$, $J := 0 \propto E$ is an ideal of B. Clearly, $f^{-1}(J) = 0$. Therefore, from [9, Proposition 5.1 (3)], $f(A) + J = A \propto 0 + 0 \propto E = A \propto E = B \simeq A \bowtie^f J$. Since A is an integral domain, $f^{-1}(J) = 0$ is a radical ideal of A. Hence, by Theorem 3.1, we have the desired result. $\square$

As a consequence of Theorem 3.1, we give a complete characterization of those $D + M$ rings such that every proper ideal is (m,n)-closed.

Corollary 3.9. *Let M be a maximal ideal of an integral domain T, and D be a subring of T such that $D \cap M = \{0\}$. Then every proper ideal of $D+M$ is an (m,n)-closed ideal of $D+M$ if and only if every proper ideal of D is an (m,n)-closed ideal of D for some integers $1 \leq n < m$.*

Proof. Let $f : D \hookrightarrow T$ be the natural embedding and $J := M$ is the maximal ideal of T. Since $f^{-1}(J) = D \cap M = \{0\}$ (which is radical ideal of D, as 0 is prime ideal of D), it is easy to see that $D + M \simeq D \bowtie^f M$. Therefore, by Theorem 3.1, it follows that every proper ideal of $D + M$ is an (m,n)-closed ideal of $D + M$ if and only if every proper ideal of $D \bowtie^f M$ is an (m,n)-closed ideal if and only if every proper ideal of D is an (m,n)-closed ideal of D, for some integers $1 \leq n < m$. $\square$

As an application Theorem 3.1, we give a new characterization for the amalgamation $A \bowtie^f J$ to be von Neumann regular. Recall that a ring R is von Neumann regular if and only if every proper ideal of R is a radical ideal [2, Theorem 2.13 (2)]. A combination of this fact and Corollary 3.5(1) establishes the transfer of von Neumann regular property to the amalgamation $A \bowtie^f J$.

Corollary 3.10. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then $A \bowtie^f J$ is von Neumann regular if and only if A and $f(A) + J$ are von Neumann regular.*

The next result is an application of Corollary 3.10 on the transfer of π-regular property to the amalgamation. Recall that a ring R is π-regular if and only if $R/Nil(R)$ is von Neumann regular.

Corollary 3.11. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then $A \bowtie^f J$ is π-regular if and only if A and $f(A) + J$ are π-regular.*

Proof. Set $\overline{A} = A/Nil(A)$, $\overline{B} = B/Nil(B)$, $p : B \to \overline{B}$ the canonical projection and $\overline{J} = p(J)$ and let $\overline{f} : \overline{A} \to \overline{B}$, defined by $\overline{f}(\overline{x}) = \overline{f(x)}$. Observe that $\overline{f}$ is well defined. From [6, Proof of Theorem 2.9], we get $A \bowtie^f J/Nil(A \bowtie^f J) \simeq \overline{A} \bowtie^{\overline{f}} \overline{J}$. Consequently, $A \bowtie^f J$ is a π-regular ring if and only if $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is von Neumann regular if and only if $\overline{A}$ and $\overline{f(A) + J}$ are von Neumann regular (by Corollary 3.10). Hence, the conclusion is straightforward. $\square$

As another application of Theorem 3.1, we get necessary and sufficient conditions for an amalgamation to be semisimple.

Corollary 3.12. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then $A \bowtie^f J$ is semisimple if and only if A and $f(A) + J$ are semisimple.*

Proof. It is well known that semisimple rings collapse with von Neumann regular rings that are Noetherian. A combination of this fact with Corollary 3.10 and [9, Proposition 5.6] (on the transfer of the Noetherian property) leads to the conclusion. $\square$

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