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Every 5-connected planar triangulation is 4-ordered Hamiltonian*

Research Article

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Abstract: A graph G is said to be 4-*ordered* if for any ordered set of four distinct vertices of G, there exists a cycle in G that contains all of the four vertices in the designated order. Furthermore, if we can find such a cycle as a Hamiltonian cycle, G is said to be 4-*ordered Hamiltonian*. It was shown that every 4-connected planar triangulation is (i) Hamiltonian (by Whitney) and (ii) 4-ordered (by Goddard). Therefore, it is natural to ask whether every 4-connected planar triangulation is 4-ordered Hamiltonian. In this paper, we give a partial solution to the problem, by showing that every 5-connected planar triangulation is 4-ordered Hamiltonian.

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1. Introduction

A graph G is said to be k-*ordered* for an integer $3 \leq k \leq |V(G)|$, if for any ordered set of k distinct vertices of G, there exists a cycle in G that contains all the k vertices in the designated order. Furthermore, if we can find such a cycle as a Hamiltonian cycle, G is said to be k-*ordered Hamiltonian*. These topics have been extensively studied; see the survey [2].

In this paper, we focus on 4-connected planar triangulations. In fact, it is known that such graphs have good properties;

Theorem 1.1. *Let G be a 4-connected planar triangulation. Then*

(i) G is Hamiltonian. (Whitney [12])

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(ii) G is 4-ordered. (Goddard [3])

Note that Theorem 1.1 (i) was improved to 4-connected planar graphs (by Tutte [11]) and 4-connected projective planar graphs (by Thomas and Yu [8]). However, we cannot lower the assumption on 4-connectedness to 3-connectedness, since there exist infinitely many 3-connected planar triangulations that are not Hamiltonian (see [4]). On the other hand, by using ideas of Goddard [3], we can construct infinitely many 3-connected planar triangulations that are not 4-ordered, and infinitely many 5-connected planar graphs that are not 4-ordered. Therefore, both of the assumptions "4-connected" and "triangulation" are needed for the property of being "4-ordered".

Recall that 4-ordered Hamiltonian graphs are definitely Hamiltonian and 4-ordered. It follows from Theorems 1.1 (i) and (ii) that every 4-connected planar triangulation satisfies both properties, and hence it is natural to pose the following conjecture (Conjecture 1.2). The main purpose of this paper is to show Theorem 1.3, which is a partial solution to it.

Conjecture 1.2. *Every 4-connected planar triangulation is 4-ordered Hamiltonian.*

Theorem 1.3. *Every 5-connected planar triangulation is 4-ordered Hamiltonian.*

This paper is organized as follows; in the next section, we will give terminologies and a known result, used in the proof of Theorem 1.3 in Section 3. In the last section, we will give a conclusion of this paper, together with some open problems.

2. Preliminaries

For a graph G, the order of G is denoted by $|G|$. Let H_1 and H_2 be two subgraphs of a graph G. Then $H_1 \cup H_1$ denotes the subgraph of G with $V\big(H_1 \cup H_2\big) = V(H_1) \cup V(H_2)$ and $E\big(H_1 \cup H_2\big) = E(H_1) \cup E(H_2)$, and $H_1 \cap H_1$ denotes the subgraph of G with $V\big(H_1 \cap H_2\big) = V(H_1) \cap V(H_2)$ and $E\big(H_1 \cap H_2\big) = E(H_1) \cap E(H_2)$. We use a similar notation also for a vertex subset U or an edge subset P of G; so, $H_1 \cup U$ is the subgraph of G with $V\big(H_1 \cup U\big) = V(H_1) \cup U$ and $E\big(H_1 \cup U\big) = E(H_1)$, and $H_1 \cup P$ is the subgraph of G with $V\big(H_1 \cup P\big) = V(H_1) \cup V(P)$ and $E\big(H_1 \cup P\big) = E(H_1) \cup P$, where $V(P)$ is the set of vertices that are end vertices of some edges in P. A pair (H_1, H_2) is a *separation* of G if $H_1 \cup H_2 = G$ and $E(H_1) \cap E(H_2) = \emptyset$.

For a path P and two vertices $x, y \in V(P)$, $P[x, y]$ denotes the subpath of P between x and y. Furthermore, let $P(x, y] = P[x, y] - \{x\}$, $P[x, y) = P[x, y] - \{y\}$, and $P(x, y) = P[x, y] - \{x, y\}$.

Let G be a connected plane graph. A *facial walk* in G is the boundary walk of some face of G. Furthermore, if it is a cycle, then we call it a *facial cycle* in G.

Let T be a subgraph of a graph G. A T-*bridge* of G is either (i) an edge of $G - E(T)$ with both ends on T or (ii) a subgraph of G induced by the edges in a component of $G - V(T)$ and all edges from that component to T. A T-bridge satisfying (i) is said to be *trivial*; otherwise it is *non-trivial*. For a T-bridge B of G, the vertices in $B \cap T$ are the *attachments* of B (on T), and any vertex of B that is not an attachment is a *non-attachment*. We say that T is a *Tutte subgraph* in G if every T-bridge of G has at most three attachments on T. For another subgraph C of G, T is a C-*Tutte* subgraph in G if T is a Tutte subgraph in G and every T-bridge of G containing an edge of C has at most two attachments on T. When T is a path or a cycle, we call T a C-*Tutte path* or a C-*Tutte cycle*, respectively.

Note that if G is 4-connected and T is a Tutte subgraph in G with $|T| \geq 4$, then T must contain all vertices in G; otherwise, there exists a T-bridge in G whose attachments form a cut set in G of order at most three, contradicting that G is 4-connected. Indeed, the concept of "Tutte subgraphs" was first introduced by Tutte [11] in order to prove his seminal result; every 4-connected planar graph is Hamiltonian. Since then it has been extended by several researchers, see [6–9, 13]. The following theorem is a main tool to prove Theorem 1.3. See also the paper [9] by Thomassen.

Theorem 2.1 (Sanders [7])**.** *Let G be a connected plane graph, let C be a facial walk in G, let $x, y \in V(G)$ with $x \neq y$, and let $e \in E(C)$. Assume that G contains a path from x to y through e. Then G has a C-Tutte path from x to y through e.*

Note that originally Sanders [7] showed only the 2-connected case, but we can easily show Theorem 2.1 using a block decomposition. Hence, we omit the proof of Theorem 2.1.

3. Proof of Theorem 1.3

Let G be a 5-connected planar triangulation, and let v_1, v_2, v_3 and v_4 be four distinct vertices in G. We will show that G has a Hamiltonian cycle passing through v_1, v_2, v_3 and v_4 in this order. It follows from Theorem 1.1 (ii) that G has a cycle passing through those vertices in that order. This implies the following; $G - v_4$ has a path P from v_1 to v_3 through v_2 such that

(P1) $G - V\big(P(v_1, v_3)\big)$ contains a path from v_3 to v_1 through v_4.

In addition, by taking a path satisfying property (P1) as short as possible, we can also consider the following condition. Here a *chord* of P is an edge e not in P such that both of end vertices of e are contained in P.

(P2) For any chord of P, one end vertex of it is contained in $P[v_1, v_2)$ and the other is contained in $P(v_2, v_3]$.

Indeed, if there exists a chord xy of P such that both end vertices are contained in $P[v_1, v_2]$ or in $P[v_2, v_3]$, then we can detour P by xy instead of $P[x, y]$. It is easy to see that the new path also satisfies condition (P1) and is shorter than P. Therefore, a path that is as short as possible, subject to (P1), also satisfies condition (P2).

Let $G^1 = G - V\big(P(v_1, v_3)\big)$, and let C^1 be the unique facial walk of G^1 that is not facial in G. Note that $v_1, v_3 \in V(C^1)$. Now we consider a separation (H_1, H_2) of G^1 such that $|H_1 \cap H_2| \leq 2$, $v_1, v_3 \in V(H_1)$, and $v_4 \in V(H_2) - V(H_1)$. When H_1' consists of only the two vertices v_1 and v_3 and no edges and $H_2' = G^1$, a pair (H_1', H_2') is a separation of G^1 satisfying all of the above conditions. Therefore, such a separation (H_1, H_2) of G^1 must exist. Take such a separation (H_1, H_2) of G^1 so that $|H_2|$ is as small as possible. If H_2 does not contain any edge in C^1, then $H_1 \cap H_2$ forms a cut set in G of order at most 2, contradicting that G is 5-connected. Hence H_2 contains an edge in C^1. Then it follows from condition (P1) that $|H_1 \cap H_2| = 2$ and there exists an edge e^1 in $H_2 \cap C^1$ such that G^1 has a path from v_3 to v_1 through e^1. (In fact, take an edge in $H_2 \cap C^1$ such that it is incident with a vertex in $H_1 \cap H_2$.)

It follows from Theorem 2.1 that G^1 has a C^1-Tutte path T^1 from v_3 to v_1 through e^1. Note that by the choice of e^1, T^1 passes through both of the two vertices in $H_1 \cap H_2$. In addition, it satisfies the following property.

Claim 3.1. *T^1 contains v_4, but does not contain v_2.*

Proof. Since $v_2 \notin V(G^1)$, the second statement is trivial. So we only show the first one.

Suppose not, and let B be a T^1-bridge of G^1 such that B contains v_4 as a non-attachment. Let S_B be the set of attachments of B on T^1. If B has no neighbors in $P(v_1, v_3)$, then S_B would be a cut set of G such that S_B separates $B - S_B$ from other vertices and $|S_B| \leq 3$, which contradicts that G is 5-connected. Therefore, B has neighbors in $P(v_1, v_3)$. This implies that B contains an edge in C^1. Then since T^1 is a C^1-Tutte path in G^1, we have $|S_B| \leq 2$. Let $\overline{B} = G^1 - V\big(B - S_B\big)$. Then $(\overline{B}, B)$ is a separation of G^1 such that $|\overline{B} \cap B| = |S_B| \leq 2$, $v_1, v_3 \in V(\overline{B})$ and $v_4 \in V(B) - V(\overline{B})$. Furthermore, since T^1 passes through e^1 and B is a T^1-bridge of G^1, we have $e^1 \in E(\overline{B})$, which implies that $V(H_2) - V(B) \neq \emptyset$. Since T^1 passes through both of the two vertices in $H_1 \cap H_2$, we see that $V(B) \subset V(H_2)$, which contradicts the choice of (H_1, H_2). This completes the proof of Claim 3.1. $\square$

Let $G^2 = G - V\big(T^1(v_3, v_1)\big)$, and let C^2 be the unique facial walk of G^2 that is not facial in G. Note that $v_1, v_3 \in V(C^2)$. Then we consider a separation (R_1, R_2) of G^2 such that $|R_1 \cap R_2| \leq 2$, $v_1, v_3 \in V(R_1)$, and $v_2 \in V(R_2) - V(R_1)$. When R'_1 consists of only the two vertices v_1 and v_3 and no edges and $R'_2 = G^2$, a pair (R'_1, R'_2) is a separation of G^2 satisfying all of the above conditions. Therefore, such a separation (R_1, R_2) of G^2 must exist. Take such a separation (R_1, R_2) so that $|R_2|$ is as small as possible. Since G is 5-connected, R_2 contains an edge in C^2. Note that P is contained in G^2, and hence G^2 has a path from v_1 to v_3 through v_2. This implies $|R_1 \cap R_2| = 2$ and there exists an edge e^2 in $R_2 \cap C^2$ such that G^2 has a path from v_1 to v_3 through e^2.

It follows from Theorem 2.1 that G^2 has a C^2-Tutte path T^2 from v_1 to v_3 through e^2. Note that by the choice of e^2, T^2 passes through both of the two vertices in $R_1 \cap R_2$. Notice also that $T^1 \cup T^2$ is a cycle in G, and it satisfies the following, which is crucial in the proof of Theorem 1.3.

Claim 3.2. *There exist no non-trivial $\big(T^1 \cup T^2\big)$-bridges in G. In particular, $T^1 \cup T^2$ is a Hamiltonian cycle in G.*

Proof. Suppose that there exists a non-trivial $\big(T^1 \cup T^2\big)$-bridge D in G. Let S_D be the set of attachments of D on $T^1 \cup T^2$.

Suppose first that $S_D \cap V\big(T^1(v_3, v_1)\big) = \emptyset$. This condition implies that D is a T^2-bridge of G^2. Since T^2 is a C^2-Tutte path in G^2, we have $|S_D| \leq 3$, which implies that S_D is a cut set in G of order at most three, contradicting that G is 5-connected. Therefore, we may assume that $S_D \cap V\big(T^1(v_3, v_1)\big) \neq \emptyset$. By the same argument, we also see that $S_D \cap V\big(T^2(v_1, v_3)\big) \neq \emptyset$.

These conditions, together with the planarity, imply that D contains an edge in C^1 and an edge in C^2. Then since $D - V\big(T^1(v_3, v_1)\big)$ is a T^2-bridge of G^2 containing an edge in C^2, we have

$$|S_D \cap V(T^2)| \;\leq\; 2. \tag{1}$$

Suppose that D contains no vertices in P as non-attachments. See Figure 1. This condition implies that there exists a $\big(T^1 \cup P\big)$-bridge, say B_D, such that $D \subseteq B_D$. Note that $B_D - V\big(P(v_1, v_3)\big)$ is connected and a T^1-bridge of G^1 containing an edge in C^1, and hence $B_D - V\big(P(v_1, v_3)\big)$ has at most two attachments on T^1. Since any vertex in $S_D \cap V\big(T^1(v_3, v_1)\big)$ is an attachment of $B_D - V\big(P(v_1, v_3)\big)$ on T^1, we have

$$|S_D \cap V\big(T^1(v_3, v_1)\big)| \;\leq\; \Big|\Big(B_D - V\big(P(v_1, v_3)\big)\Big) \cap V(T^1)\Big| \;\leq\; 2.$$

Then it follows from inequality (1) that $|S_D| \leq 4$, which contradicts that G is 5-connected.

Therefore, we may assume that D contains vertices in P as non-attachments. See Figure 2. Since P is a path in G^2 from v_1 to v_3 and $v_1, v_3 \in V(T^2)$, D has at least two attachments on P. Then it follows from inequality (1) that $S_D \cap V(T^2) \subseteq V(P)$ and $|S_D \cap V(T^2)| = 2$. Let $\{x, y\} = S_D \cap V(T^2)$. Note that $P[x, y]$ is contained in D. Consider the region bounded by $P[x, y] \cup T^2[x, y]$. Since $S_D \cap V(T^2) = S_D \cap V(P)$ is the set of attachments of D on T^2, there are no edges between vertices in $P(x, y)$ and those in $T^2(x, y)$. Thus, since G is a triangulation, there exists an edge in G connecting x and y. Note that xy is a chord of P. It follows from condition (P2) and symmetry that we may assume that x is contained in $P[v_1, v_2)$ and y is contained in $P(v_2, v_3]$. Then $(\overline{D}, D)$ is a separation of G^2, where $\overline{D} = G^2 - V\big(D - S_D\big)$, such that $|\overline{D} \cap D| = |S_D| = 2$, $v_1, v_3 \in V(\overline{D})$, and $v_2 \in V(D)$. Furthermore, since T^2 passes through e^2 and D is a $\big(T^1 \cup T^2\big)$-bridge of G, we have $e^2 \in E(\overline{D})$, which implies that $V(R_2) - V(D) \neq \emptyset$. Since T^2 passes through both of the two vertices in $R_1 \cap R_2$, we see that $V(D) \subset V(R_2)$, which contradicts the choice of (R_1, R_2).

Therefore, there exist no non-trivial $\big(T^1 \cup T^2\big)$-bridges in G, which easily implies that $T^1 \cup T^2$ is a Hamiltonian cycle in G. This completes the proof of Claim 3.2. □

By Claim 3.1, v_2 appears in T^2 and v_4 appears in T^1, which implies that $T^1 \cup T^2$ contains v_1, v_2, v_3 and v_4 in this order. By Claim 3.2, $T^1 \cup T^2$ is a Hamiltonian cycle in G. These complete the proof of Theorem 1.3.

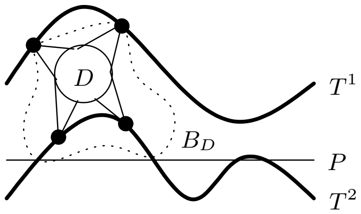

Figure 1. **The case when D contains no vertices in P as non-attachments.**

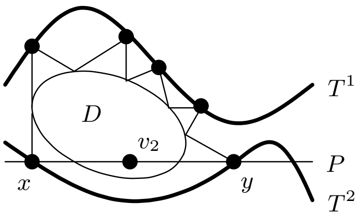

Figure 2. **The case when D contains vertices in P as non-attachments.**

4. Conclusion

In this paper, we have focused on the property of being 4-ordered Hamiltonian. In fact, considering known results (Theorem 1.1) on 4-connected planar triangulations, it is natural to pose Conjecture 1.2. We gave a partial solution to it, by showing that every 5-connected planar triangulation is 4-ordered Hamiltonian.

In the rest, we would like to put some problems related to k-ordered Hamiltonian. The first one is Conjecture 1.2, which already appeared in Section 1.

The second problem is the property of being 4-ordered Hamiltonian of graphs on non-spherical surfaces. In fact, there are some results that are the counterparts of Theorem 1.1. Recall that for a graph G on a non-spherical surface F^2, the *edge-width* of G is the length of a shortest non-contractible cycle in G.

Theorem 4.1 (Mukae and Ozeki [5]). *Let G be a 4-connected triangulation on a surface. Then G is 4-ordered.*

Theorem 4.2 (Yu [13]). *For any surface F^2, there exists an integer $N = N(F^2)$ satisfying the following; for any 5-connected triangulation G of F^2, if the edge-width of G is at least N, then G is Hamiltonian.*

Note that the assumptions on 5-connectedness and edge-width in Theorem 4.2 are both best possible, in some sense. In fact, Theorem 4.2 cannot be improved to 4-connected graphs (see [10]) and to the statement without the edge-width assumption (see [1]).

Considering these two theorems, the following seems also a natural conjecture. Because of the facts mentioned above, the assumptions on the edge-width and 5-connectedness are best possible, if the conjecture is true. We leave it to readers as an open problem.

Problem 4.3. *For any surface F^2, there exists an integer $N = N(F^2)$ satisfying the following; for any 5-connected triangulation G of F^2, if the edge-width of G is at least N, then G is 4-ordered Hamiltonian.*

Goddard [3] also mentioned about the property of being 5-ordered; no planar graph can be 5-ordered. However, his idea cannot work for graphs on non-spherical surfaces, and hence the following might also hold. Those are the last problems in this paper.

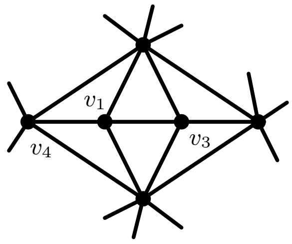

Figure 3. **Two adjacent vertices of degree** 4**.**

Problem 4.4. *Any* 5*-connected triangulation of a non-spherical surface* F^2 *is* 5*-ordered.*

Problem 4.5. *For any surface* F^2*, there exists an integer* $N = N(F^2)$ *satisfying the following; for any* 5*-connected triangulation* G *of* F^2*, if the edge-width of* G *is at least* N*, then* G *is* 5*-ordered Hamiltonian.*

Note that if a 4-connected triangulation G of a surface has two adjacent vertices of degree 4, then G cannot be 5-ordered. In fact, if v_1, v_3 and v_4 are specified as in Figure 3 and v_2 and v_5 are specified as vertices outside of the structure, then the graph cannot have a cycle containing v_1, v_2, v_3, v_4 and v_5 in this order. Hence the assumption on 5-connectedness in Problems 4.4 and 4.5 are best possible, in a sense.

References

[1] D. Archdeacon, N. Hartsfield, C. H. C. Little, *Nonhamiltonian triangulations with large connectivity and representativity*, J. Combin. Theory Ser. B, 68, 45-55, 1996.
[2] R. J. Faudree, *Survey of results on k-ordered graphs*, Discrete Math., 229, 73-87, 2001.
[3] W. Goddard, 4*-connected maximal planar graphs are* 4*-ordered*, Discrete Math., 257, 405-410, 2002.
[4] J. W. Moon and L. Moser, *Simple paths on polyhedra*, Pacific J. Math., 13, 629-631, 1963.
[5] R. Mukae and K. Ozeki, 4*-connected triangulations and* 4*-orderedness*, Discrete Math.,310, 2271-2272, 2010.
[6] K. Kawarabayashi and K. Ozeki, *4-connected projective planar graphs are hamiltonian-connected*, (to appear in) J. Combin. Theory Ser. B.
[7] D. P. Sanders, *On paths in planar graphs*, J. Graph Theory, 24, 341-345, 1997.
[8] R. Thomas and X. Yu, 4*-connected projective-planar graphs are Hamiltonian*, J. Combin. Theory Ser. B, 62, 114-132, 1994.
[9] C. Thomassen, *A theorem on paths in planar graphs*, J. Graph Theory, 7, 169-176, 1983.
[10] C. Thomassen, *Trees in triangulations*, J. Combin. Theory Ser. B, 60, 58-62, 1994.
[11] W. T. Tutte, *A theorem on planar graphs*, Trans. Amer. Math. Soc., 82, 99-116, 1956.
[12] H. Whitney, *A theorem on graphs*, Ann. of Math., 32, 378-390, 1931.
[13] X. Yu, *Disjoint paths, planarizing cycles, and spanning walks*, Trans. Amer. Math. Soc., 349, 1333-1358, 1997.