Araştırma makalesi

Prime Numbers in Layers

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Abstract

In this article, unlike the known deterministic and probabilistic methods used in determining prime numbers, a new type of deterministic method based on only matrice analysis will be proven without creating a designed equation and without writing preliminary assumptions and prerequisites. For this, it will be shown that the places 30k+1, 30k+7, 30k+11, 30k+13, 30k+17, 30k+19, 30k+23, 30k+29 form eqvvailance classes of the elements of the group U(30). These eight locations will be referred to as eight layers and it will also be proven that the prime numbers are distributed equally across eight layers (equidistribution principle) in the closed system. With this new matrices method, it is also reduced the processing load considerably.

Keywords: *Prime numbers in eqvvailance classes, regular primes, equal distribution, unconditional new deterministic matrices method.*

Özet

Bu yazıda, asal sayıların belirlenmesinde kullanılan, bilinen deterministik ve olasılıkçı yöntemlerden farklı olarak, tasarlanan bir denklem oluşturmadan, ön kabuller ve ön şartlar yazmadan, sadece matris analizlerine dayanan, yeni bir tip deterministik yöntemi kanıtlamaktır. Bunun için, 30k+1, 30k+7, 30k+11, 30k+13, 30k+17, 30k+19, 30k+23, 30k+29, U(30) grubunun eşitlik sınıflarından birisini oluşturduğu gösterilecektir. Bu sekiz yer, sekiz katman olarak adlandırılıp, kapalı sistem içinde asal sayıların sekiz katmana eşit olarak dağıldığı da (eşit dağılım ilkesi/ equidistribution principle) kanıtlanacaktır. Bu yeni matrisler yöntemiyle, işlem yükünün oldukça azaltıldığı da gösterilecektir.

Anahtar Kelimeler: *Eşitlik sınıflarındaki asal sayılar, düzenli asallar, eşit dağılım, şartsız yeni deterministik matrisler yöntemi.*

1. Introduction

For centuries, in mathematics one of the most important problems in the field of number theory is the determination of prime numbers. For this purpose, the process that started with the Sieve of Eratosthenes (240 BC) and gained momentum with Fermat's

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Little Theorem. For the prime number p, which does not divide a number a; it is based on the satisfaction of the equality $a^{p-1} = 1 \pmod p$. Although this method is not exactly correct, it has formed the basis of many new deterministic algebraic methods in the future. Some of these are the AKS [1] primality test in 2004, the Miller [2] primality test in 1975 and the Rabin [3] primality test in 1980. Also in 1974, Solovay and Strassen [4] developed the primality test that could work under the extended Riemann's Hypothesis. In 1983, Adleman, Pomerance and Rumely [5] created a method based on Miller's primality test. In 1986, Goldwasser and Killian [6] found a primality method by using elliptic curves. In1992, Adleman and Huang [7] developed this method.

Besides all these deterministic tests, in 1996 Ronald Joseph Burthe, Jr. [8], in 1995 F. Arnault [9] and in 1998 J. Grantham [10] developed some of the probabilistic tests. However, with the contributions of various complexity theorems that emerged in the 1960s, primality tests had more effective and productive ways of usage. In 1967, Blum [11] and Hartley Rogers Jr. [12], and in 1972, Borodin [13] published some of the studies about complexity theories.

The purpose of this article is to introduce a new type of unconditional deterministic primality test, different from all known deterministic and probabilistic primality tests. For this purpose, in section 2, the layers system that forms the basis of the new type of primality test and the results of this system will be shown. In section 3, a new type of primilaty test will be introduced.

2.   Preliminaries: The Foundatin of Eight Layers

The basic idea of the article starts with the knowledge that primes after 7 cannot be placed in places where multiples of the primes 2, 3, 5. Since the products of the primes 2, 3 and 5 is 30, other primes can be only located in the places $30k + 1, 30k + 7,\ 30k + 11,$ $30k + 13, 30k + 17, 30k + 19, 30k + 23, 30k + 29$. After that, each section of $30k$ and $30k + 29$ will be called 'package' (б). (The section 1-30 taken as the zeroth package.) We named it this way because the matrice method is based on determining in which 30 section each prime number is. Also, there are two reasons why the eight places are called 'layers'. The first reason is; there are more than one equvailance classes of the elements of the group U(30). For example; even numbers form an equvailance class in modula 30 or also odd numbers do. Furthermore, it can be added other examples. But, in modula 30, only one equvailance class which is formed by the places 30k+1, 30k+7, 30k+11, 30k+13, 30k+17, 30k+19, 30k+23, 30k+29 is significant for distribution of prime numbers. So it has to be emphasized the difference of this equvailance class from others. Then the second reason is that if it is thought that the sections of 30's are placed side by side, the numbers in the form of 1, 31, 61, 91,... ($30k$+1) are arranged side by side. This situation horizontally forms a layer. This is class [1] which is also K_1. [1]= K_1. For the numbers 7, 37, 67, 97,..., ($30k + 7$) are arranged side by side. They form class [7] which is also K_2. [7]= K_2. Similar sequences occur for the other 11, 13, 17, 19, 23 and 29 primes. Thus, the values of these eight prime numbers arranged side by side in 30 steps form their classes. So it is obtained [11]= K_3, [13]= K_4, [17]= K_5, [19]= K_6, [23]= K_7, [29]= K_8. These classes

compose equvailance class that is proved in Lemma 2.1. Furthermore, when the prime number is mentioned from now on, only the prime numbers, except the primes 2, 3 and 5, which are placed in layers, will be meant. Also, after that, eight layers will be used instead of eight classes, $[1]= K_1$, $[7]= K_2$, $[11]= K_3$, $[13]= K_4$, $[17]= K_5$, $[19]= K_6$, $[23]= K_7$, $[29]= K_8$.

Lemma 2.1. The set $S = \{1, 7, 11, 13, 17, 19, 23, 29\}$ form a group under the operation of multiplication in mod 30.

Proof. The Cayley table of set S in U(30) proves Lemma 2. 1.

U(30)	1	7	11	13	17	19	23	29	
1	1	7	11	13	17	19	23	29	
7	7	19	17	1	29	13	11	23	
11	11	17	1	23	7	29	13	19	
13	13	1	23	19	11	7	29	17	(1)
17	17	29	7	11	19	23	1	13	
19	19	13	29	7	23	1	17	11	
23	23	11	13	29	1	17	19	7	
29	29	23	19	17	13	11	7	1	

Furthermore, according to lemma 2.1 and its proof, corollary 2.2 and corollary 2.3 are true.

Corollary 2.2. The set S forms an equvailance class.

After that, 'pseudo prime (PP)' expression will be used for numbers that are not prime numbers in the closed system. This expression was defined by many mathematicians in the process that started with Fermat's little theorem. The reason for using the same expression, this known expression is suitable for the purpose of understanding non-prime numbers in layers.

Corollary 2.3. When one of the layer is multiplied with all layers, it is again obtained all layers one by one.

Theorem 2.4. Let $n \in N$, $n \geq 4$, p_n be the nth prime number and $C_n = 2.3.5.7.11...p_n$

For every p_n in the range of C_n, there are only

$$PP_n = (p_{n-1} - 1).(p_{n-2} - 1).(p_{n-3} - 1)....(17 - 1).(13 - 1).(11 - 1).(7 - 1).8 \qquad (2)$$

number of multiples which contains itself and the primes after itself as a multiplier.

Proof. The number of total packages $(Б_T)$ up to C_n is

$$Б_T = \frac{C_n}{30} = \frac{2.3.5.7.11....p_n}{30} = 7.11...p_n$$

Then total place of layers up to C_n is

$$8.\mathit{Б}_T = 8.7.11.13.17 \ldots p_n \tag{3}$$

In these layer places, the number of multiple of 7 is;

$$PP_4 = \frac{p_n \cdot p_{n-1} \cdot p_{n-2} \ldots 13.11.7.8}{7} = p_n \cdot p_{n-1} \cdot p_{n-2} \ldots 13.11.8$$

The number of multiples of 11 is;

$$PP_5 = \frac{8.\mathit{Б}_T - PP_7}{11} = \frac{p_n \ldots 13.11.7.8 - \frac{p_n \cdot p_{n-1} \ldots 13.11.7.8}{7}}{11} = \frac{p_n \cdot p_{n-1} .13.11.7.8.(1 - \frac{1}{7})}{11} =$$

$$p_n \cdot p_{n-1} \cdot p_{n-2} \ldots 13.6.8 \quad PP_5 = p_n \cdot p_{n-1} \cdot p_{n-2} \ldots 13.(7-1).8$$

The number of multiples of 13 is;

$$PP_6 = \frac{8.\mathit{Б}_T - PP_7 - PP_{11}}{13}$$

$$PP_6 = \frac{p_n \cdot p_{n-1} \ldots 13.11.7.8 - p_n \cdot p_{n-1} \cdot p_{n-2} \ldots 13.11.8 - p_n \cdot p_{n-1} \ldots 13.(7-1).8}{13}$$

$$PP_6 = p_n \cdot p_{n-1} \ldots 19.17.10.6.8 = p_n \cdot p_{n-1} \ldots 19.17.(11-1).(7-1).8$$

$$\vdots$$

$$PP_{n-k} = p_n \cdot p_{n-1} \ldots p_{n-k+1} \cdot (p_{n-k-1} - 1) \ldots (17-1).(13-1).(11-1).(7-1).8$$

$$\vdots$$

$$PP_n = (p_{n-1} - 1).(p_{n-2} - 1).(p_{n-3} - 1) \ldots (17 - 1).(13 - 1).(11 - 1).(7 - 1).8.$$

Corollary 2.5. Let $n \in N$, $n \geq 4$, p_n be the nth prime number and $C_n = 2.3.5.7.11 \ldots p_n$. For all prime numbers from 7 to p_n, the multiples that cannot contain primes before itself, are always placed in layers with equal distribution in the width of C_n.

Proof. The first prime in the layers is 7 (n = 4 and p_4 = 7 and C_4 = 2.3.5.7 = 210), by Theorem 2.4 the number of multiples of 7 in the width of C_4 is equal to $PP_4 = \dfrac{7.8}{7} =$

8 . When 7 is multiplied with layers in an interval $[30p - 30(p+1)]$, it gives 8 results in an interval $[210p - 210(p+1)]$. According to the Cayley table of layers (proof of Lemma 2.1), these 8 results are equally distributed in 8 layers. Only in the range of (1-210), there are 7 multiples of prime 7. Because the number 1 appears as an exception. But all the other 210 pieces up to infinity includes 8 multiples of prime 7. Within 210 pieces (210/30 = 7 packages), there are 7.8 = 56 layer places are in equal distribution. When we subtract the 8 multiples of prime 7 with equal distribution, 48 places with equal distribution remains. Although these places are not in the standard layer order (1,7,11,13,17,19,23,29), when a multiple of 7 is subtracted from every layer, there are 6 places remain in every layers. These 48 places are also found when Theorem 2.4 is applied for n = 5. After that , when prime 11 is multiplied by remaining 48 places in an interval $[210p - 210(p+1)]$, the results expand in an interval $[2310p - 2310(p+1)]$. So, these new 48 results is in equal distribution, in the width of all 11.210 = 2310

pieces   except the first one. Also, these multiples do not includes 7. After that, the interval of 13.2310 = 30030 is reached. When multiples of 7 in every 210 pieces and multiples of   11 (not includes 7) in every pieces of 2310 are subtracted from the total layer places in the closed system, which are equally distributed, only remains places for multiples of 13 with equal distribution. That is, when equal distribution parts are removed from the total equal distribution, the number of places remains in the equal distribution. If we generalize, as a result of all primes before p_n are placed by equal distribution, the multiples whose count is defined according to the Theorem 2.4, placed with equal distribution in the width of C_n. According to this, all primes and multiples from 7 to p_n are placed with equal distribution in the width of C_n. Also, while every sequence of multiples of primes continue equal distribution separately in themselves, even if the first primes are removed from every equidistributive sequence of multiples of each prime numbers, the equal distribution property of the sequence of multiples of prime numbers continue. It means that prime numbers go to equal distribution in closed layers system because of equal distribution of pseudo-primes. Furthermore, the proof of the theorem 2.4 gives the algebraic explanation of the corollary 2.5 and corollary 2.13 will give the algebraic proof of the corollary 2.5.

Corollary 2.6. Let $n \in N$, n≥4, p_n be the nth prime number and C_n = 2.3.5.7.11. . . p_n. The ratio (δ_n^{+}) of p_n regular prime and its pseudo primes in the layers becomes

$$\delta_n^{+} = \frac{(p_{n-1}-1).(p_{n-2}-1).(p_{n-3}-1)....(13-1).(11-1).(7-1)}{p_n \cdot p_{n-1} \cdot p_{n-2}....13.11.7} \qquad (4)$$

Proof. Total places in the layers up to C_n boundary is $8.\mathbb{B}_s = 8.7.11.13.17. . . p_n$. Generalization of δ^+ will be reached by proportioning the multiples of a prime of p_n to the total layer places in C_n. The number of multiples of each prime is obtained from Theorem 2.4.

For 7; $\delta_4^{+} = \dfrac{PP_4}{8.\mathbb{B}_T} = \dfrac{p_n \cdot p_{n-1} \cdot p_{n-2}....13.11.8}{p_n \cdot p_{n-1} \cdot p_{n-2}....13.11.7.8} = \dfrac{1}{7}$.

For 11; $\delta_5^{+} = \dfrac{PP_5}{8.\mathbb{B}_T} = \dfrac{p_n \cdot p_{n-1} \cdot p_{n-2}....13.6.8}{p_n \cdot p_{n-1} \cdot p_{n-2}....13.11.7.8} = \dfrac{6}{7.11}$.

For 13; $\delta_6^{+} = \dfrac{PP_6}{8.\mathbb{B}_T} = \dfrac{p_n \cdot p_{n-1} \cdot p_{n-2}....17.10.6.8}{p_n \cdot p_{n-1} \cdot p_{n-2}....17.13.11.7.8} = \dfrac{6.10}{7.11.13}$.

For 17; $\delta_7^{+} = \dfrac{PP_7}{8.\mathbb{B}_T} = \dfrac{p_n \cdot p_{n-1} \cdot p_{n-2}...19.12.10.6.8}{p_n \cdot p_{n-1} \cdot p_{n-2}....19.17.13.11.7.8} = \dfrac{6.10.12}{7.11.13.17}$.

$$\vdots$$

For p_n ; $\delta_n^+ = \dfrac{PP_n}{8.6_T} = \dfrac{(p_{n-1}-1).(p_{n-2}-1).(p_{n-3}-1)....(13-1).(11-1).(7-1)}{p_n \cdot p_{n-1} \cdot p_{n-2}....13.11.7}$.

Corollary 2.7.: When the sum of the proportions of the multiples of all regular primes in the layers is; $\delta^+ = \delta_4^+ + \delta_5^+ + \delta_6^+ +...+ \delta_n^+$, so δ^+ become

$$\delta^+ = \sum_{k=4}^{n}\left[\prod_{r=4}^{k}\left(1-\frac{1}{p_{r-1}}\right) - \prod_{r=4}^{k}\left(1-\frac{1}{p_r}\right)\right] \tag{5}$$

Also in the term $\left(1-\dfrac{1}{p_{r-1}}\right)$ when $p_{r-1} = p_3$ for $r = 4$, $p_3 = 5$ and $\left(1-\dfrac{1}{p_3}\right) = 1$ is used because the prime 5 is not in layers.

Proof. If the sum $\delta^+ = \delta_4^+ + \delta_5^+ + \delta_6^+ +...+ \delta_n^+$ is written in detailed, it is obtained that

$$\delta^+ = \frac{1}{7} + \frac{(7-1)}{7.11} + \frac{(7-1).(11-1)}{7.11.13} + \frac{(7-1).(11-1).(13-1)}{7.11.13.17} + ... + \frac{(p_{n-1}-1).(p_{n-2}-1)...(13-1).(11-1).(7-1)}{p_n.p_{n-1}...13.11.7}$$

$$\delta^+ = \sum_{k=4}^{n}\left[\left(\frac{1}{p_k}\right).\prod_{r=4}^{k}\left(1-\frac{1}{p_{r-1}}\right)\right] = \sum_{k=4}^{n}\left[\left(-\left(1-\frac{1}{p_k}\right)+1\right).\prod_{r=4}^{k}\cdot\left(1-\frac{1}{p_{r-1}}\right)\right]$$

$$\delta^+ = \sum_{k=4}^{n}\left[\prod_{r=4}^{k}\left(-\left(1-\frac{1}{p_r}\right)\right)\left(1-\frac{1}{p_{r-1}}\right) + \prod_{r=4}^{k}\left(1-\frac{1}{p_{r-1}}\right)\right]$$

$$\delta^+ = \sum_{k=4}^{n}\left[\prod_{r=4}^{k}\left(1-\frac{1}{p_{r-1}}\right) - \prod_{r=4}^{k}\left(1-\frac{1}{p_r}\right)\right].$$

Corollary 2.8. Let $n \in N$, $n \geq 4$, p_n be the nth prime number and $C_n = 2.3.5.7.11...p_n$. Up to C_n the sum of the ratios of the primes that couldn't get regular in layers (irregular primes) and the multiples they have created among themselves is

$$\delta_n^- = \delta^- = \prod_{k=4}^{n}\left(1-\frac{1}{p_k}\right) \tag{6}$$

Proof. In the closed layer system, it is obvious that the sum of the total ratio of multiples of regular primes and the total ratio of multiples of irregular primes becomes 1. That is

$$\delta^+ + \delta^- = 1 \tag{7}$$

Then $\delta^- = 1-\delta^+$. Let's generalize the ratio of multiples of irregular primes by finding them separately for each term. Let δ_n^- be after n-th prime starting with the $(n + 1)$th prime up to C_n the total ratio of irregular primes and the pseudo primes formed between themselves. Then we obtain

$$\delta_4^- = 1 - \delta_4^+ = 1 - \frac{1}{7} = \frac{6}{7}$$

$$\delta_5^- = 1 - \delta_4^+ - \delta_5^+ = \frac{6}{7} - \frac{6}{7.11} = \frac{6}{7}\left(1 - \frac{1}{11}\right) = \frac{6}{7} \cdot \frac{10}{11} = \left(1 - \frac{1}{7}\right) \cdot \left(1 - \frac{1}{11}\right)$$

$$\delta_6^- = 1 - \delta_4^+ - \delta_5^+ - \delta_6^+$$

$$\delta_6^- = \frac{6}{7} \cdot \frac{10}{11} - \frac{6.10}{7.11.13} = \frac{6}{7} \cdot \frac{10}{11} \cdot \left(1 - \frac{1}{13}\right) = \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} = \left(1 - \frac{1}{7}\right) \cdot \left(1 - \frac{1}{11}\right) \cdot \left(1 - \frac{1}{13}\right)$$

$$\vdots$$

$$\delta_n^- = 1 - \delta_4^+ - \delta_5^+ - \delta_6^+ - \ldots - \delta_n^+ = \left(1 - \frac{1}{7}\right) \cdot \left(1 - \frac{1}{11}\right) \cdot \left(1 - \frac{1}{13}\right) \ldots \left(1 - \frac{1}{p_n}\right)$$

Then when the limit goes to $C_n = 2.3.5.7.11\ldots p_n$, $\delta_n^- = \delta^- = \prod_{k=4}^{n}\left(1 - \frac{1}{p_k}\right)$ is obtained.

Corollary 2.9. In the closed layers system, within the conditions of the Corollary 2.7., Corollary 2.8. and the equation (7), the total ratio of all primes and pseudo primes is

$$\sum_{k=4}^{n}\left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=4}^{k}\left(1 - \frac{1}{p_{r-1}}\right)\right] + \prod_{k=4}^{n}\left(1 - \frac{1}{p_k}\right) = 1 \qquad (8)$$

The distribution of all prime numbers except the primes of 2,3,5 occurs according to the equality of (8).

Corollary 2.10. Let $n \in N$, n≥4, p_n be the nth prime number and $C_n = 2.3.5.7.11. . . p_n$. According to the Corollary 2.5., all regular primes and multiples from 7 to p_n are in equal distribution, when this regular region is subtracted from the total equal distribution in the closed system, the primes in the remaining irregular region and the pseudo primes they create among themselves are also placed in layers with equal distribution. In other words, like the sum of the regular region, the sum of the irregular region is placed on the layers with equal distribution.

Corollary 2.11. Let $n \in N$, n≥4, p_n be the nth prime number and $C_n = 2.3.5.7.11. . . p_n$. The placement of irregular primes in the layers maintains its equal distribution character within the irregular region.

 Proof. (1) Primes and multiples from 7 to p_n, forming the regular region, are equally distributed in C_n parts according to the Corollary 2.5.

(2) According to Corollary 2.10., like the sum of the regular region, the sum of the irregular region is placed on the layers with equal distribution.

(30)According to Lemma 2.1. and Corollary 2.4, pseudo primes maintain their equal distribution order.

(4) Let k irregular primes be from the prime p_n to C_n. There is a C_{n+k} boundary where all of these primes will become regular. As n goes to infinity, for all irregular primes in the interval $[p_n, C_n]$ there exist a C_{n+k} boundary such that these irregular primes must be in absolute equal distribution. The algebraic proof and result of this item is given in Corollary 2.13.

After all; according to the items (1), (2), (3) and (4), irregular primes within each C_n limit have the character of maintaining equal distribution, even if they are not in absolute equal distribution before reaching their boundaries. The main difference of equal distribution between regular and irregular regions is; in each C_n boundary while regular primes and multiples are in absolutely equal distribution, there is no absolute equal distribution in the primes of the irregular section. While in some layer the number of primes are equal, in some other layers there may be minor differences. At each new boundary, the number of primes in any layer cannot be dominant to the number of primes in other layers, that is, the number of primes in one continuous layer is more or less than the number of primes in other layers, they maintain equal distribution by fluctuating. In the equivalence classes (layers system), since the existence of a C_n boundary in which each prime will become regular is certain and the formation of pseudo primes maintains their equal distribution, all regular and irregular primes maintain equal distribution.

Corollary 2.12. Let $n \in N$, $n \geq 4$, p_n be the nth prime number and $C_n = 2.3.5.7.11...p_n$. The total number of places(D^-) of the primes $(D_p{}^-)$ in irregular region and the pseudo primes $(D_{pp}{}^-)$ that they created among themselves in the C_n range is $D^- = (D_p{}^-) + (D_{pp}{}^-)$ and

$$D^- = 8.(7\text{-}1).(11\text{-}1).(13\text{-}1)...(p_n - 1). \tag{9}$$

Proof. The result is obtained by the product of equation (3) which is the total number of places and equation (6) which is the ratio of the irregular region in the C_n range.

Corollary 2.13. Let n ∈ N, p_n be the nth prime number, then

$$lim_{n\to\infty} \left[\sum_{k=4}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=4}^{k} \left(1 - \frac{1}{p_{r-1}}\right) \right] \right] = 1. \tag{10}$$

Proof. According to the equation (8),

$$\sum_{k=4}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=4}^{k} \left(1 - \frac{1}{p_{r-1}}\right) \right] + \prod_{k=4}^{n} \left(1 - \frac{1}{p_k}\right) = 1$$

If we take the limit of both sides, we obtain

$$\lim_{n\to\infty} \left[\sum_{k=4}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=4}^{k} \left(1 - \frac{1}{p_{r-1}}\right) \right] + \prod_{k=4}^{n} \left(1 - \frac{1}{p_k}\right) \right] = \lim_{n\to\infty} 1$$

$$\lim_{n\to\infty} \left[\sum_{k=4}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=4}^{k} \left(1 - \frac{1}{p_{r-1}}\right) \right] \right] + \lim_{n\to\infty} \left[\prod_{k=4}^{n} \left(1 - \frac{1}{p_k}\right) \right] = 1$$

In the Euler product, let $\zeta(s) = \prod_{k=1}^{\infty} \left(1 - \frac{1}{p_k{}^s}\right)^{-1}$; $\quad p_k$ be k-th prime. Then it is obtained

$$\zeta(1)^{-1} = \prod_{k=1}^{\infty} \left(1 - \frac{1}{p_k}\right).$$

Since $p_k = p_n$, it gets;

$$\prod_{k=1}^{n} \left(1 - \frac{1}{p_n}\right) = \prod_{k=1}^{\infty} \left(1 - \frac{1}{p_k}\right) \text{ and}$$

$$\prod_{k=4}^{n} \left(1 - \frac{1}{p_n}\right) < \prod_{k=1}^{\infty} \left(1 - \frac{1}{p_n}\right) = \zeta(1)^{-1}.$$

Since $\zeta(1) = \infty$ [14], $\zeta(1)^{-1} = 0$ Then it is obtained

$$\lim_{n\to\infty} \left[\prod_{k=4}^{n} \left(1 - \frac{1}{p_k}\right)\right] \le \lim_{n\to\infty}\left[\zeta(1)^{-1}\right] = 0.$$

Since all terms of $\prod_{k=4}^{n} \left(1 - \frac{1}{p_k}\right)$ are non-negative,

$$\lim_{n\to\infty} \left[\prod_{k=4}^{n} \left(1 - \frac{1}{p_k}\right)\right] = 0. \tag{11}$$

Then it is obtained,

$$\lim_{n\to\infty} \left[\sum_{k=4}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=4}^{k} \left(1 - \frac{1}{p_{r-1}}\right)\right]\right] + \lim_{n\to\infty} \left[\prod_{k=4}^{n} \left(1 - \frac{1}{p_k}\right)\right] = 1$$

$$\lim_{n\to\infty} \left[\sum_{k=4}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=4}^{k} \left(1 - \frac{1}{p_{r-1}}\right)\right]\right] + 0 = 1$$

$$\lim_{n\to\infty} \left[\sum_{k=4}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=4}^{k} \left(1 - \frac{1}{p_{r-1}}\right)\right]\right] = 1$$

Corollary 2.13 is the algebraic proof of the equiditribution of pseudo-primes which are in regular region as mentioned in corollary 2.5 and pseudo-primes in equivalence classes as mentioned in forth item of proof of corollary 2.11.

Corollary 2.14. Let $n \in N$, A_n be the nth prime number, then

$$\sum_{k=4}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=4}^{k} \left(1 - \frac{1}{p_{r-1}}\right)\right] = \sum_{k=1}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=1}^{k} \left(1 - \frac{1}{p_{r-1}}\right)\right] = 1 \tag{12}$$

Proof. According to the Corollary 2.7 and Corollary 2.13, it is obtained

$$\sum_{k=4}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=4}^{k} \left(1 - \frac{1}{p_{r-1}}\right)\right] = \frac{1}{7} + \frac{6}{7.11} + \frac{6.10}{7.11.13} + \frac{6.10.12}{7.11.13.17} + \dots = 1$$

Then if the right part of the equation (12) written explicitly,

$$\sum_{k=1}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=1}^{k} \left(1 - \frac{1}{p_{r-1}}\right) \right] = \frac{1}{2} + \frac{1}{2.3} + \frac{2}{2.3.5} + \frac{2.4}{2.3.5.7} + \frac{2.4.6}{2.3.5.7.11} +$$

$$\frac{2.4.6.10}{2.3.5.7.11.13} + \frac{2.4.6.10.12}{2.3.5.7.11.13.17} + \dots$$

$$= \frac{1}{2} + \frac{1}{6} + \frac{1}{15} + \frac{4}{15.7} + \frac{4.6}{15.7.11} + \frac{4.6.10}{15.7.11.13} + \frac{4.6.10.12}{15.7.11.13.17} + \dots$$

$$= \frac{1}{2} + \frac{1}{6} + \frac{1}{15} + \frac{4}{15} \cdot \left(\frac{1}{7} + \frac{6}{7.11} + \frac{6.10}{7.11.13} + \dots \right)$$

In paranthesis;

$$\sum_{k=4}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=4}^{k} \left(1 - \frac{1}{p_{r-1}}\right) \right] = \frac{1}{7} + \frac{6}{7.11} + \frac{6.10}{7.11.13} + \frac{6.10.12}{7.11.13.17} + \dots = 1$$

So, it is obtained;

$$\sum_{k=1}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=1}^{k} \left(1 - \frac{1}{p_{r-1}}\right) \right] = \frac{1}{2} + \frac{1}{6} + \frac{1}{15} + \frac{4}{15} \cdot (1)$$

$$\sum_{k=1}^{n} \left[\left(\frac{1}{p_k}\right) \cdot \prod_{r=1}^{k} \left(1 - \frac{1}{p_{r-1}}\right) \right] = \frac{1}{2} + \frac{1}{6} + \frac{5}{15} = = \frac{1}{2} + \frac{1}{6} + \frac{1}{3} = 1$$

3. The Method of Package Numbers Matrıxes

3.1. Preliminaries: The Linear Algebra of Pseudo-Prime Numbers

Lemma 3.1.1. Let $(Б_1, Б_2, Б_3) \in N$, x, y, s = 1, 2, 3, 4, 5, 6, 7, 8. K_x, K_y, K_s be layers and B is number checked. I n the product; K_s is result layer and layer of checked number B. Also, $Б_s$ is result package number and package number of checked number B.

$$(30Б_1 + K_x)(30Б_2 + K_y) = (30Б_s + K_s) = B \qquad (13)$$

The values of $Б_s$ form a matrix which has all rows and columns are arithmetic sequences.

Proof: In product (1), it is obtained;

$$(30 Б_1 + K_x) \cdot (30 Б_2 + K_y) = 30 Б_1.30 Б_2 + 30 Б_1.K_y + 30 Б_2.K_x + K_x.K_y \qquad (14)$$

According to Lemma 2.1, $K_x.K_y = 30 Б' + K_s$. $\qquad (15)$

So the equation (14) turns into;

$$(30 Б_1 + K_x) \cdot (30 Б_2 + K_y) = 30 Б_1.30 Б_2 + 30 Б_1.K_y + 30 Б_2.K_x + 30 Б' + K_s \qquad (16)$$

the first four terms on the right are taken into 30 parenthesis and it is obtained;

$$(30\ Б_1 + K_x).(30\ Б_2 + K_y) = 30(\ 30\ Б_1.\ Б_2 + Б_1.K_y + Б_2.K_x + Б') + K_s \qquad (17)$$

So; $\quad Б_s = 30\ Б_1.\ Б_2 + Б_1.K_y + Б_2.K_x + Б' \qquad (18)$

It is assumed that $Б_1$ values are rows and $Б_2$ values are columns.

For rows (r_i) ;
$(\ r_1 = 1,\ Б_1 = 0),\ (\ r_2 = 2,\ Б_2 = 1),\ (\ r_3 = 3,\ Б_3 = 2),\ldots,(\ r_i = i,\ Б_x = (i\text{-}1)\ ),\ldots \qquad (19)$

For columns (c_j);
$(\ c_1 = 1,\ Б_1 = 0),\ (\ c_2 = 2,\ Б_2 = 1),\ (\ c_3 = 3,\ Б_3 = 2),\ldots,(\ c_j = j,\ Б_y = (j\text{-}1)\ ),\ldots \qquad (20)$

It will be composed matrixes using either $Б_1$ values or $Б_2$ values in equation (18). Also, both can be used at the same time.

In addition, when used $Б_1$, $Б'$, K_x and K_y are constants, value of each row changes according to $Б_2$ as follows;

Subracting consecutive elements of a row, it is obtained;

$$\left[30Б_1.(Б_2 + 1) + Б_1.K_y + (Б_2 + 1).K_x + Б'\right] - \left[30Б_1.Б_2 + Б_1.K_y + Б_2.K_x + Б'\right]$$

$$= 30Б_1.Б_2 + 30Б_1 + Б_1.K_y + Б_2.K_x + K_x + Б' - 30Б_1.Б_2 - Б_1.K_y - Б_2.K_x - Б'$$

$$= 30Б_1 + K_x \qquad (21)$$

Or, when used $Б_2$, $Б'$, K_x and K_y are constants, value of each column changes according to $Б_1$ like obtaining equation (21), it is accessed that;

$$\left[30Б_2.(Б_1 + 1) + Б_2.K_x + (Б_1 + 1).K_y + Б'\right] - \left[30Б_1.Б_2 + Б_1.K_y + Б_2.K_x + Б'\right]$$
$$= 30Б_2 + K_y \qquad (22)$$

Consequently, according to equations (21) and (22), all rows and columns of the matrixes are arithmetic sequence. That is, all rows and columns of the matrixes have linear relations within themselves.

The Result Matrix (23)

According to equalities (18), (19), (20), (21), (22) and equation $K_x.K_y = 30\ Б' + K_s$; $Б_1$'s are rows and $Б_2$'s are columns.

$$\begin{bmatrix}
Б' & Б' + 1.K_x & Б' + 2.K_x & \ldots & Б' + n.K_x & \ldots \\
Б' + 1.K_y & Б' + 1.K_y + 1.1.30 + 1.K_x & Б' + 1.K_y + 1.2.30 + 2.K_x & \ldots & Б' + 1.K_y + 1.n.30 + n.K_x & \ldots \\
Б' + 2.K_y & Б' + 2.K_y + 2.1.30 + 1.K_x & Б' + 2.K_y + 2.2.30 + 2.K_x & \ldots & Б' + 2.K_y + 2.n.30 + n.K_x & \ldots \\
Б' + 3.K_y & Б' + 3.K_y + 3.1.30 + 1.K_x & Б' + 3.K_y + 3.2.30 + 2.K_x & \ldots & Б' + 3.K_y + 3.n.30 + n.K_x & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
Б' + m.K_y & Б' + m.K_y + m.1.30 + 1.K_x & Б' + m.K_y + m.2.30 + 2.K_x & \ldots & Б' + m.K_y + m.n.30 + n.K_x & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

$$(23)$$

Corollary 3.1.2. According to the matrix (23) and equalities (21) and (22);

$$(Б_s)_{ixj} = Б' + m.K_y + m.n.30 + n.K_x ,$$

so $m = (i - 1) = Б_1$ and $n = (j - 1) = Б_2$ (24)

Corollary 3.1.3. There are 36 multiplication possibilities of equation (15); $K_x.K_y = 30\ Б' + K_s$.

Proof:

Table 1. Multiplication Possibilities of Equation $K_x.K_y = 30\ Б' + K_s$

K_s	$K_x.K_y$
1	(1.1), (7.13), (11.11), (17.23), (19.19), (29.29)
7	(1.7), (11.17), (13.19), (23.29)
11	(1.11), (7.23), (13.17), (19.29)
13	(1.13), (7.19), (11.23), (17.29)
17	(1.17), (7.11), (19.23), (13.29)
19	(1.19), (7.7), (11.29), (13.13), (17.17), (23.23)
23	(1.23), (11.13), (17.19), (7.29)
29	(1.29), (7.17), (13.23), (11.19)

In 8 layers, for the K_s layers 7, 11, 13, 17, 23 and 29, there are 4 different multiplication possibilities giving as a result. There are 6 different multiplication possibilities, giving 1 and 19 layers as a result. Consequently, there are, totally, 36 multiplication possibilities of equation (15); $K_x.K_y = 30\ Б' + K_s$.

Corollary 3.1.4. There are 36 result matrix possibilities according to corollary 3.1.3 and its proof.

Corollary 3.1.5. When it is taken $K_x = 1$ and $Б_1 = 0$ in result matrices of multiplication possibilities of (1,7), (1,11), (1,13), (1,17), (1,19), (1,23), (1,29), some elements of the first rows of these seven results matrices are package numbers of the pseudo-prime numbers and some are package numbers of the prime numbers. Also, when it is taken $K_x = 1$, $K_y = 1$, $Б_1 = 0$ and $Б_2 = 0$ in result matrice of multiplication possibility of (1,1), some elements of the first row and the first column of this result matrice are package numbers of the pseudo-prime numbers and some are package numbers of the prime numbers.

Proof: $K_x = 1$ and $Б_1 = 0$, so $30Б_1 + K_x = 1$. The values of $Б_2$ as integers are multiplied by 1. So these multiplications do not turn the package numbers of prime numbers to the package numbers of pseudo-prime numbers. The first rows of these seven results matrices are composed 0,1,2,3,4,... as integers. In conclusion, some elements of the first rows of these seven results matrices are package numbers of the pseudo-prime numbers and some are package numbers of the prime numbers. In addition, the first row and the first column of result matrice of

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multiplication possibility of (1,1) is in same situation. These rows or/and columns are exeptional first rows and columns.

Corollary 3.1.6. Exept for the elements of the exeptional rows and columns in corollary 3.1.5, all elements of 36 matrix possibilities of the result matrix(23) are package numbers of the pseudo-prime numbers according to corollaries 3.1.2, 3.1.3 and 3.1.4.

Theorem 3.1.7. $Б' \in$ N, K_s is the layer and B = 30 $Б'$ + K_s. According to corollary 3.1.5 and corollary 3.1.6, the number B is a prime number if the value $Б'$ is not an element of one of the resulting matrices that yields K_s without using exeptional rows or/and columns.

Theorem 3.1.8. The determinant of a square matrix formed by four elements with locations ((a+1),(b+1)), ((a+1),(d+1)), ((c+1),(b+1)) and ((c+1),(d+1)) in any 36 result matrices is:

$$\det \begin{bmatrix} Б' + a.K_y + a.b.30 + b.K_x & Б' + a.K_y + a.d.30 + d.K_x \\ Б' + c.K_y + c.b.30 + b.K_x & Б' + c.K_y + c.d.30 + d.K_x \end{bmatrix} = - K_s . (c\text{-}a). (d\text{-}b). \quad (25)$$

Proof: According to equalities (19) and (20), For rows (r_i) ; r_1 = (a+1) , p_1 = a and r_2 = (c+1), p_2 = c

For columns (c_j); c_1 = (b+1), p_3 = b and c_2 = (d+1), p_4 = d. When a,b,c,d values are substituted in a 2x2 square marix according to equation (18), it is formed;

$$\begin{bmatrix} Б' + a.K_y + a.b.30 + b.K_x & Б' + a.K_y + a.d.30 + d.K_x \\ Б' + c.K_y + c.b.30 + b.K_x & Б' + c.K_y + c.d.30 + d.K_x \end{bmatrix} \quad (26)$$

When it is calculated the determinant of the matrix (26);

$\det[(26)] = (Б' + a.K_y + a.b.30 + b.K_x).(Б' + c.K_y + c.d.30 + d.K_x)$
$\qquad -(Б' + a.K_y + a.d.30 + d.K_x).(Б' + c.K_y + c.b.30 + b.K_x)$

$\det[(26)] = Б'.Б' + Б'.c.K_y + Б' . c.d.30 + Б' . d.K_x + a.K_y . Б' + a.K_y . c.K_y + a.K_y.c.d.30 + a.K_y.d.K_x + a.b.30 . Б' + a.b.30 . c.K_y + a.b.30 . c.d.30 + a.b.30. d.K_x + b.K_x. Б' + b.K_x. c.K_y + b.K_x.c.d.30 + b.K_x. d.K_x - Б'.Б'- Б'.c.K_y - Б' . c.b.30 - Б' . b.K_x - a.K_y . Б' - a.K_y . c.K_y - a.K_y . c.b.30 - a.K_y . b.K_x - a.d.30. Б' - a.d.30. c.K_y - a.d.30.c.b.30 - a.d.30. b.K_x - d.K_x. Б' - d.K_x. c.K_y - d.K_x. c.b.30 - d.K_x. b.K_x$

$\det[(26)] = Б'. c.d.30 + a.K_y. d.K_x + a.b.30. Б' + b.K_x. c.K_y - Б'.c.b.30 - a.K_y . b.K_x - a.d.30. Б' - d.K_x. c.K_y$

det[(26)] = 30. Б′.(c.d +a.b – cb –a.d) - K_x. K_y .(- a.d - b.c +a.b + c.d)

det[(26)] = (30. Б′ - K_x. K_y). (c.d +a.b – cb –a.d)

According to equation (15) $K_x.K_y$ = 30 Б′ + K_s , so; - K_s = 30 Б′ - $K_x.K_y$

det[(26)] = - K_s . (c.d +a.b – cb –a.d) = - K_s .[c. (d − b) − a. (d − b)]

det[(26)] = - K_s . (c-a). (d-b)

3.2. The Method of Finding Prime Numbers by Using Matrixes of Package Numbers of Pseudo- Prime Numbers:

Definition 3.2.1. The method of finding prime numbers by using matrices of package numbers of pseudo- prime numbers will be defined step by step:

Step 1: Using equation (13), it is reached Б$_s$, K_s values.

Step 2: In equation ; $K_x.K_y$ = 30 Б′ + K_s and according to the table 1, it is obtained K_x , K_y and Б′ values.

Step 3: Substituting the values obtained from step 1 and 2 to the matrix (23). So, it is composed all matrice possibilites (4 or 6, according to table 1) of K_s .

Step 4: It is checked whether Б$_s$ value is an element of one of the possible matrices exept for the elements of the exeptional first rows or/and columns in corollary 3.1.5.

Step 5: If it is not an element of any possible matrices, B is a prime number (Theorem 3.1.7).

Step 6: If , at least, it is one of an element of any possible result matrices (without exeptional rows or/and columns), B is a pseudo- prime number.

Step 7: For Б$_s$ values which is found in all possible matrixes, it is obtained all multipliers of number B according to equalities (24) and equation (13).

$$(30(i-1) + K_x).(30(j-1) + K_y) = (30Б_1 + K_x)(30Б_2 + K_y) = (30Б_s + K_s) = B \quad (27)$$

Examples 3.2.2. :

1) B = 553243, is it a prime number?

553243 = 30.18441 + 13, so Б$_s$ = 18441 and K_s = 13.

It has 4 matrice possibilities which are (1.13), (7.19), (11.23), (17.29).

Only the element of matrice possibility of (17.29 = 30.16 +13) gives Б$_s$ = 18441 value. Also, Б′ = 16.

Using matrice (23), with respect to equation $(30Б_1 + 17)(30Б_2 + 29) = (30Б_s + 13)$=B, the equality (18) which gives $(Б_s = 30Б_1 . Б_2 + Б_1 . 29 + Б_2 . 17 + 16)$ generates the matrix where $Б_1$'s are rows and $Б_2$'s are columns.

$$\begin{bmatrix} 16 & 33 & 50 & 67 & ... \\ 45 & 92 & 139 & 186 & ... \\ 74 & 151 & 228 & 305 & ... \\ \vdots & \vdots & \vdots & \vdots & ... \\ (9064)_{313,1} & (18441)_{313,2} & (27818)_{313,3} & (37195)_{313,4} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

i = 313, j = 2, so, according to equation (26);

$(30(i-1) + K_x) . (30(j-1) + K_y) = (30.312 + 17)(30.1 + 29) = 9377.59 = 553243 = B$

Proof of example 1: The number 18441 is in the second column. According to equation (22), the difference between consecutive elements of second column is; 30.(2-1) +29 = 59. It must be 33 + 59.m =18441 and m must be an integer, too.

$m = \frac{18441 - 33}{59} = 312$; m is an integer. m = (i -1) = 312 and i = 313. So, 553243 is not a prime number.

   2) B = 105817, is it a prime number?

   105817 = 30.3527 + 7, so $Б_s$ = 3527 and K_s = 7.

It has 4 matrice possibilities which come from (1.7), (11.17), (13.19), (23.29). When these possible result matrices are composed according to equality (18) and matrice (23), any of 4 matrices do not include 3527 as an element. Consequently, 105817 is a prime number according to the Theorem 3.1.7.

3.3. General Using of the Matrice Method

According to matrice algorithm which is matrice (23), computers compose 36 matrices. Package numbers are asked to computers one by one as integers goes to infinity. It is determined in which matrices an searched integer is found or not. Furthermore, the first rows or/and columns of matrices which are composed by 1 are not included searching. There are pseudo primes in the result layers (K_s) defined by the matrices containing the searched integer and there are primes in the result layers (K_s) defined by the matrices which do not contain the searched integer. As a result, what needs to be done is to search an integer in which 36 matrices have as elements or not. That is, one number is asked and eight answers are received. It is not necessary to repeatedly compose 36 matrices to search prime numbers. Matrices are formed and they are used to check prime numbers infinitely. Also, for very large elements of matrices, some arrangements can be made within the matrices using linear algebra. Besides all of these, the most important

result of this matrice method is that for finding a prime number, it does not have to been searched prime numbers. Thus, it is reduced the processing load considerably.

4. Conclusion

In this study, it was proved that all prime numbers except the first three prime numbers form equivalence class and that the placement of prime numbers in this system is realized with an equal distribution. In addition, it has been proved algebraically that the non-prime numbers formed by the prime numbers in this system are equally distributed. Furthermore, the new type of prime number determination method obtained depending on the advantage provided by the layer system and the linear algebra also has a much more development potential.

Acknowledgement

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