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Graphical sequences of some family of induced subgraphs

Research Article

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Abstract: The subdivision graph $S(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. The S_{vertex} or S_{ver} join of the graph G_1 with the graph G_2, denoted by $G_1 \dot{\vee} G_2$, is obtained from $S(G_1)$ and G_2 by joining all vertices of G_1 with all vertices of G_2. The S_{edge} or S_{ed} join of G_1 and G_2, denoted by $G_1 \bar{\vee} G_2$, is obtained from $S(G_1)$ and G_2 by joining all vertices of $S(G_1)$ corresponding to the edges of G_1 with all vertices of G_2. In this paper, we obtain graphical sequences of the family of induced subgraphs of $S_J = G_1 \vee G_2$, $S_{ver} = G_1 \dot{\vee} G_2$ and $S_{ed} = G_1 \bar{\vee} G_2$. Also we prove that the graphic sequence of S_{ed} is potentially $K_4 - e$-graphical.

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Keywords: Graphical sequences, Subdivision graph, Join of graphs, Split graph

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with n vertices and m edges having vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. The set of all non-increasing non-negative integer sequences $\pi = (d_1, d_2, \cdots, d_n)$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. There are several famous results, Havel and Hakimi [6–8] and Erdös and Gallai [2] which give necessary and sufficient conditions for a non-negative sequence $\pi = (d_1, d_2, \cdots, d_n)$ to be the degree sequence of a simple graph G. A graphical sequence π is potentially H-graphical if there is a realization of π containing H as a subgraph, while π is forcibly H graphical if every realization of π contains H as a subgraph. If π has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1}-graphic. We know that a graphic sequence π is potentially K_{k+1}-graphic if and only if π is potentially A_{k+1}-graphic [10, 11]. The disjoint union of the graphs G_1 and G_2 is defined by $G_1 \bigcup G_2$. If $G_1 = G_2 = G$, we abbreviate $G_1 \bigcup G_2$ as $2G$. Let K_k, C_k and P_k respectively denote a complete graph on k vertices, a cycle on k vertices and a path on $k+1$ vertices.

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A sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be potentially K_{r+1}- graphic if there is a realization G of π containing K_{r+1} as a subgraph. If π is a graphic sequence with a realization G containing H as a subgraph, then in [4], it is shown that there is a realization G of π containing H with the vertices of H having $|V(H)|$ largest degree of π. In 2014 [1], Bu, Yan, X. Zhou and J. Zhou obtained Resistance distance in the subdivision vertex join and edge join type of graphs. Also conditions for r-graphic sequences to be potentially $K^{(r)}_{m+1}$-graphic can be seen in [12].

2. Definitions and preliminary results

In the simple graph G, let d_i be the degree of v_i for $1 \leq i \leq n$. Then $\pi = (d_1, d_2, \cdots, d_n)$ is the degree sequence of G. We note that the vertices have been labelled so that π is in increasing order. The degree sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be potentially A_{r+1}-graphic if it has a realization $H = (V(H), E(H))$, where $V(H) = \{u_1, u_2, \cdots, u_n\}$ and the degree of u_i is d_i for $1 \leq i \leq n$, such that the subgraph induced by $\{u_1, u_2, \cdots, u_{r+1}\}$ is K_{r+1}. For $\pi = (d_1, d_2, \cdots, d_n) \in NS_n$ and $1 \leq k \leq n$, let

$$\begin{aligned} \pi' &= (d_1 - 1, \cdots, d_{k-1} - 1, \cdots, d_k + 1 - 1, d_k + 2, \cdots, d_n), \quad if \quad d_k \geq k, \\ &= (d_1 - 1, \cdots, d_k - 1, \cdots, d_k + 1, \cdots, d_{k-1}, d_{k+1}, d_n), \quad if \quad d_k < k. \end{aligned}$$

Denote $\pi'_k = (d_1^{i'}, d_2^{i'}, \cdots, d_{n-1}^{i'})$ where $1 \leq i' \leq n$ and $d_1^{i'}, d_2^{i'}, \cdots, d_{n-1}^{i'}$ is a rearrangement of the $n-1$ terms of π'. Then π' is called the residual sequence obtained by laying off d_k from π.

Gould, Jacobson and Lehel [4] obtained the following result.

Theorem 2.1. *If $\pi = (d_1, d_2, \cdots, d_n)$ is the graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.*

Throughout this paper, we take $\pi_1 = (d_1^1, d_2^1, \cdots, d_m^1)$ and $\pi_2 = (d_1^2, d_2^2, \cdots, d_n^2)$ respectively to be the graphical sequence of the graphs G_1 and G_2. Let $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$ and d^t in the graphic sequence π means d occurs t-times.

The following definitions will be required for obtaining the main results.

Definition 2.2. *The join of G_1 and G_2 is a graph $S_J = G_1 \vee G_2$ with vertex set $V(G_1) \cup V(G_2)$ and an edge set consisting of all edges of G_1 and G_2, together with the edges joining each vertex of G_1 with every vertex of G_2.*

Definition 2.3. *The subdivision graph $S(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. Equivalently, each edge of G is replaced by a path of length 2. Figure 1 shows the subdivision graph $S(G)$ of the graph G. The vertices inserted are denoted by open circles.*

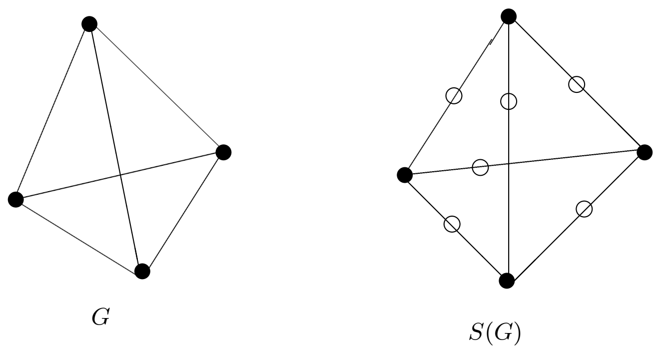
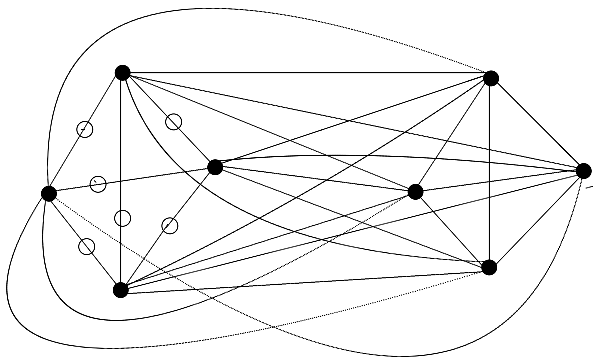


Figure 1.

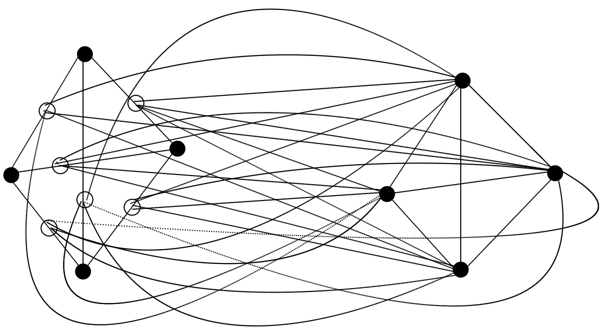
Definition 2.4. *The subdivision vertex join of* G_1 *onto* G_2*, denoted by* $S_{ver} = G_1 \dot{\vee} G_2$*, is the graph obtained from* $S(G_1) \cup G_2$ *by joining every vertex of* $V(G_1)$ *to every vertex of* $V(G_2)$*. Figure 2 gives* $S_{ver} = K_4 \dot{\vee} K_4$*.*



$S_{ver} = K_4 \dot{\vee} K_4$

Figure 2.

Definition 2.5. *Let* G_1 *and* G_2 *be two graphs, let* $S(G_1)$ *be the subdivision graph of* G_1 *and let* $I(G_1)$ *be the set of new inserted vertices of* $S(G_1)$*. The subdivision edge join of* G_1 *and* G_2 *denoted by* $S_{ed} = G_1 \bar{\vee} G_2$*, is the graph obtained from* $S(G_1) \cup G_2$ *by joining every vertex of* $I(G_1)$ *to every vertex of* $V(G_2)$*. Figure 3 below illustrates this operation by taking* $S_{ed} = K_4 \bar{\vee} K_4$*.*



$S_{ed} = K_4 \bar{\vee} K_4$

Figure 3.

Definition 2.6. *If the vertex set of a graph can be partitioned into a clique and an independent set, then it is called a split graph [5]. Let* K_r *and* K_s *be complete graphs on* r *and* s *vertices. Clearly* $K_r \vee \overline{K}_s$ *is one type of split graph on* $r + s$ *vertices and is denoted by* $S_{r,s}$*.*

Pirzada and Bilal [9] defined new types of graphical operations and obtained graphical sequences of some derived graphs.

Definition 2.7. *Let* K_r *and* K_s *be any two graphs. Let* $K_{\dot{r}}$ *be the subdivision graph of* K_r *and* $\overline{K}_s$ *be the complement of* K_s*. The graphs* $\left(B_{\dot{r},s}\right) = K_{\dot{r}} \dot{\vee} \overline{K}_s$ *is called the* r*-vertex sub-division-*$S_{r,s}$*-graph and the graph* $\left(B_{\bar{r},s}\right) = K_{\dot{r}} \bar{\vee} \overline{K}_s$ *is called the* r*-edge sub-division-*$S_{r,s}$*-graph. These are illustrated in Figures 4, 5, 6 and 7 below by taking the graphs* K_4 *and* K_2*.*

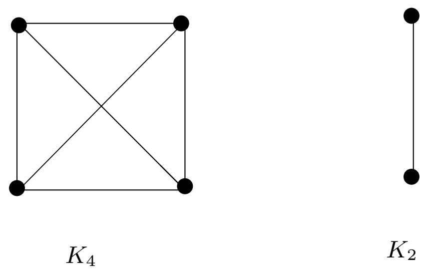


Figure 4.

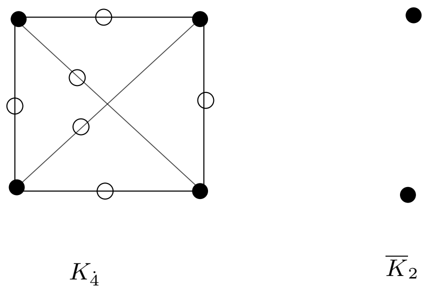


Figure 5.

In 2014, Pirzada and Bilal [9] proved the following assertion.

Theorem 2.8. *If G_1 is a realization of $\pi_1 = (d_1^1, \ldots, d_m^1)$ containing K_p as a subgraph and G_2 is a realization of $\pi_2 = (d_1^2, \ldots, d_n^2)$ containing K_q as a subgraph, then the degree sequence $\pi = (d_1, \ldots, d_{m+n})$ of the join of G_1 and G_2 is K_{p+q}-graphic.*

The purpose of this paper is to find the graphic sequence of the family of induced subgraphs of S_J, S_{ver} and S_{ed}. We also give the characterization for a graphic sequence of S_{ed} to be potentially $K_4 - e$-graphical.

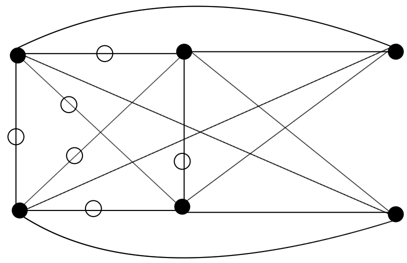
Now we have the following observations.

Remark 2.9. *Let $L_r = K_{a_1,a_2,\cdots,a_r}$ and $M_r = K_{b_1,b_2,\cdots,b_r}$ respectively be the r-partite graphs on $\sum\limits_{i=1}^{r} a_i$ and $\sum\limits_{i=1}^{r} b_i$ vertices. Let $l_1 = \sum\limits_{i=1}^{r} a_i$ and $l_1' = \sum\limits_{i=1}^{r} b_i$ and define*

$$l = \sum_{i=1}^{r} \big(a_i + b_i\big), \quad m = \sum_{i=1}^{r} \big(a_i^2 + b_i^2\big).$$

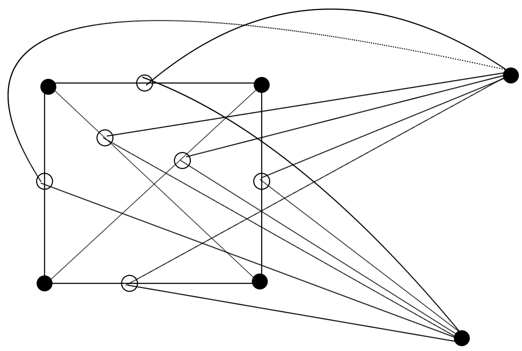
Clearly, the number of edges in $K_{a_1,a_2,\cdots,a_r} = |E_{L_r}| = \sum\limits_{i,j=1, i\neq j}^{\binom{r}{2}} a_i a_j$ and the number of edges in $K_{b_1,b_2,\cdots,b_r} = |E_{M_r}| = \sum\limits_{i,j=1,i\neq j}^{\binom{r}{2}} b_i b_j$.

Remark 2.10. *Let cl_{2m} be the circular ladder with $2m$ vertices, $m \geq 3$. The circular ladder is the graph formed by taking two copies of the cycle C_m with corresponding vertices from each copy of C_m being adjacent. A ladder graph on 8 vertices is shown in Figure 8. Let $(K_{2m} - cl_{2m})$ be the graph obtained from K_{2m} by removing the edges of cl_{2m}. For $m \geq 3$, it can be easily seen that $(K_{2m} - cl_{2m})$ is a $2m - 4$ regular graph on $2m$ vertices and the number of edges in this graph being $2m(m-2)$.*



$(B_{\dot{4},2}) = K_{\dot{4}}\dot{\vee}\overline{K_2}$

Figure 6.



$(B_{\bar{4},2}) = K_{\dot{4}}\bar{\vee}\overline{K_2}$

Figure 7.

Remark 2.11. *In the split graph $S_{r,s}$, the number of vertices is $r+s$ and number of edges is $\frac{r(r-1)}{2}+rs$. That is, $|V(S_{r,s})| = r+s$ and $|E(S_{r,s})| = \frac{r(r-1)}{2}+rs$. Figure 9 ilustrates $S_{3,2}$ and $S_{4,1}$.*

Remark 2.12. *If π_1 and π_2 respectively are the graphic sequences of G_1 and G_2, the graphic sequence of $S_{ver} = G_1\dot{\vee}G_2$ is $\pi = (d_1^1+n, d_2^1+n, \cdots, d_m^1+n, d_1^2+m, \cdots, d_n^2+m, 2^{|E_1|})$ and the graphic sequence of $S_{ed} = G_1\bar{\vee}G_2$ is $\pi = (d_1^1, d_2^1, \cdots, d_m^1, d_1^2+|E_1|, \cdots, d_n^2+|E_1|, (2+n)^{|E_1|})$.*

3. Main results

In the following result, we find the graphic sequence of the induced subgraph $(K_{2m'}-cl_{2m'})\dot{\vee}(K_{2n'}-cl_{2n'})$ of the graph $S_{ver} = G_1\dot{\vee}G_2$.

Theorem 3.1. *If π_1 and π_2 respectively are potentially $(K_{2m'}-cl_{2m'})$ and $(K_{2n'}-cl_{2n'})$-graphic sequences, $m' \geq 3$, $n' \geq 3$, $m \geq 2m'$, $n \geq 2n'$, then the graphic sequence of $(K_{2m'}-cl_{2m'})\dot{\vee}(K_{2n'}-cl_{2n'})$ is $\big((2(m'+n'-2))^{2(m'+n')}, 2^{2m'(m'-2)}\big)$.*

Proof. By Remark 2.12 and Theorem 2.1, the graphic sequence of S_{ver} is $\pi = (d_1^1+n, d_2^1+n, \cdots, d_m^1+n, d_1^2+m, \cdots, d_n^2+m, 2^{|E_1|})$. Now let π^* be the graphic sequence of the induced subgraph $(K_{2m'}-cl_{2m'})\dot{\vee}(K_{2n'}-cl_{2n'})$ of S_{ver}. By taking $|E(K_{2m'}-cl_{2m'})| = 2m'(m'-2)$, we have

$$\pi^* = \big(d_1^{1'}+2n', d_2^{1'}+2n', \cdots, d_{2m'}^{1'}+2n', d_1^{2'}+2m', d_2^{2'}+2m',$$

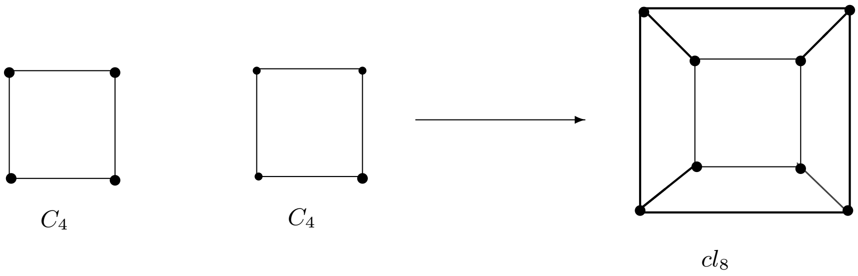


Figure 8.

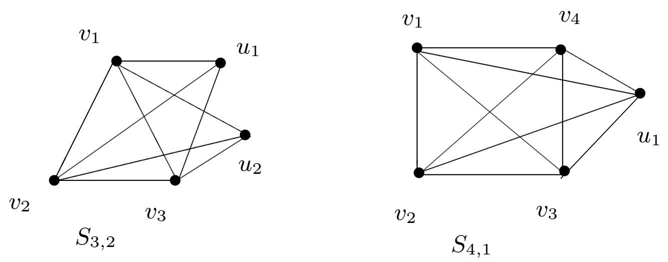


Figure 9.

$$\begin{aligned}&\cdots, d_{2n'}^{2'} + 2m', 2^{2m'(m'-2)})\\&= (2m' - 4 + 2n', 2m' - 4 + 2n', \cdots, 2m' - 4 + 2n', 2n' - 4 + 2m',\\&\cdots, 2n' - 4 + 2m', 2^{2m'(m'-2)})\\&= \big((2(m' + n' - 2))^{2(m'+n')}, 2^{2m'(m'-2)}\big).\end{aligned}$$

□

Corollary 3.2. *If π_1 and π_2 are potentially $(K_{2m'} - cl_{2m'})$-graphic sequences, where $m' \geq 3,\ m \geq 2m',\ n \geq 2m'$, then the graphic sequence of the induced subgraph $(K_{2m'} - cl_{2m'})\dot{\vee}(K_{2m'} - cl_{2m'})$ of S_{ver} is $\big((4(m'-1))^{4m'}, 2^{2m'(m'-2)}\big)$ and $\sigma(\pi^*) = 4m'(5m'-6)$.*

Proof. Put $m' = n'$ in Theorem 3.1, we get

$$\begin{aligned}\pi^* &= (2m' - 4 + 2m', 2m' - 4 + 2m', \cdots, 2m' - 4 + 2m', 2m' - 4 + 2m',\\&\cdots, 2m' - 4 + 2m', 2^{2m'(m'-2)})\\&= \big((4(m'-1))^{4m'}, 2^{2m'(m'-2)}\big)\end{aligned}$$

Also $\sigma(\pi^*) = 4m'(4(m'-1)) + 2m'(m'-2)2 = 4m'(5m'-6)$. □

The following result shows that the graphic sequence π of $S_{ed} = G_1 \bar{\nabla} G_2$ is potentially $K_4 - e$-graphical.

Theorem 3.3. *If π_1 and π_2 respectively are potentially K_{p_1} and K_{p_2}-graphic sequences, where $m \geq 3,\ n \geq 2,\ p_1 \leq m$ and $p_2 \leq n$, then the graphical sequence π of S_{ed} is potentially $K_4 - e$-graphical.*

Proof. Let π_1 and π_2 respectively be potentially K_{p_1} and K_{p_2}-graphic sequences, where $m \geq 3,\ n \geq 2,\ p_1 \leq m$ and $p_2 \leq n$. Let $S_{ed} = G_1 \bar{\nabla} G_2$ and π be its graphic sequence. Then $\pi = \big(d_1^1, d_2^1, \cdots, d_m^1, d_1^2 +$

$|E_1|, d_2^2+|E_1|, \cdots, d_n^2+|E_1|)$. Clearly there are at-least three vertices and at least two edges in G_1 and there are at least two vertices and at least one edge in G_2, since G_1 and G_2 are connected. Let v_i, v_j and v_k be any three vertices in G_1 and u_i and u_j be any two vertices in G_2. Since there are at least two edges in G_1 and at least one edge in G_2, without loss of generality we take $v_iv_j, v_jv_k \in E(G_1)$ and $u_iu_j \in E(G_2)$. By construction, it can easily be seen that the graph G formed from G_1 and G_2 contains a subgraph on u_i', u_j', u_i and u_j vertices (where u_i' and u_j' are the two inserted vertices in v_iv_j and v_jv_k of G_1) which is K_4-e. Thus π is potentially K_4-e graphical. □

Now we obtain the graphic sequence of the induced subgraph $S_{r_1,s_1}\bar{\nabla}S_{r_2,s_2}$ of $S_{ed}=G_1\bar{\nabla}G_2$.

Theorem 3.4. *If π_1 and π_2 respectively are potentially S_{r_1,s_1} and S_{r_2,s_2}-graphic, then the graphic sequence of the induced subgraph $S_{r_1,s_1}\bar{\nabla}S_{r_2,s_2}$ of S_{ed} is*

$$\begin{aligned}\pi^* =\Bigg(&\left(\frac{2(r_2+s_2-1)+r_1(2s_1+r_1-1)}{2}\right)^{r_2}, \left(\frac{2r_2+r_1(2s_1+r_1-1)}{2}\right)^{s_2},\\ &(2+r_2+s_2)^{\frac{r_1(2s_1+r_1-1)}{2}}, (r_1+s_1-1)^{r_1}, r_1^{s_1}\Bigg).\end{aligned}$$

Proof. Let π^* be the graphic sequence of the induced subgraph $S_{r_1,s_1}\bar{\nabla}S_{r_2,s_2}$ of S_{ed}. By Remark 2.11 and Theorem 2.1, we have

$$\begin{aligned}\pi^* &= \big(d_1^{1'}, d_2^{1'}, \cdots, d_{r_1}^{1'}, d_{r_1+1}^{1'}, d_{r_1+2}^{1'}, \cdots, d_{r_1+s_1}^{1'}, d_1^{2'}, d_2^{2'}, \cdots, d_{r_2+s_2}^{2'}, (2+r_2+s_2)^{|E(S_{r_1,s_1})|}\big)\\
&= \big(r_1+s_1-1, r_1+s_1-1, \cdots, r_1+s_1-1, r_1, r_1, \cdots, r_1, r_2+s_2-1+\frac{r_1(r_1-1)}{2}+(r_1s_1),\\
&\cdots, r_2+s_2-1+\frac{r_1(r_1-1)}{2}+(r_1s_1), (r_2+|E(S_{r_1,s_1})|),\\
&\cdots, (r_2+|E(S_{r_1,s_1})|), (2+r_2+s_2)^{|E(S_{r_1,s_1})|}\big)\\
&=\Bigg((r_1+s_1-1)^{r_1}, r_1^{s_1}, \left(\frac{2(r_2+s_2-1)+r_1(r_1-1)+2r_1s_1}{2}\right)^{r_2},\\
&\left(\frac{2r_2+r_1(r_1-1)+2r_1s_1}{2}\right)^{s_2}, (2+r_2+s_2)^{\frac{r_1(r_1-1)+2r_1s_1}{2}}\Bigg)\\
&=\Bigg(\left(\frac{2(r_2+s_2-1)+r_1(2s_1+r_1-1)}{2}\right)^{r_2}, \left(\frac{2r_2+r_1(2s_1+r_1-1)}{2}\right)^{s_2},\\
&(2+r_2+s_2)^{\frac{r_1(2s_1+r_1-1)}{2}}, (r_1+s_1-1)^{r_1}, r_1^{s_1}\Bigg).\end{aligned}$$

□

Next we obtain the graphic sequence of the induced subgraph $K_{p_1}\bar{\nabla}K_{p_2}$ and $S_{r_2,s_2}\bar{\nabla}S_{r_1,s_1}$ of $S_{ed}=G_1\bar{\nabla}G_2$.

Theorem 3.5. *If π_1 and π_2 respectively are potentially K_{p_1} and K_{p_2}-graphic, then the graphic sequence of the induced subgraph $K_{p_1}\bar{\nabla}K_{p_2}$ of S_{ed} is*

$$\pi^* =\left((p_1-1)^{p_1}, \left(\frac{p_1(p_1-1)+2(p_2-1)}{2}\right)^{p_2}, (2+p_2)^{\frac{p_1(p_1-1)}{2}}\right),$$

where $p_1 \geq 2$, $p_2 \geq 1$.

Proof. By Theorem 2.1, in the graphic sequence of the induced subgraph $K_{p_1}\bar{\nabla}K_{p_1}$ of S_{ed}, we have $d_1^{1'}=p_1-1, d_2^{1'}=p_1-1, \cdots, d_{p_1}^{1'}=p_1-1, d_1^{2'}=p_2-1, \cdots, d_{p_2}^{2'}=p_2-1$, $|E(K_{P_1})|=\frac{p_1(p_1-1)}{2}$ and $n=p_2$. Thus the graphic sequence π^* of the induced subgraph $K_{p_1}\bar{\nabla}K_{p_1}$ of S_{ed} is

$$\pi^* = \left(p_1-1, \cdots, p_1-1, p_2-1+\frac{p_1(p_1-1)}{2}, \cdots, p_2-1+\frac{p_1(p_1-1)}{2}, (2+p_2)^{\frac{p_1(p_1-1)}{2}}\right)$$

$$= \left((p_1 - 1)^{p_1}, \left(\frac{p_1(p_1 - 1) + 2(p_2 - 1)}{2} \right)^{p_2}, (2 + p_2)^{\frac{p_1(p_1-1)}{2}} \right).$$

□

Theorem 3.6. *If π_1 and π_2 respectively are potentially $(K_{2m'} - cl_{2m'})$ and $(K_{2n'} - cl_{2n'})$-graphic, where $m', n' \geq 3$, then the graphic sequence of the induced subgraph $(K_{2m'} - cl_{2m'})\bar{\nabla}(K_{2n'} - cl_{2n'})$ of $S_{ed} = G_1 \bar{\nabla} G_2$ is*

$$\pi^* = \left(\left(2(m' - 2)\right)^{2m'}, \left(2(n' + m'^2) - 4(n' + 1)\right)^{2n'}, (2(1 + n'))^{2m'(m'-2)} \right).$$

Proof. Let π^* is the graphic sequence of $(K_{2m'} - cl_{2m'})\bar{\nabla}(K_{2n'} - cl_{2n'})$. Then by Theorem 2.1, we have $d_1^{1'} = 2m' - 4 = d_2^{1'} = d_{2m'}^{1'}, d_1^{2'} + |E((K_{2m'} - cl_{2m'}))| = d_2^{2'} + |E((K_{2m'} - cl_{2m'}))|, \cdots, d_{2n'}^{2'} + |E((K_{2m'} - cl_{2m'}))| = 2n' - 4 + (2m' - 4)m'$. Thus the graphic sequence π^* of the required induced subgraph $(K_{2m'} - cl_{2m'})\bar{\nabla}(K_{2n'} - cl_{2n'})$ of S_{ed} becomes $\left(\left(2(m' - 2)\right)^{2m'}, \left(2(n' + m'^2) - 4(n' + 1)\right)^{2n'}, (2(1 + n'))^{2m'(m'-2)} \right)$. □

Theorem 3.7. *If π_1 and π_2 respectively are potentially $L_r = K_{a_1,a_2,\cdots,a_r}$ and $M_r = K_{b_1,b_2,\cdots,b_r}$ graphic, then*
(a) the graphic sequence of induced subgraph $L_r \dot{\vee} M_r$ of $S_{ver} = G_1 \dot{\vee} G_2$ is

$$\pi^* = \left(\left(l - a_1\right)^{a_1}, \left(l - a_2\right)^{a_2}, \cdots, \left(l - a_r\right)^{a_r}, \left(l - b_1\right)^{b_1}, \cdots, \left(l - b_r\right)^{b_r}, 2^{\binom{r}{2}} \right),$$

where $\binom{r}{2}$ is the number of combinations of $a_1, a_2, \cdots, a_r$ taken two at a time.
(b) $\sigma(\pi^*) = \sum_{i=1}^{r} \left(a_i(l - a_i) + b_i(l - b_i) \right) + 2\binom{r}{2}$.

Proof. Let π_1 and π_2 respectively be potentially $L_r = K_{a_1,a_2,\cdots,a_r}$ and $M_r = K_{b_1,b_2,\cdots,b_r}$ graphic. So clearly the graphs G_1 and G_2 contain respectively L_r and M_r as a subgraph. Let $S_{ver} = G_1 \dot{\vee} G_2$ be the graph obtained by sub-division vertex join of graphs and let π be the graphic sequence of S_{ver}. We have

$$\pi = \left(d_1^1 + n, d_2^1 + n, \cdots, d_m^1 + n, d_1^2 + m, \cdots, d_n^2 + m, 2^{|E_1|} \right) \tag{1}$$

where $|E_1|$ is the size of G_1.

Let π^* be the graphic sequence of the induced subgraph $L_r \dot{\vee} M_r$ of S_{ver}. To prove (a) we use induction on r. For $r = 1$, the result is obvious. For $r = 2$, we have $G_2' = K_{a_1,a_2} \dot{\vee} K_{b_1,b_2}$. Let π_2' be the graphic sequence of G_2'. Therefore, by Remark 2.12, we have

$$\pi_2' = \Bigg(\left(a_2 + b_1 + b_2\right)^{a_1}, \left(a_1 + b_1 + b_2\right)^{a_2}, \left(b_2 + a_1 + a_2\right)^{b_1},$$
$$\left(b_1 + a_1 + a_2\right)^{b_2}, 2^{\sum_{i,j=1, i \neq j}^{\binom{2}{2}} a_i a_j} \Bigg)$$
$$= \Bigg(\left(\sum_{i=1}^{2} (a_i + b_i) - a_1 \right)^{a_1}, \left(\sum_{i=1}^{2} (a_i + b_i) - a_2 \right)^{a_2}, \left(\sum_{i=1}^{2} (a_i + b_i) - b_1 \right)^{b_1},$$
$$\left(\sum_{i=1}^{2} (a_i + b_i) - b_2 \right)^{b_2}, 2^{\sum_{i,j=1, i \neq j}^{\binom{2}{2}} a_i a_j} \Bigg)$$
$$= \left(\left(\sum_{i=1}^{2} (a_i + b_i) - a_i \right)^{a_i}, \left(\sum_{i=1}^{2} (a_i + b_i) - b_i \right)^{b_i}, 2^{a_1 a_2} \right).$$

This proves the result for $r=2$. Assume that the result is true for $r=k-1$. Therefore, we have

$$G'_{k-1}=K_{a_1,a_2,\cdots,a_{k-1}}\dot{\vee}K_{b_1,b_2,\cdots,b_{k-1}}$$

and let π'_{k-1} be the graphic sequence of G'_{k-1}. Then we have

$$\pi'_{k-1}=\Bigg(\Big(\sum_{i=1}^{k-1}(a_i+b_i)-a_1\Big)^{a_1},\cdots,\Big(\sum_{i=1}^{k-1}(a_i+b_i)-a_{k-1}\Big)^{a_{k-1}},$$
$$\Big(\sum_{i=1}^{k-1}(a_i+b_i)-b_1\Big)^{b_1},\cdots,\Big(\sum_{i=1}^{k-1}(a_i+b_i)-b_{k-1}\Big)^{b_{k-1}},2^{\sum_{i,j,i\neq j}^{\binom{k-1}{2}}a_ia_j}\Bigg). \quad (2)$$

Now, for $r=k$, we have

$$\begin{aligned}G'_k&=K_{a_1,a_2,\cdots,a_{k-1},a_k}\dot{\vee}K_{b_1,b_2,\cdots,b_{k-1},b_k}\\&=K_{R,a_k}\dot{\vee}K_{S,b_k},\end{aligned}$$

where $R=a_1,a_2,\cdots,a_{k-1}$ and $S=b_1,b_2,\cdots,b_{k-1}$.

Since the result is proved for all $r=k-1$ and using the fact that the result is proved for each pair and since the result is already proved for $r=2$, it follows by induction hypothesis that the result holds for $r=k$ also. That is,

$$\begin{aligned}\pi^*=\pi'_k=&\Bigg(\Big(a_k+b_k+\sum_{i=1}^{k-1}(a_i+b_i)-a_1\Big)^{a_1},\cdots,\Big(a_k+b_k+\sum_{i=1}^{k-1}(a_i+b_i)-a_{k-1}\Big)^{a_{k-1}},\\&\Big(a_k+b_k+\sum_{i=1}^{k}(a_i+b_i)-a_k\Big)^{a_k},\Big(a_k+b_k+\sum_{i=1}^{k-1}(a_i+b_i)-b_1\Big)^{b_1},\cdots,\\&\Big(a_k+b_k+\sum_{i=1}^{k-1}(a_i+b_i)-b_{k-1}\Big)^{b_{k-1}},\Big(a_k+b_k+\sum_{i=1}^{k-1}(a_i+b_i)-b_k\Big)^{b_k},2^{\left(\sum_{i,j,i\neq j}^{\binom{k-1}{2}}a_ia_j+\sum_{i=1}^{k-1}a_ka_i\right)}\Bigg)\\=&\Bigg(\Big(l-a_1\Big)^{a_1},\Big(l-a_2\Big)^{a_2},\cdots,\Big(l-a_r\Big)^{a_r},\Big(l-b_1\Big)^{b_1},\cdots,\Big(l-b_r\Big)^{b_r},2^{\binom{r}{2}}\Bigg).\end{aligned}$$

This proves part (a).

Now we have

$$\begin{aligned}\sigma(\pi^*)&=a_1(l-a_1)+\cdots+a_r(l-a_r)+b_1(l-b_1)+\cdots+b_r(l-b_r)+2\left(\sum_{i,j=1,i\neq j}^{\binom{k}{2}}a_ia_j\right)\\&=\sum_{i=1}^{r}\Big(a_i(l-a_i)+b_i(l-b_i)\Big)+2\binom{r}{2}.\end{aligned}$$

□

Theorem 3.8. *If π_1 and π_2 respectively are potentially $L_r=K_{a_1,a_2,\cdots,a_r}$ and $M_r=K_{b_1,b_2,\cdots,b_r}$-graphic, then the graphic sequence of the induced subgraph $L\bar{\vee}M$ of S_{ed} is*

$$\pi^*=\Bigg(\Big((l_1-a_i)^{a_i},(l'_1+|E_L|-b_i)^{b_i}\Big)_{i=1}^{r},(2+l'_1)^{|E_L|}\Bigg)$$

and $\sigma(\pi^*)=l_1^2+{l'_1}^2-m+2(1+l'_1)|E_L|$, where $l_1=\sum_{i=1}^{r}a_i$ and $l'_1=\sum_{i=1}^{r}b_i$.

Proof. Let π_1 and π_2 respectively be potentially L_r and M_r graphic. Then the graphs G_1 and G_2 contain L_r and M_r as a subgraph. Let $S_{ed} = G_1 \bar{\vee} G_2$ be the graph obtained by sub-division edge join of graphs and let π be the graphic sequence of S_{ed}. Then, we have

$$\pi = \Big(d_1^1, d_2^1, \cdots, d_m^1, d_1^2 + |E_1|, \cdots, d_n^2 + |E_1|, (2+n)^{|E_1|}\Big) \tag{3}$$

where $|E_1|$ is the size of G_1. Let π^* be the graphic sequence of the induced subgraph $L_r \bar{\vee} M_r$ of S_{ed}. To prove the result we use induction on r. For $r = 1$, the result follows by Theorem 3.5. For $r = 2$, we have $G_2' = K_{a_1,a_2} \bar{\vee} K_{b_1,b_2}$. Let π_2' be the graphic sequence of G_2'. Therefore, by Remark 2.12, we have

$$\begin{aligned}
\pi_2' &= \left(a_2^{a_1}, a_1^{a_2}, \Big(a_1a_2 + b_2\Big)^{b_1}, \Big(a_1a_2 + b_1\Big)^{b_2}, \Big(2 + b_1 + b_2\Big)^{a_1a_2}\right) \\
&= \left(\Big(\sum_{i=1}^{2} a_i - a_1\Big)^{a_1}, \Big(\sum_{i=1}^{2} a_i - a_2\Big)^{a_2}, \Big(\sum_{i,j,i\neq j}^{\binom{2}{2}} a_ia_j + b_2\Big)^{b_1}, \right. \\
&\qquad \left. \Big(\sum_{i,j,i\neq j}^{{}^2C_2} a_ia_j + b_1\Big)^{b_2}, \Big(2 + \sum_{i=1}^{2} b_i\Big)^{\sum_{i,j,i\neq j}^{\binom{2}{2}} a_ia_j}\right) \\
&= \left(\Big(l_1^* - a_1\Big)^{a_1}, \Big(l_1^* - a_2\Big)^{a_2}, \Big(|E(L_2)| + b_2\Big)^{b_1}, \Big(|E(L_2)| + b_1\Big)^{b_2}, \Big(2 + l_1'\Big)^{|E(L_2)|}\right) \\
&= \left(\left(\Big(l_1^* - a_i\Big)^{a_i}, \Big(|E(L_2)| + l_1' - b_i\Big)^{b_i}\right)_{i=1}^{2}, \Big(2 + l_1'\Big)^{|E(L_2)|}\right)
\end{aligned}$$

where $l_1^* = \sum_{i=1}^{2} a_i$ and $|E(L_2)| = |E(K_{a_1,a_2})| = a_1a_2$. This proves the result for $r = 2$. Assume that the result is true for $r = k-1$, therefore, we have

$$G_{k-1}' = K_{a_1,a_2,\cdots,a_{k-1}} \bar{\vee} K_{b_1,b_2,\cdots,b_{k-1}}.$$

Let π_{k-1}' be the graphic sequence of G_{k-1}', then we have

$$\pi_{k-1}' = \left(\left(\Big(l_1^{**} - a_i\Big), \Big(|E(L_{k-1})| + l_1' - b_i\Big)^{b_i}\right)_{i=1}^{k-1}, \Big(2 + l_1'\Big)^{|E(L_{k-1})|}\right)$$

where $l_1^{**} = \sum_{i=1}^{k-1} a_i$.

Now we show that the result holds for $r = k$. We have

$$\begin{aligned}
G_k' &= K_{a_1,a_2,\cdots,a_{k-1},a_k} \bar{\vee} K_{b_1,b_2,\cdots,b_{k-1},b_k} \\
&= K_{R,a_k} \bar{\vee} K_{S,b_k}
\end{aligned}$$

where $R = a_1, a_2, \cdots, a_{k-1}$ and $S = b_1, b_2, \cdots, b_{k-1}$.

Since the result is proved for every $r = k-1$ and using the fact that the result is proved for each pair and since the result is already proved for $r = 2$, it follows by induction hypothesis that the result holds for $r = k$ also. That is,

$$\pi^* = \pi_k' = \left(\Big(a_k + \sum_{i=1}^{k-1} a_i - a_1\Big)^{a_1}, \Big(a_k + \sum_{i=1}^{k-1} a_i - a_2\Big)^{a_2}, \cdots, \Big(a_k + \sum_{i=1}^{k-1} a_i - a_k\Big)^{a_k}, \right.$$

$$\Big(a_ka_1 + a_ka_2 + \cdots + a_ka_{k-1} + \sum_{i,j,i\neq j}^{\binom{k-1}{2}} a_ia_j + b_2\Big)^{b_1},$$

$$\Big(a_ka_1 + a_ka_2 + \cdots + a_ka_{k-1} + \sum_{i,j,i\neq j}^{\binom{k-1}{2}} a_ia_j + b_2\Big)^{b_2}, \cdots,$$

$$\Big(a_ka_1 + a_ka_2 + \cdots + a_ka_{k-1} + \sum_{i,j,i\neq j}^{\binom{k-1}{2}} a_ia_j + b_k\Big)^{b_k}, \Big(2 + b_k + \sum_{i=1}^{k-1} b_i\Big)^{\big(\sum_{i,j,i\neq j}^{\binom{k-1}{2}} a_ia_j + \sum_{i=1}^{k-1} a_ka_i\big)}\Bigg)$$

$$= \Bigg(\bigg(\Big(l_1 - a_i\Big)^{a_i}, \Big(|E_{L_r}| + l_1' - b_1\Big)^{b_i}\bigg)_{i=1}^{k}, \big(2 + l_1'\big)^{|E_{L_r}|}\Bigg)$$

Also, we have

$$\begin{aligned}\sigma(\pi^*) &= a_1(l_1 - a_1) + \cdots + a_r(l_1 - a_r) + b_1(l_1' + |E_L| - b_1) \\ &+ \cdots + b_r(l_1' + |E_L| - b_r) + \sum_{i,j=1,i\neq j}^{{}^kC_2} a_ia_j(2 + 2l_1') \\ &= l_1^2 + {l_1'}^2 - m + 2(1 + l_1')|E_L|.\end{aligned}$$

This completes the proof. □

Let G_1 and G_2 be any two graphs. Let $S_J = G_1 \vee G_2$ and let $S_{J^*} = \big(B_{\dot{m}_1,n_1}\big) \vee \big(B_{\dot{m}_2,n_2}\big)$ be the induced subgraph of S_J and let π^* be the graphic sequence of S_J^*.

Theorem 3.9. *If π_1 and π_2 respectively be potentially $B_{\dot{m}_1,n_1}$ and $B_{\dot{m}_2,n_2}$, then (a) the graphic sequence π^* of induced subgraph $\big(B_{\dot{m}_1,n_1}\big) \vee \big(B_{\dot{m}_2,n_2}\big)$ of S_J is*

$$\pi^* = \Bigg(\Big(A + |E(K_{m_2})| - 1\Big)^{m_1}, \Big(A + |E(K_{m_1})| - 1\Big)^{m_2}, \Big(A + |E(K_{m_2})| + 2 - (m_1 + n_1)\Big)^{|E(K_{m_1})|},$$

$$\Big(A + |E(K_{m_1})| + 2 - (m_2 + n_2)\Big)^{|E(K_{m_2})|}, \Big(A + |E(K_{m_2})| - n_1\Big)^{n_1}, \Big(A + |E(K_{m_1})| - n_2\Big)^{n_2}\Bigg)$$

and

$$(b)\ \ \sigma(\pi^*) = A\Big(A + \sum_{i=1}^{2} |E(K_{m_i})|\Big) + \prod_{i,j=1,i\neq j}^{2} (m_i + n_i)|E(K_{m_j})|$$
$$+ 2\Big(|E(K_{m_1})||E(K_{m_2})| + \sum_{i}^{2} |E(K_{m_i})|\Big) - \sum_{i=1}^{2} \Big(m_i + \big(m_i + n_i\big)|E(K_{m_i})|\Big) - \sum_{i=1}^{2} n_i^2.$$

where $A = \sum_{i=1}^{2}(m_i + n_i)$ and $|E(K_{m_i})| = \frac{m_i(m_i-1)}{2}$.

Proof. The graphic sequence of $B_{\dot{m}_1,n_1}$ and $B_{\dot{m}_2,n_2}$ respectively are

$$\pi_1' = \Big(\Big(m_1 + n_1 - 1\Big)^{m_1}, 2^{\frac{m_1(m_1-1)}{2}}, m_1^{n_1}\Big) \tag{4}$$

$$\pi_2' = \Big(\Big(m_2 + n_2 - 1\Big)^{m_2}, 2^{\frac{m_2(m_2-1)}{2}}, m_2^{n_2}\Big) \tag{5}$$

Clearly from (4) and (5), the graphic sequence of S_J^* is

$$\pi^* = \Bigg(\Big(m_1+m_2+n_1+n_2-1+\frac{m_2(m_2-1)}{2}\Big)^{m_1}, \Big(m_1+m_2+n_1+n_2-1+\frac{m_1(m_1-1)}{2}\Big)^{m_2},$$
$$\Big(2+m_2+n_2+\frac{m_2(m_2-1)}{2}\Big)^{\frac{m_1(m_1-1)}{2}}, \Big(2+m_1+n_1+\frac{m_1(m_1-1)}{2}\Big)^{\frac{m_2(m_2-1)}{2}}$$
$$\Big(m_1+m_2+n_2+\frac{m_2(m_2-1)}{2}\Big)^{n_1}, \Big(m_1+m_2+n_1+\frac{m_1(m_1-1)}{2}\Big)^{n_2}\Bigg)$$
$$= \Bigg(\big(A+|E(K_{m_2})|-1\big)^{m_1}, \big(A+|E(K_{m_1})|-1\big)^{m_2}, \Big(A+|E(K_{m_2})|+2-(m_1+n_1)\Big)^{|E(K_{m_1})|},$$
$$\Big(A+|E(K_{m_1})|+2-(m_2+n_2)\Big)^{|E(K_{m_2})|}, \Big(A+|E(K_{m_2})|-n_1\Big)^{n_1}, \Big(A+|E(K_{m_1})|-n_2\Big)^{n_2}\Bigg)$$

This proves (a).

Further

$$\begin{aligned}
\sigma(\pi^*) = {} & m_1(A+|E(K_{m_2})|-1)+m_2(A+|E(K_{m_1}|-1) \\
& +|E(K_{m_1})|\big(A+|E(K_{m_2})|+2-m_1-n_1\big) \\
& +|E(K_{m_2})|\big(A+|E(K_{m_1})|+2-m_2-n_2\big) \\
& +n_1\big(A+|E(K_{m_2}|-n_1\big)+n_2\big(A+|E(K_{m_1})|-n_2\big) \\
= {} & m_1A+m_1|E(K_{m_2})|-m_1+|E(K_{m_2})|A+m_2|E(K_{m_1})|-m_2 \\
& +|E(K_{m_1})|A+|E(K_{m_1})||E(K_{m_2})|+2|E(K_{m_1})|-|E(K_{m_1})|m_1-n_1|E(K_{m_1})| \\
& +A|E(K_{m_2})|+|E(K_{m_2})||E(K_{m_1})|+2|E(K_{m_2})|-m_2|E(K_{m_2})|-n_2|E(K_{m_2})| \\
& +n_1A+n_1|E(K_{m_2})|-n_1^2+n_2A+n_2|E(K_{m_1})|-n_2^2 \\
= {} & (m_1+n_1+m_2+n_2)A+\big(|E(K_{m_1})|+|E(K_{m_2})|\big)A+m_1|E(K_{m_2})| \\
& +m_2|E(K_{m_1})|+n_2|E(K_{m_1})|+n_1|E(K_{m_2})|-m_1|E(K_{m_1})|-n_1|E(K_{m_1})| \\
& -m_2|E(K_{m_2})|-n_2|E(K_{m_2})|-(m_1+m_2) \\
& +2(|E(K_{m_1})||E(K_{m_2})|)+2\big(|E(K_{m_1})|+|E(K_{m_1})|\big)-(n_1^2+n_2^2) \\
= {} & A^2+A\sum_{i=1}^{2}|E(K_{m_i})|+\prod_{i,j=1,i\neq j}^{2} m_i|E(K_{m_j})|+\prod_{i,j=1,i\neq j}^{2} n_i|E(K_{m_j})| \\
& -\sum_{i=1}^{2}(m_i+n_i)|E(K_{m_i})|-\sum_{i=1}^{2}m_i+2\big(|E(K_{m_1})||E(K_{m_2})|+\sum_{i=1}^{2}|E(K_{m_i})|\big)-\sum_{i=1}^{2}n_i^2 \\
= {} & A\Big(A+\sum_{i=1}^{2}|E(K_{m_i})|\Big)+\prod_{i,j=1,i\neq j}^{2}(m_i+n_i)|E(K_{m_j})| \\
& +2\Big(|E(K_{m_1})||E(K_{m_2})|+\sum_{i}^{2}|E(K_{m_i})|\Big)-\sum_{i=1}^{2}\Big(m_i+\big(m_i+n_i\big)|E(K_{m_i})|\Big)-\sum_{i=1}^{2}n_i^2.
\end{aligned}$$

which proves (b). □

Let G_1 and G_2 be two graphs. Let $S_J = G_1 \vee G_2$ and let $S_J^{**} = (B_{\bar{m}_1,n_1}) \vee (B_{\bar{m_2},n_2})$ be the induced subgraph of S_J and let π^{**} be the graphic sequence of S_J^{**}.

Theorem 3.10. *If π_1 and π_2 respectively are potentially $\left(B_{\bar{m}_1,n_1}\right)$ and $\left(B_{\bar{m}_2,n_2}\right)$, then the graphic sequence of induced subgraph $\left(B_{\bar{m}_1,n_1}\right) \vee \left(B_{\bar{m}_1,n_1}\right)$ of S_J is*

$$\begin{aligned}\pi^{**} =&\Bigg(\Big(A+|E(K_{m_2})|-(n_1+1)\Big)^{m_1}, \Big(A+|E(K_{m_2}|+2-m_1\Big)^{|E(K_{m_1}|},\\ &\Big(A+\sum_{i=1}^{2}|E(K_{m_i})|-(m_1+n_1)\Big)^{n_1}, \Big(A+|E(K_{m_1}|-(n_2+1)\Big)^{m_2}\\ &\Big(A+|E(K_{m_1}|+2-m_2\Big)^{|(K_{m_2}|}, \Big(A+\sum_{i=1}^{2}|E(K_{m_i})|-(m_2+n_2)\Big)^{n_2}\Bigg).\end{aligned}$$

and

$$\begin{aligned}\sigma(\pi^{**}) =&A\big(A+\sum_{i=1}^{2}|E(K_{m_i})|\big)+\prod_{i,j=1,i\neq j}^{2}(m_i+n_i)|E(K_{m_j})|+\prod_{i,j=1,i\neq j}^{2}|E(K_{m_i})||E(K_{m_j})|\\ &+\sum_{i=1}^{2}(2+n_i)|E(K_{m_i})|-\sum_{i=1}^{2}\big(2n_i+1+|E(K_{m_i})|\big)m_i-\sum_{i=1}^{2}n_i^2.\end{aligned}$$

Proof. The graphic sequence of $B_{\bar{m}_1,n_1}$ and $B_{\bar{m}_2,n_2}$ respectively are

$$\pi_1' = \left(\Big(m_1-1\Big)^{m_1}, \Big(\frac{m_1(m_1-1)}{2}\Big)^{n_1}, (2+n_1)^{\frac{m_1(m_1-1)}{2}}\right) \tag{6}$$

$$\pi_2' = \left(\Big(m_2-1\Big)^{m_2}, \Big(\frac{m_2(m_2-1)}{2}\Big)^{n_2}, (2+n_2)^{\frac{m_2(m_2-1)}{2}}\right). \tag{7}$$

Then by (6), (7) and by Definition 2.2, we have

$$\begin{aligned}\pi^{**} =&\Bigg(\Big(m_1-1+m_2+n_2+\frac{m_2(m_2-1)}{2}\Big)^{m_1}, \Big(m_2-1+m_1+n_1+\frac{m_1(m_1-1)}{2}\Big)^{m_2},\\ &\Big(\frac{m_1(m_1-1)}{2}+m_2+n_2+\frac{m_2(m_2-1)}{2}\Big)^{n_1}, \Big(\frac{m_2(m_2-1)}{2}+m_1+n_1+\frac{m_1(m_1-1)}{2}\Big)^{n_2},\\ &\Big(2+n_1+m_2+n_2+\frac{m_2(m_2-1)}{2}\Big)^{\frac{m_1(m_1-1)}{2}}, \Big(2+n_2+m_1+n_1+\frac{m_1(m_1-1)}{2}\Big)^{\frac{m_2(m_2-1)}{2}}\Bigg)\\ =&\Bigg(\Big(A+|E(K_{m_2})|-(n_1+1)\Big)^{m_1}, \Big(A+|E(K_{m_2}|+2-m_1\Big)^{|E(K_{m_1}|},\\ &\Big(A+\sum_{i=1}^{2}|E(K_{m_i})|-(m_1+n_1)\Big)^{n_1}, \Big(A+|E(K_{m_1}|-(n_2+1)\Big)^{m_2}\\ &\Big(A+|E(K_{m_1}|+2-m_2\Big)^{|(K_{m_2}|}, \Big(A+\sum_{i=1}^{2}|E(K_{m_i})|-(m_2+n_2)\Big)^{n_2}\Bigg).\end{aligned}$$

Further

$$\begin{aligned}\sigma(\pi^{**}) =&\Bigg(\Big(A+|E(K_{m_2})|-(n_1+1)\Big)^{m_1}+\Big(A+|E(K_{m_2}|2-m_1\Big)^{|E(K_{m_1}|}+\\ &+\Big(A+\sum_{i=1}^{2}|E(K_{m_i})|-(m_1+n_1)\Big)^{n_1}+\Big(A+|E(K_{m_1}|-(n_2+1)\Big)^{m_2}\end{aligned}$$

$$\Big(A+|E(K_{m_1}|+2-m_2\Big)^{|(K_{m_2}|}+\Big(A+\sum_{i=1}^{2}|E(K_{m_i})|-(m_2+n_2)\Big)^{n_2}\Bigg).$$

$$\begin{aligned}
&= m_1\big(A+|E(K_{m_2})|-n_1-1\big)+|E(K_{m_1})|\big(A+|E(K_{m_2})|+2-m_1\big)+\\
&\quad n_1\big(A+|E(K_{m_1})|+|E(K_{m_2})|-m_1-n_1\big)+m_2\big(A+|E(K_{m_1})|-n_2-1\big)+\\
&\quad |E(K_{m_2})|\big(A+|E(K_{m_1})|+2-m_2\big)+n_2\big(A+|E(K_{m_1})|+|E(K_{m_2})|-m_2-n_2\big)\\
&= m_1A+m_1|E(K_{m_2})|-n_1m_1-m_1+|E(K_{m_1})|A+|E(K_{m_1})||E(K_{m_2})|+2|E(K_{m_1})|\\
&\quad -|E(K_{m_1})|m_1+n_1A+n_1|E(K_{m_1})|+n_1|E(K_{m_2})|-n_1m_1-n_1^2+m_2A+m_2|E(K_{m_1})|\\
&\quad -m_2n_2-m_2+|E(K_{m_2})|A+|E(K_{m_2})||E(K_{m_1})|+2|E(K_{m_2})|-|E(K_{m_2})|m_2\\
&\quad +n_2A+n_2|E(K_{m_2})|-n_2m_2-n_2^2+n_2|E(K_{m_1})|\\
&= (m_1+n_1+m_2+n_2)A+\big(|E(K_{m_1})|+|E(K_{m_2})|\big)A+m_1|E(K_{m_2})|\\
&\quad +m_2|E(K_{m_1})|+n_1|E(K_{m_1})|+n_2|E(K_{m_2})|+|E(K_{m_1})||E(K_{m_2})|\\
&\quad +|E(K_{m_2})||E(K_{m_1})|+2(|E(K_{m_1})|+|E(K_{m_2})|)+n_1|E(K_{m_2})|\\
&\quad +n_2|E(K_{m_1})|-2n_1m_1-2n_2m_2-(m_1+m_2)\\
&\quad -\big(|E(K_{m_1})|m_1+|E(K_{m_2})|m_2\big)\\
&= A^2+A\sum_{i=1}^{2}|E(K_{m_i})|+\prod_{i,j=1,i\neq j}^{2}m_i|E(K_{m_j})|+\prod_{i,j=1,i\neq j}^{2}n_i|E(K_{m_j})|\\
&\quad +\prod_{i,j=1,i\neq j}^{2}|E(K_{m_i})||E(K_{m_j})|+2\sum_{i=1}^{2}|E(K_{m_i})|+\sum_{i=1}^{2}n_i|E(K_{m_i})|\\
&\quad -2\sum_{i=1}^{2}n_im_i-\sum_{i=1}^{2}m_i-\sum_{i=1}^{2}|E(K_{m_i})|m_i-\sum_{i=1}^{2}n_i^2\\
&= A\big(A+\sum_{i=1}^{2}|E(K_{m_i})|\big)+\prod_{i,j=1,i\neq j}^{2}(m_i+n_i)|E(K_{m_j})|+\sum_{i=1}^{2}(2+n_i)|E(K_{m_i})|\\
&\quad -\sum_{i=1}^{2}\big(2n_i+1+|E(K_{m_i})|\big)m_i-\sum_{i=1}^{2}n_i^2.
\end{aligned}$$

This completes the proof. □

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