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Finitely Essential Supplemented Modules

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Abstract

In this work, finitely essential supplemented modules are defined and some properties of these modules are investigated. Let *M* be a finitely essential supplemented module. If *M* is noetherian, then *M* is essential supplemented. Let *M* be a finitely essential supplemented *R*-module and *N* be a finitely generated submodule of *M*. Then *M*/*N* is finitely essential supplemented. Let *M* be a finitely essential supplemented *R*-module and *N* be a finitely generated submodule of *M* with $RadM\leq N$. Then *M*/*N* have no proper finitely generated essential submodules. Let *M* be an *R*-module and $V\leq M$. If *V* is a supplement of a finitely generated essential submodule in *M*, then *V* is called an fe-suppplement submodule in *M*. Let *M* be an *R*-module. If every finitely generated essential submodule of *M* is β* equivalent to an fe-supplement submodule in *M* or *M* have no finitely generated essential submodules, then *M* is finitely essential supplemented.

Keywords: Small submodules, radical, essential submodules, supplemented modules.

Sonlu Büyük Tümlenmiş Modüller

Öz

Bu çalışmada sonlu büyük tümlenmiş modül kavramı tanımlandı ve bu kavramla ilgili birtakım özellikler incelendi. *M* bir sonlu büyük tümlenmiş modül olsun. Eğer *M* noetherian ise *M* büyük tümlenmiştir. *M* bir sonlu büyük tümlenmiş *R*-modül ve *N*, *M*'nin sonlu üretilmiş bir alt modülü olsun. Bu durumda *M*/*N* sonlu büyük tümlenmiştir. *M* bir sonlu büyük tümlenmiş *R*-modül ve *N*, *M*'nin sonlu üretilmiş ve $RadM\leq N$ koşuluna uyan bir alt modülü olsun. Bu durumda *M*/*N*'nin hiçbir öz sonlu üretilmiş büyük alt modülü yoktur. *M* bir *R*-modül ve $V\leq M$ olsun. Eğer *V*, *M*'de sonlu üretilmiş büyük bir alt modülün bir tümleyeni ise *V*'ye *M*'de bir fe-tümleyen alt modül denir. *M* bir *R*-modül olsun. Eğer *M*'nin her sonlu üretilmiş büyük alt modülü *M*'de β* bağıntısı ile bir fe-tümleyen alt modüle denkse veya *M*'nin hiçbir sonlu üretilmiş büyük alt modülü yoksa, *M* sonlu büyük tümlenmiştir.

Anahtar Kelimeler: Küçük alt modüller, radikal, büyük alt modüller, tümlenmiş modüller.

1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let *R* be a ring and *M* be an *R*-module. We will denote a submodule *N* of *M* by $N\leq M$. Let *M* be an *R*-module and $N\leq M$. If $L=M$ for every submodule *L* of *M* such that $M=N+L$, then *N* is called a *small* (or *superfluous*) submodule of *M* and denoted by $N\ll M$. A submodule *N* of an *R*-module

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M is called an *essential* submodule of M and denoted by $N\unlhd M$ in case $K\cap N\neq 0$ for every submodule $K\neq 0$, or equivalently, $N\cap L=0$ for $L\leq M$ implies that $L=0$. Let M be an R-module and $U,V\leq M$. If $M=U+V$ and V is minimal with respect to this property, or equivalently, $M=U+V$ and $U\cap V\ll V$, then V is called a *supplement* of U in M. M is called a *supplemented* module if every submodule of M has a supplement in M. M is said to be *essential supplemented* (briefly, *e-supplemented*) if every essential submodule of M has a supplement in M. M is said to be *finitely supplemented* (briefly, *f-supplemented*) if every finitely generated submodule of M has a supplement in M. Let M be an R-module and $U\leq M$. If for every $V\leq M$ such that $M=U+V$, U has a supplement V' with $V'\leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in M, then M is called an *amply supplemented* module. M is said to be *amply essential supplemented* if every essential submodule of M has ample supplements in M. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM=M$. Let M be an R-module and $U,V\leq M$. If $M=U+V$ and $U\cap V\ll M$, then V is called a *weak supplement* of U in M. If every submodule of M has a weak supplement in M, then M is called a *weakly supplemented* module. Let M be an R-module. We say submodules X and Y of M are β* *equivalent*, $X\beta^*Y$, if and only if $Y+K=M$ for every $K\leq M$ such that $X+K=M$ and $X+T=M$ for every $T\leq M$ such that $Y+T=M$. A module M is said to be *noetherian* if every submodule of M is finitely generated. Let M be an R-module and $X\leq Y\leq M$. if $Y/X\ll M/X$, then we say *Y lies above X* in M. More informations about (amply) supplemented modules are in Clark et al. (2006), Wisbauer (1991) and (Zöschinger, 1974). More details about weakly supplemented modules are in (Clark et al., 2006). More details about (amply) essential supplemented modules are in (Nebiyev et al., 2018a and b). The definition of β* relation and some properties of this relation are in (Birkenmeier et al., 2010).

2. Material and Methods

2.1. Finitely Essential Supplemented Modules

Definition 2.1. Let M be an R-module. If every finitely generated essential submodule of M has a supplement in M or M have no finitely generated essential submodules, then M is called a *finitely essential supplemented* (or briefly *fe-supplemented*) module (See also (Nebiyev and Ökten, 2020))

Lemma 2.2. Every f-supplemented module is fe-supplemented (See also (Nebiyev and Ökten, 2020))
Proof. Let M be an f-supplemented module. Then every finitely generated submodule of M has a supplement in M. By this, every finitely generated essential submodule of M has a supplement in M. Hence M is fe-supplemented, as required.

Corollary 2.3. Let M be an R-module and $L\ll M$. If M is f-supplemented, then M/L is fe-supplemented (See also (Nebiyev and Ökten, 2020))

Proof. Since M is f-supplemented and $L \ll M$, by Wisbauer (1991) 41.3(2)(i), M/L is f-supplemented. Then by Lemma 2.2, M/L is fe-supplemented.

Corollary 2.4. Let M be an R-module and L be a finitely generated submodule of M. If M is f-supplemented, then M/L is fe-supplemented (See also (Nebiyev and Ökten, 2020)).
Proof. Since M is f-supplemented and L is a finitely generated submodule of M, by Wisbauer (1991) 41.3(2)(i), M/L is f-supplemented. Then by Lemma 2.2, M/L is fe-supplemented.

Proposition 2.5. Let M be an fe-supplemented R-module. If every nonzero finitely generated submodule of M is essential in M, then M is f-supplemented (See also (Nebiyev and Ökten, 2020)).
Proof. Let U be a finitely generated submodule of M. If $U=0$, then M is a supplement of U in M. If $U \neq 0$, then by hypothesis, U is a finitely generated essential submodule of M and since M is fe-supplemented, U has a supplement in M. Hence M is f-supplemented.

Proposition 2.6. Every essential supplemented module is fe-supplemented.
Proof. Let M be an essential supplemented module. Then every essential submodule of M has a supplement in M. Hence every finitely generated essential submodule of M has a supplement in M. Therefore, M is fe-supplemented.

Corollary 2.7. Every amply essential supplemented module is fe-supplemented.
Proof. Clear from Proposition 2.6, since every amply essential supplemented module is essential supplemented.

Proposition 2.8. Let M be an fe-supplemented module. If M is noetherian, then M is essential supplemented.
Proof. Let $U \trianglelefteq M$. Since M is noetherian, U is finitely generated. Since M is fe-supplemented, U has a supplement in M. Hence M is essential supplemented.

Lemma 2.9. Let M be an fe-supplemented R-module and N be a finitely generated submodule of M. Then M/N is fe-supplemented (See also (Nebiyev and Ökten, 2020))
Proof. Let K/N be a finitely generated essential submodule of M/N. Since $K/N \trianglelefteq M/N$, by Wisbauer (1991) 17.3(3), $K \trianglelefteq M$. Since K/N and N are finitely generated, K is also finitely generated. Since M is fe-supplemented, K has a supplement T in M. Then by Wisbauer (1991) 41.1(7), $(T+N)/N$ is a supplement of K/N in M/N. Hence M/N is fe-supplemented.

Corollary 2.10. Let M be an fe-supplemented R-module and N be a cyclic submodule of M. Then M/N is fe-supplemented (See also (Nebiyev and Ökten, 2020)).
Proof. Since N is cyclic, then N is finitely generated. Then by Lemma 2.9, M/N is fe-supplemented, as desired.

Corollary 2.11. Let $f: M \to N$ be an R-module epimorphism and $Kerf$ be finitely generated. If M is fe-supplemented, then N is also fe-supplemented (See also (Nebiyev and Ökten, 2020)).
Proof. Clear from Lemma 2.9.

Corollary 2.12. Let $f: M \to N$ be an R-module epimorphism with cyclic kernel. If M is fe-supplemented, then N is also fe-supplemented (See also (Nebiyev and Ökten, 2020)).
Proof. Clear from Corollary 2.11.

Corollary 2.13. Let $M=M_1 \oplus M_2$ with M_1 finitely generated. If M is fe-supplemented, then M_2 is fe-supplemented.
Proof. Since M is fe-supplemented, by Lemma 2.9, M/M_1 is fe-supplemented. Then by $M/M_1=(M_1 \oplus M_2)/M_1 \cong M_2/(M_1 \cap M_2)=M_2/0 \cong M_2$, M_2 is fe-supplemented.

Corollary 2.14. Let $M=M_1 \oplus M_2$ with M_1 cyclic. If M is fe-supplemented, then M_2 is fe-supplemented.
Proof. Clear from Corollary 2.13.

Lemma 2.15. Let M be an fe-supplemented R-module and $L \ll M$. If $K \trianglelefteq M$ for every nonzero finitely generated submodule K of M with $K+L \trianglelefteq M$, then M/L is fe-supplemented.

Proof. Let U/L be a finitely generated essential submodule of M/L. Since $U/L \trianglelefteq M/L$, by Wisbauer (1991) 17.3(3), $U \trianglelefteq M$. Since U/L is finitely generated, there exists a finitely generated submodule K of M such that $U=K+L$. If $K=0$, then $U/L=(K+L)/L=L/L=0$ and M/L is a supplement of U/L in M/L. Suppose that $K \neq 0$. Then by hypothesis, $K \trianglelefteq M$ and since M is fe-supplemented, K has a supplement V in M. Since V is a supplement of K in M and $L \ll M$, by Wisbauer (1991) 41.1(4), V is a supplement of $U=K+L$ in M . Then by Wisbauer (1991) 41.1(7), $(V+L)/L$ is a supplement of U/L in M/L. Hence M/L is fe-supplemented.

Corollary 2.16. Let M be an fe-supplemented R-module and $f: M \to N$ be an R-module epimorphism with $Kerf \ll M$. If $K \trianglelefteq M$ for every nonzero finitely generated submodule K of M with $K+Kerf \trianglelefteq M$, then N is fe-supplemented.

Proof. By Lemma 2.15, $M/Kerf$ is fe-supplemented. Since $M/Kerf \cong Imf=N$, N is also fe-supplemented.

Lemma 2.17. Let M be an fe-supplemented R-module and N be a finitely generated submodule of M with $RadM \leq N$. Then M/N have no proper finitely generated essential submodules.

Proof. Let U/N be a finitely generated essential submodule of M/N. Since $U/N \trianglelefteq M/N$, by Wisbauer (1991) 17.3(3), $U \trianglelefteq M$. Since N and U/N are finitely generated, U is also finitely generated. Then by M being fe-supplemented, U has a supplement V in M. Here $M=U+V$ and $U \cap V \ll V$. Since $M=U+V$, $M/N=U/N+(V+N)/N$. Since $U \cap V \ll V$, $U \cap V \leq RadM \leq N$. Hence $(U/N) \cap ((V+N)/N)=(U \cap V+N)/N=0$ and $M/N=U/N \oplus (V+N)/N$. Since $U/N \trianglelefteq M/N$, $U/N=M/N$. Hence M/N have no proper finitely generated essential submodules.

Corollary 2.18. Let M be an fe-supplemented R-module and $RadM$ be finitely generated. Then $M/RadM$ have no proper finitely generated essential submodules.
Proof. Clear from Lemma 2.17.

Corollary 2.19. Let $f : M \rightarrow N$ be an R-module epimorphism with $Kerf$ finitely generated and $RadM \leq Kerf$. If M is fe-supplemented, then N have no proper finitely generated essential submodules.
Proof. Clear from Lemma 2.17.

Definition 2.20. Let M be an R-module and $V \leq M$. If V is a supplement of a finitely generated essential submodule in M, then V is called an *fe-supplement* submodule in M.

Lemma 2.21. Let M be an R-module. If every finitely generated essential submodule of M is β^* equivalent to an fe-supplement submodule in M or M have no finitely generated essential submodules, then M is fe-supplemented.
Proof. Let U be a finitely generated essential submodule of M. By hypothesis, there exists an fe-supplement submodule X in M with $U\beta^*X$. Since X is an fe-supplement submodule in M, there exists a finitely generated essential submodule Y of M such that X is a supplement of Y in M. Since Y is a finitely generated essential submodule of M, by hypothesis, there exists an fe-supplement submodule V in M with $Y\beta^*V$. Since X is a supplement of Y in M, Y is a weak supplement of X in M. Since $U\beta^*X$, by Birkenmeier et al. (2010) Theorem 1.6(ii), Y is a weak supplement of U in M. Hence U is a weak supplement of Y in M and since $Y\beta^*V$, by Birkenmeier et al. (2010) Theorem 1.6(ii), U is a weak supplement of V in M. By the last sentence, $M=U+V$ and $U \cap V \ll M$. Since V is a supplement submodule in M, by Wisbauer (1991) 41.1(5), $U \cap V \ll V$ and hence V is a supplement of U in M . Therefore, M is fe-supplemented.

Corollary 2.22. Let M be an R-module. If every finitely generated essential submodule of M lies above an fe-supplement submodule in M or M have no finitely generated essential submodules, then M is fe-supplemented.
Proof. Clear from Lemma 2.21.

Lemma 2.23. Let M be an R-module. If every finitely generated essential submodule of M is small in M or M have no finitely generated essential submodules, then M is fe-supplemented.

Proof. Let U be a finitely generated essential submodule of M. By hypothesis, $U \ll M$ and hence M is a supplement of U in M. Therefore, M is fe-supplemented.

Example 2.24. Consider the $\mathbb{Z}$-module ${}_{\mathbb{Z}}\mathbb{Q}$. Since every finitely generated essential submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ is small in ${}_{\mathbb{Z}}\mathbb{Q}$, by Lemma 2.23, ${}_{\mathbb{Z}}\mathbb{Q}$ is fe-supplemented. But it is well known that ${}_{\mathbb{Z}}\mathbb{Q}$ is not essential supplemented.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

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