THE GROUP OF UNITS OF GROUP ALGEBRAS OF GROUPS
$D_{30}$ AND $C_3 \times D_{10}$ OVER A FINITE FIELD OF CHARACTERISTIC 3

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Abstract. Let $F$ be a finite field of characteristic $p$. There are three non-isomorphic non-abelian groups of order 30. The structure of $U(F(C_5 \times D_6))$ for $p = 3$ is given in [J. Gildea and R. Taylor, Int. Electron. J. Algebra, 24 (2018), 62-67]. In this article, we give the structure of $U(FD_{30})$ and $U(F(C_3 \times D_{10}))$ for $p = 3$.

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1. Introduction

Let $U(FG)$ be the group of units of the group algebra $FG$ of a group $G$ over a finite field $F$ of characteristic $p$ having $q = p^k$ elements. Let $J(FG)$ be the Jacobson radical of $FG$ and let $V = 1 + J(FG)$. We denote by $D_n$ the dihedral group of order $n$. In this paper, we study the structure of $U(FG)$ where $G = D_{30}$ and $C_3 \times D_{10}$, for $p = 3$.

Let $V_1(FG) = \{ \sum_{g \in G} r_g g \in U(FG) \mid \sum_{g \in G} r_g = 1 \}$ be the group of normalized units of $FG$. It is well known that $U(FG) = V_1(FG) \times F^*$. If $G$ is a finite abelian $p$-group, then $V_1(FG)$ is a finite $p$-group of order $|F|^{|G|^{-1}}$. In [19], Sandling provides a basis for $V_1(FG)$. The map $*: FG \rightarrow FG$ defined by $(\sum_{g \in G} a_g g)^* = \sum_{g \in G} a_g g^{-1}$ is an antiautomorphism of $FG$ and its order is 2. An element $v \in V_1(FG)$ is called a unitary unit if $v^* = v^{-1}$. Unitary units of some modular group algebras have been studied in [2,3]. In [16,17], the structure of the unitary subgroup of the group algebra $FD_{2^n}$ and $F(QD_{16})$, where $QD_{16}$ is the quasi-dihedral group of order 16 and $p = 2$, has been obtained.

Describing the group of units of group algebras is, in general, a hard task and it is more difficult when the group algebra is not semi-simple. Many authors have

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studied the structure of $U(FG)$ for the non-semisimple case, see ([5]-[10]). In [4], Creedon provides a list of presentations of the unit groups $U(FG)$ of all group algebras $FG$ with $|FG| < 1024$. The structure of $U(FD_6)$ for $p = 3$, is established in terms of split extensions of elementary abelian groups in [5]. The structure of $U(FS_5)$ for $p > 5$ is given in [12], where $S_5$ is symmetric group of degree 5. For $p = 3$, Monaghan [15], studied the structure of $U(FG)$ where $G$ is a non-abelian group of 24 such that $G$ has a normal subgroup of order 3. The structure of $U(FG)$ where $G$ is a group of order 12 has been studied in [20] and [21]. Recently, Ansari and Sahai in [1], obtained the structure of $U(FG)$ where $G = C_20$, $C_{10} \times C_2$ and $GA(1,5)$ where $GA(1,5)$ is the general affine group of order 20. In the same paper, the structure of the unit groups $U(FQ_{2^n})$ of the finite group algebras of the generalized quaternion groups $Q_{2^n}$, $p > 2$.

In [13], Makhijani et al. obtained the structure of the unit group of $FD_{2n}$ for any odd $n \geq 3$ and $p = 2$. This is an extension of [10] in which they have studied the unit group of $FD_{2p}$, where $p$ is a prime number. There are three non-abelian groups of order 30, namely, $D_{30}$, $C_5 \times D_{10}$ and $C_5 \times D_6$. In 2018, Gildea and Taylor [9] described the structure of $U(F(C_n \times D_6))$ for $p = 3$ which is an extension of [7]. In 2015, Makhijani et al. [14] studied the structure of $U(FD_{30})$, but for $p = 3$ they provided only a preliminary description of the $U(FD_{30})$. Here in Section 1, we provide a complete characterization of $U(FD_{30})$ for $p = 3$. In Section 2, we give the structure of $U(F(C_3 \times D_{10}))$, again for $p = 3$ only.

2. Unit group of $FD_{30}$

**Theorem 2.1.** Let $F$ be a finite field of characteristic 3 with $|F| = q = 3^k$ and let $G = D_{30}$.

1. If $q \equiv \pm 1 \mod 5$, then $U(FG) \cong (C_3^{15k} \rtimes C_3^{3k}) \times (C_{3^{k-1}}^2 \times GL(2,F))^2$.
2. If $q \equiv \pm 3 \mod 5$, then $U(FG) \cong (C_3^{15k} \rtimes C_3^{5k}) \times (C_{3^{k-1}}^2 \times GL(2,F_2))^2$.

**Proof.** Let $G = \langle x, y \mid x^{15} = y^2 = 1, yxy = x^{-1} \rangle$. Let $K$ be the normal subgroup of $G$ generated by $x^5$. Then $G/K \cong H \cong \langle x^3, y \rangle$. Thus from the ring epimorphism $FG \to FH$ given by

$$\sum_{j=0}^{4} \sum_{i=0}^{2} x^{5i+3j}(a_{i+3j} + a_{i+3j+15}y) \rightarrow \sum_{j=0}^{4} \sum_{i=0}^{2} x^{3j}(a_{i+3j} + a_{i+3j+15}y),$$

we have

1. If $q \equiv \pm 1 \mod 5$, then $U(FG) \cong (C_3^{15k} \rtimes C_3^{3k}) \times (C_{3^{k-1}}^2 \times GL(2,F))^2$.
2. If $q \equiv \pm 3 \mod 5$, then $U(FG) \cong (C_3^{15k} \rtimes C_3^{5k}) \times (C_{3^{k-1}}^2 \times GL(2,F_2))^2$.
we get a group epimorphism $\phi: U(FG) \to U(FH)$ and $ker\phi \cong 1 + J(FG) \cong V$. Further, from the ring monomorphism $FH \to FG$ given by

$$
\sum_{i=0}^{4} x^{3i}(b_i + b_{i+5}y) \to \sum_{i=0}^{4} x^{3i}(b_i + b_{i+5}y),
$$

we get a group monomorphism $\psi: U(FH) \to U(FG)$. Clearly, $\phi\psi = 1_{U(FH)}$ and $U(FG) \cong V \rtimes U(FD_{10})$.

If $u = \sum_{j=0}^{4} \sum_{i=0}^{2} x^{5i+3j}(a_{i+3j} + a_{i+3j+15}) \in U(FG)$, then $u \in V$ if and only if

$$
\sum_{i=0}^{2} a_i = 1 \quad \text{and} \quad \sum_{i=0}^{2} a_{i+3k} = 0 \quad \text{for} \quad k = 1, 2, \ldots, 9.
$$

Hence

$$
V = \{ 1 + \sum_{j=0}^{4} \sum_{i=1}^{2} (x^{5i} - 1) x^{3j}(b_{i+2j} + b_{i+2j+10}) \mid b_i \in F \},
$$

$V^3 = 1$ and $|V| = 3^{20k}$.

Now we show that $V \cong C_3^{15k} \rtimes C_3^{5k}$. The centralizer of $x^5$ in $V$ is

$$
C_V(x^5) = \{ v \in V \mid vx^5 = x^5 v \}.
$$

If $v = 1 + \sum_{j=0}^{4} \sum_{i=1}^{2} (x^{5i} - 1) x^{3j}(b_{i+2j} + b_{i+2j+10}) \in V$, then

$$
vx^5 = x^5 v = x^5 \sum_{j=0}^{4} (b_{1+2j} - b_{12+2j}) x^{3j} y.
$$

Thus $v \in C_V(x^5)$ if and only if $b_i = b_{i+1}$ for $i = 11, 13, 15, 17$ and $19$ and

$$
C_V(x^5) = \{ 1 + \sum_{j=0}^{4} \sum_{i=1}^{2} (x^{5i} - 1) c_{i+2j} x^{3j} + \hat{x}^5 \sum_{j=0}^{4} c_{j+11} x^{3j} y \mid c_i \in F \}.
$$

Let $W$ be a subset of $V$ given by

$$
W = \{ 1 + \sum_{j=0}^{4} \sum_{i=0}^{2} x^{5i+3j}(a_{j+1} + ia_{j+6}) \mid a_i \in F \}.
$$

It can easily be shown that $W$ is an abelian group and $W \cong C_3^{10k}$. If

$$
c = 1 + \sum_{j=0}^{4} \sum_{i=1}^{2} (x^{5i} - 1) c_{i+2j} x^{3j} + \hat{x}^5 \sum_{j=0}^{4} c_{j+11} x^{3j} y \in C_V(x^5)
$$

and

$$
w = 1 + \sum_{j=0}^{4} \sum_{i=0}^{2} x^{5i+3j}(d_{j+1} + id_{j+6}) \in W,
$$

then

$$
c^w = 1 + \sum_{j=0}^{4} \sum_{i=1}^{2} (x^{5i} - 1) c_{i+2j} x^{3j} + \hat{x}^5 \sum_{j=0}^{4} (c_{j+11} - s_{j+1}) x^{3j} y \in C_V(x^5).
$$
where
\[ s_1 = 2(c_1 - c_2)d_6 + ((c_3 - c_4) + (c_9 - c_{10}))(d_7 + d_{10}) + ((c_5 - c_6) + (c_7 - c_8))(d_6 + d_9) \]
\[ s_2 = 2(c_1 - c_2)d_7 + ((c_3 - c_4) + (c_9 - c_{10}))(d_6 + d_8) + ((c_5 - c_6) + (c_7 - c_8))(d_9 + d_{10}) \]
\[ s_3 = 2(c_1 - c_2)d_8 + ((c_3 - c_4) + (c_9 - c_{10}))(d_7 + d_9) + ((c_5 - c_6) + (c_7 - c_8))(d_6 + d_{10}) \]
\[ s_4 = 2(c_1 - c_2)d_9 + ((c_3 - c_4) + (c_9 - c_{10}))(d_8 + d_{10}) + ((c_5 - c_6) + (c_7 - c_8))(d_6 + d_7) \]
\[ s_5 = 2(c_1 - c_2)d_{10} + ((c_3 - c_4) + (c_9 - c_{10}))(d_6 + d_9) + ((c_5 - c_6) + (c_7 - c_8))(d_7 + d_8). \]

Now
\[ R = C_V(x^5) \cap U = \{1 + x^5 \sum_{j=0}^{4} d_{j+1} x^{3j} \mid d_i \in F\} \cong C_3^{5k}. \]

So, for some subgroup \( T \cong C_3^{5k} \) of \( W \), \( W = R \times T \cong C_3^{5k} \times C_3^{5k} \). Obviously, \( C_V(x^5) \cap T = 1 \). Thus \( V \cong C_V(x^5) \times T \cong C_3^{15k} \times C_3^{5k} \).

By [11, Theorem 2.1],
\[ U(FD_{10}) \cong \begin{cases} C_{3^k-1}^2 \times GL(2, F)^2, & \text{if } q \equiv \pm 1 \text{ mod } 5; \\ C_{3^k-1}^2 \times GL(2, F_2), & \text{if } q \equiv \pm 3 \text{ mod } 5. \end{cases} \]

Hence
\[ U(FG) \cong (C_{3^k}^{5k} \times C_3^{5k}) \times (C_{3^k-1}^2 \times GL(2, F)^2), \text{ if } q \equiv \pm 1 \text{ mod } 5 \]
and
\[ U(FG) \cong (C_{3^k}^{15k} \times C_3^{5k}) \times (C_{3^k-1}^2 \times GL(2, F_2)), \text{ if } q \equiv \pm 3 \text{ mod } 5. \]

3. Unit group of \( F(C_3 \times D_{10}) \)

**Theorem 3.1.** Let \( F \) be a finite field of characteristic 3 with \( |F| = q = 3^k \) and let \( G = C_3 \times D_{10} \).

(1) If \( q \equiv \pm 1 \text{ mod } 5 \), then \( U(FG) \cong V \times (C_{3^k-1}^2 \times GL(2, F)^2) \),

(2) If \( q \equiv \pm 3 \text{ mod } 5 \), then \( U(FG) \cong V \times (C_{3^k-1}^2 \times GL(2, F_2)) \)

where \( V \cong (((C_{3^k}^{15k} \times C_3^{5k}) \times C_3^{5k}) \times C_3^{5k}) \times C_3^{5k}). \)

**Proof.** Let \( G = \langle x, y, z \mid x^2 = y^2 = z^3 = 1, xyx = y^{-1}, xz = zx, yz = zy \rangle \). Let \( K \) be the normal subgroup of \( G \) generated by \( z \). Then \( G/K \cong H \cong \langle x, y \rangle \). Thus from the ring epimorphism \( FG \rightarrow FH \) given by
\[ \sum_{j=0}^{4} \sum_{i=0}^{2} z^i y^j (a_{i+3j} + a_{i+3j+15} x) \rightarrow \sum_{j=0}^{4} \sum_{i=0}^{2} y^j (a_{i+3j} + a_{i+3j+15} x), \]
we get a group epimorphism \( \phi: U(FG) \to U(FH) \) and \( \ker \phi \cong 1 + J(FG) \cong V \). Further, from the ring monomorphism \( FH \to FG \) given by
\[
\sum_{i=0}^{4} y^i(b_i + b_{i+5}x) \to \sum_{i=0}^{4} y^i(b_i + b_{i+5}x),
\]
we get a group monomorphism \( \psi: U(FH) \to U(FG) \). Clearly, \( \phi \psi = 1_{U(FH)} \) and \( U(FG) \cong V \times U(FD_{10}) \).

If \( u = \sum_{j=0}^{4} \sum_{i=0}^{2} z^iy^j(a_{i+3j} + a_{i+3j+15}x) \in U(FG) \), then \( u \in V \) if and only if \( \sum_{i=0}^{2} a_i = 1 \) and \( \sum_{i=0}^{2} a_{i+3k} = 0 \) for \( k = 1, 2, \ldots, 9 \). Hence
\[
V = \{1 + \sum_{j=0}^{4} \sum_{i=1}^{2} (z^i - 1)y^j(b_{i+2j} + b_{i+2j+10}x) \mid b_i \in F\},
\]
\( V^3 = 1 \) and \( |V| = 3^{20k} \). Now we complete the proof in following steps:

**Step 1:** Let \( H_1 \) be the subgroup of \( V \) given by
\[
H_1 = \{1 + \sum_{i=1}^{2} (z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + a_{i+10}x) + \hat{z} \sum_{i=0}^{4} a_{i+11}y^i x \mid a_i \in F\}.
\]
Then \( H_1 \cong C_3^{15k} \times C_3^k \).

Let \( P_1 \) and \( Q_1 \) be the abelian subgroups of \( H_1 \) given by
\[
P_1 = \{1 + b_1 \hat{z} + b_2 z(1 - z)x \mid b_i \in F\}
\]
and
\[
Q_1 = \{1 + \sum_{j=0}^{4} \sum_{i=1}^{2} (z^i - 1)a_{i+2j}y^j + \hat{z} \sum_{i=0}^{4} a_{i+11}y^i x \mid a_i \in F\}.
\]
If
\[
p_1 = 1 + b_1 \hat{z} + b_2 z(1 - z)x \in P_1
\]
and
\[
q_1 = 1 + \sum_{j=0}^{4} \sum_{i=1}^{2} (z^i - 1)a_{i+2j}y^j + \hat{z} \sum_{i=0}^{4} a_{i+11}y^i x \in Q_1,
\]
then
\[
q_1^2 = 1 + \sum_{j=0}^{4} \sum_{i=1}^{2} (z^i - 1)a_{i+2j}y^j + \hat{z}(a_{11} + (a_{12} + t_1)y + (a_{13} + t_2)y^2
\]
\[+ (a_{14} - t_2)y^3 + (a_{15} - t_1)y^4) x \in Q_1
\]
where \( t_1 = b_2 \{(a_4 - a_3) - (a_{10} - a_9)\} \) and \( t_2 = b_2 \{(a_6 - a_5) - (a_8 - a_7)\} \).

Now
\[
R_1 = P_1 \cap Q_1 = \{1 + b_1 \hat{z} \mid b_1 \in F\} \cong C_3^k.
\]
So, for some subgroup $S_1 \cong C_3^k$ of $P_1$, $P_1 = R_1 \times S_1$. Clearly $Q_1 \cap S_1 = 1$. Hence

$$H_1 \cong Q_1 \times S_1 \cong C_3^{15k} \times C_3^k.$$ 

**Step 2:** Let $H_2$ be the subgroup of $V$ given by

$$H_2 = \{1 + \sum_{i=1}^{2}(z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + \sum_{j=0}^{1} a_{i+2j+10y^j}x) + \sum_{i=1}^{4} a_{i+13y^i}x \mid a_i \in F\}.$$

Then $H_2 \cong (C_3^{15k} \times C_3^k) \times C_3^k$.

Let $P_2$ be the abelian subgroup of $H_2$ given by

$$P_2 = \{1 + b_1z + b_2z(1 - z)yx \mid b_i \in F\}.$$

If

$$p_2 = 1 + b_1z + b_2z(1 - z)yx \in P_2$$

and

$$h_1 = 1 + \sum_{i=0}^{2}(z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + a_{i+10}x) + \sum_{i=1}^{4} a_{i+12y^i}x \in H_1,$$

then

$$h_1^{p_2} = 1 + \sum_{i=1}^{2}(z^i - 1)\{a_i + (a_{i+2} - t_0)y + a_{i+4}y^2 + a_{i+6}y^3 + (a_{i+8} + t_0)y^4 + (a_{i+10} - t_1)x + \hat{z}\{a_{i+13}y + (a_{i+14} + t_1)y^2 + (a_{i+15} + t_2)y^3 + (a_{i+16} - t_2)y^4\}x \in H_1$$

where

$$t_0 = b_2(a_{12} - a_{11}),$$

$$t_1 = b_2\{(a_4 - a_3) - (a_{10} - a_9)\},$$

$$t_2 = b_2\{(a_6 - a_5) - (a_8 - a_7)\}.$$

Now

$$R_2 = P_2 \cap H_1 = \{1 + b_1z \mid b_1 \in F\} \cong C_3^k.$$ 

So, for some subgroup $S_2 \cong C_3^k$ of $P_2$, $P_2 = R_2 \times S_2$. Clearly $H_1 \cap S_2 = 1$. Hence

$$H_2 \cong H_1 \times S_2 \cong (C_3^{15k} \times C_3^k) \times C_3^k.$$ 

**Step 3:** Let $H_3$ be the subgroup of $V$ given by

$$H_3 = \{1 + \sum_{i=1}^{2}(z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + \sum_{j=0}^{2} a_{i+2j+10y^j}x) + \hat{z}\sum_{i=3}^{4} a_{i+14y^i}x \mid a_i \in F\}.$$

Then $H_3 \cong ((C_3^{15k} \times C_3^k) \times C_3^k) \times C_3^k$.

Let $P_3$ be the abelian subgroup of $H_3$ given by

$$P_3 = \{1 + b_1z + b_2z(1 - z)y^2x \mid b_i \in F\}.$$
If
\[ p_3 = 1 + b_1 \hat{z} + b_2 z(1 - z)y^2 x \in P_3 \]
and
\[ h_2 = 1 + \sum_{i=0}^{2} (z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + \sum_{j=0}^{1} a_{i+2j+10}y^j x) + \hat{z} \sum_{i=2}^{4} a_{i+13}y^i x \in H_2, \]
then
\[ h_2^{p_3} = 1 + \sum_{i=1}^{2} (z^i - 1)(a_i + (a_{i+2} - t_3)y + (a_{i+4} - t_0)y^2 + (a_{i+6} + t_0)y^3 + (a_{i+8} + t_3)y^4 + (a_{i+10} - t_2)x + (a_{i+12} - t_1)yx) + \hat{z}(a_{15}y^2 + (a_{16} + t_1)y^3 + (a_{17} + t_2)y^4) x \in H_2 \]
where
\[ t_0 = b_2(a_{12} - a_{11}), \quad t_1 = b_2((a_4 - a_3) - (a_{10} - a_9)), \quad t_2 = b_2((a_6 - a_5) - (a_8 - a_7)), \quad t_3 = b_2(a_{14} - a_{13}). \]

Now
\[ R_3 = P_3 \cap H_2 = \{1 + b_1 \hat{z} \mid b_1 \in F\} \cong C_3^k. \]

So, for some subgroup \( S_3 \cong C_3^k \) of \( P_3 \), \( P_3 = R_3 \times S_3 \). Clearly \( H_2 \cap S_3 = 1 \). Hence
\[ H_3 \cong H_2 \times S_3 \cong ((C_3^{15k} \times C_3^k) \times C_3^k) \times C_3^k. \]

**Step 4:** Let \( H_4 \) be the subgroup of \( V \) given by
\[ H_4 = \{1 + \sum_{i=1}^{2} (z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + \sum_{j=0}^{3} a_{i+2j+10}y^j x) + \hat{z}a_{19}y^4 x \mid a_i \in F\}. \]

Then \( H_4 \cong ((C_3^{15k} \times C_3^k) \times C_3^k) \times C_3^k \).

Let \( P_4 \) be the abelian subgroup of \( H_4 \) given by
\[ P_4 = \{1 + b_1 \hat{z} + b_2 z(1 - z)y^3 x \mid b_i \in F\}. \]

If
\[ p_4 = 1 + b_1 \hat{z} + b_2 z(1 - z)y^3 x \in P_4 \]
and
\[ h_3 = 1 + \sum_{i=0}^{2} (z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + \sum_{j=0}^{2} a_{i+2j+10}y^j x) + \hat{z} \sum_{i=3}^{4} a_{i+14}y^i x \in H_3, \]
then
\[
\begin{align*}
    h_3^{P_1} &= 1 + \sum_{i=1}^{2} (z^i - 1) \{a_i + (a_{i+2} - t_3)y + (a_{i+4} + t_4)y^2 + (a_{i+6} - t_0)y^3 \\
    &\quad + (a_{i+8} + t_3)y^4 + (a_{i+10} + t_2)x + (a_{i+12} - t_2)yx + (a_{i+14} - t_1)y^2x \} \\
    &\quad + \widehat{z}\{a_{17}y^3 + (a_{18} + t_1)y^4\}x \in H_3
\end{align*}
\]
where
\[
\begin{align*}
    t_0 &= b_2\{(a_{12} - a_{11}) - (a_{14} - a_{13})\}, \\
    t_1 &= b_2\{(a_4 - a_3) - (a_{10} - a_9)\}, \\
    t_2 &= b_2\{(a_6 - a_5) - (a_8 - a_7)\}, \\
    t_3 &= b_2(a_{16} - a_{15}).
\end{align*}
\]
Now
\[
R_4 = P_4 \cap H_3 = \{1 + b_1\widehat{z} | b_1 \in F\} \cong C_3^k.
\]
So, for some subgroup \(S_4 \cong C_3^k\) of \(P_4\), \(P_4 = R_4 \times S_4\). Clearly \(H_3 \cap S_4 = 1\). Hence
\[
H_4 \cong H_3 \times S_4 \cong (((C_3^{15k} \times C_3^k) \times C_3^k) \times C_3^k) \times C_3^k.
\]
**Step 5:** \(V \cong (((C_3^{15k} \times C_3^k) \times C_3^k) \times C_3^k) \times C_3^k\).

Let \(P_5\) be the abelian subgroup of \(V\) given by
\[
P_5 = \{1 + b_1\widehat{z} + b_2z(1 - z)y^4x | b_1 \in F\}.
\]
If
\[
p_5 = 1 + b_1\widehat{z} + b_2z(1 - z)y^4x \in P_5
\]
and
\[
h_4 = 1 + \sum_{i=0}^{2} (z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + \sum_{j=0}^{3} a_{i+2j+10}y^jx) + \widehat{z}a_{19}y^4x \in H_4,
\]
then
\[
\begin{align*}
    h_4^{P_5} &= 1 + \sum_{i=1}^{2} (z^i - 1)\{a_i + (a_{i+2} + t_3)y + (a_{i+4} + t_4)y^2 + (a_{i+6} - t_4)y^3 \\
    &\quad + (a_{i+8} - t_3)y^4 + (a_{i+10} + t_1)x + (a_{i+12} + t_2)yx + (a_{i+14} - t_1)y^2x \} \\
    &\quad + \widehat{z}\{a_{19}y^4\}x \in H_4
\end{align*}
\]
where
\[
\begin{align*}
    t_1 &= b_2\{(a_4 - a_3) - (a_{10} - a_9)\}, \\
    t_2 &= b_2\{(a_6 - a_5) - (a_8 - a_7)\}, \\
    t_3 &= b_2\{(a_{12} - a_{11}) - (a_{18} - a_{17})\}, \\
    t_4 &= b_2\{(a_{14} - a_{13}) - (a_{16} - a_{15})\}.
\end{align*}
\]
Now
\[
R_5 = P_5 \cap H_4 = \{1 + b_1\widehat{z} | b_1 \in F\} \cong C_3^k.
\]
So, for some subgroup $S_5 \cong C_3^k$ of $P_5$, $P_5 = R_5 \times S_5$. Clearly $H_4 \cap S_5 = 1$. Hence
\[ V \cong H_4 \times S_5 \cong (((((C_{3^{15k}} \times C_3^k) \times C_3^k) \times C_3^k) \times C_3^k) \times C_3^k). \]

By [11, Theorem 2.1],
\[ U(FG) \cong V \times (C_{3^{q-1}}^2 \times GL(2, F)^2), \text{ if } q \equiv \pm 1 \mod 5 \]
and
\[ U(FG) \cong V \times (C_{3^{q-1}}^2 \times GL(2, F^2)), \text{ if } q \equiv \pm 3 \mod 5 \]
where $V \cong (((((C_{3^{15k}} \times C_3^k) \times C_3^k) \times C_3^k) \times C_3^k) \times C_3^k)$.

\[ \square \]

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