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On Biharmonic Curves in 3-dimensional Heisenberg Group

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Abstract

In this paper we study the non-geodesic non-null biharmonic curves in 3-dimensional hyperbolic Heisenberg group with a semi-Riemannian metric of index 2. We prove that all of the non-geodesic non-null biharmonic curves in such a 3-dimensional hyperbolic Heisenberg group are helices. Moreover, we obtain explicit parametric equations for non-geodesic non-null biharmonic curves and non-geodesic spacelike horizontal biharmonic curves, respectively. We also show that there do not exist non-geodesic timelike horizontal biharmonic curves in 3-dimensional hyperbolic Heisenberg group with a semi-Riemannian metric of index 2.

Keywords: Biharmonic curves, Horizontal curves, Heisenberg group.

3-boyutlu Heisenberg Grubun Biharmonik Eğrileri Üzerine

Özet

Bu çalışmada indeksi 2 olan bir semi-Riemann metriğe sahip 3-boyutlu Heisenberg grubun jeodezik olmayan non-null biharmonik eğrileri çalışıldı. Bu şekildeki bir 3-boyutlu Heisenberg grubun jeodezik olmayan non-null biharmonik eğrilerinin helis olduğu ispatlandı. Ayrıca sırasıyla jeodezik olmayan non-null biharmonik eğriler ve jeodezik olmayan spacelike yatay biharmonik eğriler için açık parametrik denklemler elde edildi. İndeksi 2 olan bir semi-Riemann metriğe sahip 3-boyutlu Heisenberg grup üzerinde jeodezik olmayan timelike yatay biharmonik eğrilerin var olmadığı gösterildi.

Anahtar Kelimeler: Biharmonik eğriler, Yatay eğriler, Heisenberg Grup.

Introduction

In 1964, Eells and Sampson [8] introduced the notion of biharmonic maps as a natural generalization of the well-known harmonic maps. Thus, while a map Ψ from a compact Riemannian manifold (M, g) to another Riemannian manifold (N, h) is harmonic if it is a critical point of the energy functional $E(\Psi) = \frac{1}{2}\int_M |d\Psi|^2 v_g$, the biharmonic maps are the critical points of the bienergy functional $E_2(\Psi) = \frac{1}{2}\int_M |\tau(\Psi)|^2 v_g$.

In a different setting, Chen [6] defined biharmonic submanifolds $M \subset E^n$ of the Euclidean space as those with harmonic mean curvature vector field, that is $\Delta H = 0$, where Δ is the rough Laplacian, and stated that any biharmonic submanifold of the Euclidean space is harmonic, that is minimal.

If the definition of biharmonic maps is applied to Riemannian immersions into Euclidean space, the notion of Chen's biharmonic submanifold is obtained, so the two definitions agree.

Harmonic maps are characterized by the vanishing of the tension field $\tau(\Psi) = trace\nabla d\Psi$, where ∇ is a connection induced from the Levi-Civita connection ∇^M of M and ∇^Ψ is the pull-back connection. The first variation formula for the bienergy derived in [15, 16] shows that the Euler-Lagrange equation for the bienergy is

$$\tau_2(\Psi) = -J(\tau(\Psi)) = -\Delta\tau(\Psi) - traceR^N(d\Psi, \tau(\Psi))d\Psi = 0,$$

where $\Delta = -trace(\nabla^\Psi\nabla^\Psi - \nabla^\Psi_\nabla)$ is the rough Laplacian on the sections of $\Psi^{-1}TN$ and $R^N(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{X,Y]}$ is the curvature operator on N. From the expression of the bitension field τ_2, it is clear that a harmonic map is automatically a biharmonic map. Non-harmonic biharmonic maps are called proper biharmonic maps.

Of course, the first and easiest examples can be found by looking at differentiable curves in a Riemannian manifold. Obviously geodesics are biharmonic. So, non-geodesic biharmonic curves are more interesting. Chen and Ishikawa [5] showed non-existence of proper biharmonic curves in Euclidean 3-space E^3. Moreover they classified all proper biharmonic curves in Minkowski 3-space E_1^3 (see also [13]). Caddeo, Montaldo and Piu showed that on a surface with non-positive Gaussian curvature, any biharmonic curve is a geodesic of the surface [2]. So they gave a positive answer to generalized Chen's conjecture. Caddeo et al. in [3] studied biharmonic curves in the unit 3-sphere. More precisely, they showed that proper biharmonic curves in S^3 are circles of geodesic curvature 1 or helices which are geodesics in the Clifford minimal torus.

On the other hand, there are several classification results on biharmonic curves in arbitrary Riemannian manifolds. The biharmonic curves in the Heisenberg group are investigated in [4] by Caddeo et al. They showed that biharmonic curves in the Heisenberg group are helices, that is curves with constant geodesic curvature k_1 and geodesic torsion k_2. The authors in [17] studied non-geodesic horizontal biharmonic curves in 3-dimensional Heisenberg group. The same authors obtained some results for the Heisenberg Group with left invariant Lorentzian metric and investigated biharmonic curves in 3-dimensional Lorentzian Heisenberg group (see [18], [19]) . In [9] Fetcu studied biharmonic curves in the generalized Heisenberg group and obtained two families of proper biharmonic curves.

In contact geometry, it is well known that a simply connected 3-dimensional Sasakian space form of constant holomorphic sectional curvature 1 is isometric to S^3. So in this context J. Inoguchi classified in [14] the proper biharmonic Legendre curves and Hopf cylinders in a 3-dimensional Sasakian space form and in [10] the explicit parametric equations were obtained. In [7], the authors showed that every non-geodesic biharmonic curve in a 3-dimensional Sasakian space form of constant holomorphic sectional curvature is a helix. T. Sasahara [21], analyzed the proper biharmonic Legendre surfaces in Sasakian space forms and in the case when the ambient space is the unit 5-dimensional sphere S^5 he obtained their explicit representations. A full classification of proper biharmonic Legendre curves, explicit examples and a method to construct proper biharmonic anti-invariant submanifolds in any dimensional Sasakian space form were given in [11]. Furthermore, D. Fetcu [12] studied proper biharmonic non-Legendre curves in a Sasakian space form.

Motivated by these circumtances, in the present paper we associate a semi-Riemannian metric of index 2 with a 3-dimensional Heisenberg group and study the non-null biharmonic curves in such a 3-dimensional Heisenberg group (for short, $\mathcal{H}_2^3$). Section 1 is devoted to the some basic definitions. We also define and characterize a cross product in 3-dimensional Heisenberg group $\mathcal{H}_2^3$. In section 2 we investigate the necessary and sufficient conditions for a non-null curve in 3-dimensional Heisenberg group $\mathcal{H}_2^3$ to be non-geodesic biharmonic. In section 3 we prove that a non-geodesic non-null curve parametrized by arclenght in 3-dimensional Heisenberg group $\mathcal{H}_2^3$ with the vanishing third component of the binormal vector field cannot be biharmonic. In section 4, we study the non-geodesic non-null biharmonic helices in 3-dimensional Heisenberg group with a semi-Riemannian metric of index 2. Moreover, we obtain explicit parametric equations for non-geodesic non-null biharmonic curves in $\mathcal{H}_2^3$. In the last section, we give explicit examples of non-geodesic spacelike horizontal biharmonic curves and prove that there do not exist non-geodesic timelike horizontal biharmonic curves in 3-dimensional Heisenberg group $\mathcal{H}_2^3$.

1. Preliminaries

1.1. Biharmonic Maps

Let (M,g) and (N,h) be Riemannian manifolds and $\Psi:(M,g)\to(N,h)$ be a smooth map. The tension field of Ψ (see [8]) is given by $\tau(\Psi)=trace\nabla d\Psi$, where $\nabla d\Psi$ is the second fundamental form of Ψ defined by $\nabla d\Psi(X,Y)=\nabla_X^{\Psi}d\Psi(Y)-d\Psi(\nabla_X^M Y)$, $X,Y\in\Gamma(TM)$. For any compact domain $\Omega\subseteq M$, the bienergy is defined by [15, 16]

$$E_2(\Psi)=\frac{1}{2}\int_{\Omega}|\tau(\Psi)|^2 v_g.$$

Then a smooth map Ψ is called biharmonic map if it is a critical point of the bienergy functional for any compact domain $\Omega\subseteq M$. We have for the bienergy the following first variation formula [15, 16]:

$$\frac{d}{dt}E_2(\Psi_t;\Omega)|_{t=0}=\int_{\Omega}<\tau_2(\Psi),w>v_g,$$

where v_g is the volume element, w is the variational vector field associated to the variation $\{\Psi_t\}$ of Ψ and

$$\tau_2(\Psi)=-J(\tau_2(\Psi))=-\Delta^{\Psi}\tau(\Psi)-traceR^N(d\Psi,\tau(\Psi))d\Psi.$$

$\tau_2(\Psi)$ is called bitension field of Ψ. Here Δ^{Ψ} is the rough Laplacian on the sections of the pull-back bundle $\Psi^{-1}TN$ which is defined by

$$\Delta^{\Psi}V=-\sum_{i=1}^{m}\{\nabla_{e_i}^{\Psi}\nabla_{e_i}^{\Psi}V-\nabla_{\nabla_{e_i}^{M}e_i}^{\Psi}V\},\quad V\in\Gamma(\Psi^{-1}TN),$$

where ∇^{Ψ} is the pull-back connection on the pull-back bundle $\Psi^{-1}TN$ and $\{e_i\}_{i=1}^{m}$ is an orthonormal frame on M. When the target manifold is semi-Riemannian manifold, the bienergy and bitension field can be defined in the same way.

Let M be a semi-Riemannian manifold and $\gamma:I\to M$ be a non-null curve parametrized by arclenght. By using the definition of the tension field we have

$$\tau(\gamma)=\nabla_{\frac{\partial}{\partial s}}^{\gamma}d\gamma(\frac{\partial}{\partial s})=\nabla_T T,$$

where $T=\gamma^{'}$. In this case biharmonic equation for the curve γ reduces to (see also [20])

$$\tau_2(\gamma)=\nabla_T^3 T-R(T,\nabla_T T)T=0.$$

1.2. 3-dimensional Heisenberg group with a semi-Riemannian metric of index 2

Consider R^3 with the group law given by

$$\tilde{X}X = (\tilde{x} + x, \tilde{y} + y, \tilde{z} + z - \tilde{x}y + \tilde{y}x), \tag{1}$$

where $X = (x, y, z)$, $\tilde{X} = (\tilde{x}, \tilde{y}, \tilde{z})$.

Let $\mathcal{H}_2^3 = (R^3, g)$ be 3-dimensional Heisenberg group endowed with the semi-Riemannian metric g of index 2 which is defined by

$$g = (dx)^2 + (dy)^2 - \frac{1}{4}(dz + 2ydx - 2xdy)^2. \tag{2}$$

Note that the metric g is left invariant.

We can define an orthonormal basis for the tangent space of $\mathcal{H}_2^3$ by

$$e_1 = \frac{\partial}{\partial x} - 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + 2x\frac{\partial}{\partial z}, \quad e_3 = 2\frac{\partial}{\partial z}, \tag{3}$$

which is dual to the coframe

$$\theta^1 = dx, \quad \theta^2 = dy, \quad \theta^3 = \frac{1}{2}dz + ydx - xdy.$$

Proposition 1.2.1: *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g defined above, we have*

$$\begin{cases} \nabla_{e_1}e_1 = 0, & \nabla_{e_1}e_2 = e_3, & \nabla_{e_1}e_3 = -e_2, \\ \nabla_{e_2}e_1 = -e_3, & \nabla_{e_2}e_2 = 0, & \nabla_{e_2}e_3 = -e_1, \\ \nabla_{e_3}e_1 = -e_2, & \nabla_{e_3}e_2 = -e_1, & \nabla_{e_3}e_3 = 0, \end{cases} \tag{4}$$

where $\{e_1, e_2, e_3\}$ is the orthonormal basis for the tangent space given by (3).

Also, we have the following bracket relations

$$[e_1, e_2] = 2e_3, \quad [e_1, e_3] = [e_2, e_3] = 0. \tag{5}$$

The curvature tensor field of ∇ is given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

while the Riemannian-Christoffel tensor field is

$$R(X,Y,Z,W) = g(R(X,Y)Z, W),$$

where $X,\ Y,\ Z,\ W \in \Gamma(T\mathcal{H}_2^3)$. If we put

$$R_{abc} = R(e_a, e_b)e_c,$$

where the indices $a,\ b,\ c$ take the values $1,\ 2,\ 3$. Then the non-zero components of the curvature tensor field are

$$\begin{cases} R_{121} = 3e_2, & R_{122} = 3e_1, \quad R_{131} = -e_3, \\ R_{133} = -e_1, & R_{232} = e_3, \quad R_{233} = -e_2. \end{cases} \tag{6}$$

Now we shall define a cross product on 3-dimensional Heisenberg group $\mathcal{H}_2^3$ for later use

Definition 1.2.2: *We define a cross product* $\wedge$ *on* $\mathcal{H}_2^3$ *by*

$$X \wedge Y = -(a_2 b_3 - a_3 b_2)e_1 - (a_1 b_3 - a_3 b_1)e_2 + (a_1 b_2 - a_2 b_1)e_3,$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathcal{H}_2^3$ given by (3) and $X = a_1 e_1 + a_2 e_2 + a_3 e_3$, $Y = b_1 e_1 + b_2 e_2 + b_3 e_3 \in \Gamma(T(\mathcal{H}_2^3))$.

Theorem 1.2.3: *The cross product* $\wedge$ *on* $\mathcal{H}_2^3$ *has the following properties:*

(i) *The cross product is bilinear and anti-symmetric (*$X \wedge Y = -Y \wedge X$*).*

(ii) $X \wedge Y$ *is perpendicular both of* X *and* Y.

(iii) $e_1 \wedge e_2 = e_3$, $e_2 \wedge e_3 = -e_1$, $e_3 \wedge e_1 = e_2$.

(iv) $(X \wedge Y) \wedge Z = g(X, Z)Y - g(Y, Z)X$.

(v) *Define a mixed product by*

$$(X, Y, Z) = g(X \wedge Y, Z),$$

then we have

$$(X, Y, Z) = -det(X, Y, Z)$$

and

$$(X, Y, Z) = (Y, Z, X) = (Z, X, Y).$$

(vi) $(X \wedge Y) \wedge Z + (Y \wedge Z) \wedge X + (Z \wedge X) \wedge Y = 0$,

for all $X, Y, Z \in \Gamma(T(\mathcal{H}_2^3))$.

2. Biharmonic curves in 3-dimensional Heisenberg group with a semi-Riemannian metric of index 2

An arbitrary curve $\gamma: I \to \mathcal{H}_2^3$, $\gamma = \gamma(s)$, in 3-dimensional Heisenberg group $\mathcal{H}_2^3$ is called spacelike, timelike or null (lightlike), if all of its velocity vectors $\gamma'(s)$ are respectively spacelike, timelike or null (lightlike). If $\gamma(s)$ is a spacelike or timelike curve, we can reparametrize it such that $g\left(\gamma'(s), \gamma'(s)\right) = \varepsilon$, where $\varepsilon = 1$ if γ is spacelike and $\varepsilon = -1$ if γ is timelike, respectively. In this case $\gamma(s)$ is said to be unit speed or arclenght parametrization.

Let $\gamma: I \to \mathcal{H}_2^3$ be a non-null curve parametrized by arclenght and $\{T, N, B\}$ be the orthonormal moving Frenet frame along the curve γ in $\mathcal{H}_2^3$ such that $T=\gamma'$ is the unit vector field tangent to γ, N is the unit vector field in the direction $\nabla_T T$ normal to γ and $B=T\wedge N$. The mutually orthogonal unit vector fields T, N and B are called the tangent, the principal normal and the binormal vector fields, respectively. Then we have the following Frenet equations

$$\begin{aligned}\nabla_T T &= k_1\varepsilon_2 N,\\ \nabla_T N &= -k_1\varepsilon_1 T + k_2\varepsilon_3 B,\\ \nabla_T B &= -k_2\varepsilon_2 N,\end{aligned} \tag{7}$$

where $\varepsilon_1 = g(T,T)$, $\varepsilon_2 = g(N,N)$ and $\varepsilon_3 = g(B,B)$. Here $k_1 = |\tau(\gamma)| = |\nabla_T T|$ is the geodesic curvature of γ and k_2 is its geodesic torsion.

From (7) we have

$$\begin{aligned}\nabla_T^3 T = &\left(-3k_1k_1'\varepsilon_1\varepsilon_2\right)T + \left(k_1''\varepsilon_2 - k_1^3\varepsilon_1 - k_1k_2^2\varepsilon_3\right)N\\ &+\left(2k_1'k_2\varepsilon_2\varepsilon_3 + k_1k_2'\varepsilon_2\varepsilon_3\right)B.\end{aligned} \tag{8}$$

Using (6) one obtains

$$R(T,\nabla_T T)T = k_1\varepsilon_2[(-\varepsilon_2\varepsilon_3 - 4\varepsilon_2 B_3^2)N + (4\varepsilon_3 N_3 B_3)B], \tag{9}$$

where $T = T_1e_1 + T_2e_2 + T_3e_3$, $N = N_1e_1 + N_2e_2 + N_3e_3$ and $B = T\wedge N = B_1e_1 + B_2e_2 + B_3e_3$. Hence we get

$$\begin{aligned}\tau_2(\gamma) = &\left(-3k_1k_1'\varepsilon_1\varepsilon_2\right)T + \left(k_1''\varepsilon_2 - k_1^3\varepsilon_1 - k_1k_2^2\varepsilon_3 + k_1\varepsilon_3 + 4k_1B_3^2\right)N\\ &+\left(2k_1'k_2\varepsilon_2\varepsilon_3 + k_1k_2'\varepsilon_2\varepsilon_3 - 4k_1\varepsilon_2\varepsilon_3N_3B_3\right)B.\end{aligned} \tag{10}$$

Theorem 2.1: *Let $\gamma: I \to \mathcal{H}_2^3$ be a non-null curve parametrized by arclenght. Then γ is a non-geodesic biharmonic curve if and only if*

$$\begin{cases} k_1 = \text{constant} \neq 0,\\ k_1^2\varepsilon_1\varepsilon_3 + k_2^2 = 1 + 4\varepsilon_3 B_3^2,\\ k_2' = N_3B_3.\end{cases} \tag{11}$$

Proof. From (10) it follows that γ is biharmonic if and only if

$$\begin{cases} k_1k_1' = 0,\\ k_1''\varepsilon_2 - k_1^3\varepsilon_1 - k_1k_2^2\varepsilon_3 + k_1\varepsilon_3 + 4k_1B_3^2 = 0,\\ 2k_1'k_2 + k_1k_2' - 4k_1N_3B_3 = 0.\end{cases}$$

If we look for non-geodesic solution of the above system we complete the proof.

Corollary 2.2: *If* k_1 *=constant*$\neq 0$ *and* $k_2 = 0$ *for a non-null curve* $\gamma: I \to \mathcal{H}_2^3$ *then* γ *is a non-geodesic biharmonic curve if and only if* $k_1^2 = \varepsilon_1(\varepsilon_3 + 4B_3^2)$ *and* $N_3 B_3 = 0$.

Proposition 2.3: *Let* $\gamma: I \to \mathcal{H}_2^3$ *be a non-geodesic non-null curve parametrized by arclenght. If* k_1 *is constant and* $N_3 B_3 \neq 0$, *then* γ *is not biharmonic.*

Proof. By using (4) and (7) we have

$$\begin{aligned}\nabla_T T &= \left(T_1^{'} - 2T_2T_3\right)e_1 + \left(T_2^{'} - 2T_1T_3\right)e_2 + T_3^{'}e_3 \\ &= k_1\varepsilon_2 N,\end{aligned} \tag{12}$$

which implies that

$$T_3^{'} = k_1\varepsilon_2 N_3.$$

If we put $T_3(s) = k_1 F(s)$ and $f(s) = F^{'}(s)$ we get $f(s) = \varepsilon_2 N_3(s)$. Then we can write

$$T = \sqrt{\varepsilon_1 + k_1^2F^2}\cosh\beta\ e_1 + \sqrt{\varepsilon_1 + k_1^2F^2}\sinh\beta\ e_2 + k_1 F e_3.$$

From (12) we calculate

$$\begin{aligned}\nabla_T T = k_1\varepsilon_2 N = &\left(\frac{k_1^2Ff}{\sqrt{\varepsilon_1+k_1^2F^2}}\cosh\beta + \sqrt{\varepsilon_1 + k_1^2F^2}(\beta^{'} - k_1F)\sinh\beta\right)e_1 \\ &+\left(\frac{k_1^2Ff}{\sqrt{\varepsilon_1+k_1^2F^2}}\sinh\beta + \sqrt{\varepsilon_1 + k_1^2F^2}(\beta^{'} - k_1F)\cosh\beta\right)e_2 \\ &+(k_1\ f)e_3.\end{aligned} \tag{13}$$

By taking into account the definition of the geodesic curvature k_1 and the last equation one can see that

$$\beta^{'} - k_1 F = \pm k_1 \frac{\sqrt{-\varepsilon_1\varepsilon_2 - \varepsilon_1 f^{\,2} - \varepsilon_2 k_1^2F^2}}{\varepsilon_1 + k_1^2F^2}. \tag{14}$$

If we write (14) in (13) we get

$$\begin{aligned}\varepsilon_2 N = &\left(\pm\frac{\sqrt{-\varepsilon_1\varepsilon_2 - \varepsilon_1 f^{\,2} - \varepsilon_2 k_1^2F^2}}{\sqrt{\varepsilon_1+k_1^2F^2}}\sinh\beta + \frac{k_1Ff}{\sqrt{\varepsilon_1+k_1^2F^2}}\cosh\beta\right)e_1 \\ &+\left(\pm\frac{\sqrt{-\varepsilon_1\varepsilon_2 - \varepsilon_1 f^{\,2} - \varepsilon_2 k_1^2F^2}}{\sqrt{\varepsilon_1+k_1^2F^2}}\cosh\beta + \frac{k_1Ff}{\sqrt{\varepsilon_1+k_1^2F^2}}\sinh\beta\right)e_2 + f\ e_3.\end{aligned}$$

Since $B = T \wedge N$, from the definition of the cross product in $\mathcal{H}_2^3$ we have

$$B_3 = \pm\, \varepsilon_2 \sqrt{-\varepsilon_1 \varepsilon_2 - \varepsilon_1 f^{\,2} - \varepsilon_2 k_1^2 F^2}. \tag{15}$$

On the other hand from the Frenet equations we obtain

$$g(\nabla_T N, e_3) = k_1 \varepsilon_1 T_3 - k_2 \varepsilon_3 B_3.$$

Using (4) since $N = N_1 e_1 + N_2 e_2 + N_3 e_3$ we also have

$$g(\nabla_T N, e_3) = -N_3^{'} - B_3,$$

which implies that

$$-N_3^{'} - B_3 = k_1 \varepsilon_1 T_3 - k_2 \varepsilon_3 B_3. \tag{16}$$

By writing $N_3 = \varepsilon_2 f$, $T_3 = k_1 F$ and (15) in (16) we get

$$k_2 = \pm \frac{(f^{\,'} + k_1^2 \varepsilon_1 \varepsilon_2 F)\varepsilon_3}{\sqrt{-\varepsilon_1 \varepsilon_2 - \varepsilon_1 f^{\,2} - \varepsilon_2 k_1^2 F^2}} + \varepsilon_3 = \pm\, \varepsilon_1 \varepsilon_3 \frac{B_3^{'}}{N_3} + \varepsilon_3. \tag{17}$$

Now assume that γ is biharmonic. Then from the third equation in (11) we write $k_2^{'} = N_3 B_3 \neq 0$ which gives

$$N_3 = \frac{k_2^{'}}{B_3}\ .$$

By writing the last equation in (17) and then by integrating we obtain

$$k_2^2 = \pm\, \varepsilon_1 \varepsilon_3 B_3^2 + 2 k_2 \varepsilon_3 + c, \tag{18}$$

where c is a constant. Also, from the second equation in (11) we have

$$\varepsilon_1 \varepsilon_3 B_3^2 = k_1^2 \frac{\varepsilon_3}{4} + k_2^2 \frac{\varepsilon_1}{4} - \frac{\varepsilon_1}{4}. \tag{19}$$

By comparing (18) and (19) we get

$$k_2^2 (4 \mp \varepsilon_1) - 8 k_2 \varepsilon_3 = C,$$

where $C = \mp\ \varepsilon_1 \pm k_1^2\, \varepsilon_3 + 4c$ is a constant, which implies that k_2 is also a constant. Hence we obtain a contradiction with the assumption $k_2^{'} \neq 0$. This completes the proof.

Theorem 2.4: *Let $\gamma : I \to \mathcal{H}_2^3$ be a non-geodesic non-null curve parametrized by arclenght. Then γ is biharmonic if and only if*

$$\begin{cases} k_1 = \text{constant} \neq 0, \\ k_2 = \text{constant}, \\ N_3 B_3 = 0, \\ k_1^2 \varepsilon_1 \varepsilon_3 + k_2^2 = 1 + 4 \varepsilon_3 B_3^2. \end{cases} \tag{20}$$

3. Biharmonic helices in 3-dimensional Heisenberg group with a semi-Riemannian metric of index 2

A non-null curve in a semi-Riemannian manifold having constant both geodesic curvature and geodesic torsion is called helix. Now we shall investigate the biharmonicity conditions of a helix in 3-dimensional Heisenberg group. For any helix in $\mathcal{H}_2^3$, the system (11) reduces to

$$\begin{cases} k_1^2\varepsilon_1\varepsilon_3 + k_2^2 = 1 + 4\varepsilon_3 B_3^2, \\ N_3 B_3 = 0, \end{cases} \tag{21}$$

which implies that B_3 must be a constant.

Proposition 3.1: *Let* $\gamma: I \to \mathcal{H}_2^3$ *be a non-geodesic non-null curve parametrized by arclenght with* $B_3 = 0$. *Then we have* $\varepsilon_1 = -\varepsilon_2$ *and* B *is a timelike vector field, where* $\varepsilon_1 = g(T,T)$ *and* $\varepsilon_2 = g(N,N)$.

Proof. Assume that $\gamma: I \to \mathcal{H}_2^3$ is a non-geodesic non-null curve parametrized by arclenght and $\gamma'(s) = T(s)$. If γ is a spacelike curve then we can write

$$T = \cosh\alpha_1\cosh\beta_1 e_1 + \cosh\alpha_1\sinh\beta_1 e_2 + \sinh\alpha_1 e_3. \tag{22}$$

where $\alpha_1 = \alpha_1(s)$ and $\beta_1 = \beta_1(s)$. From (4) the covariant derivative of the unit tangent vector field T of γ, is

$$\begin{aligned} \nabla_T T &= \left(\alpha_1'\sinh\alpha_1\cosh\beta_1 + \cosh\alpha_1\sinh\beta_1\left(\beta_1' - 2\sinh\alpha_1\right)\right) e_1 \\ &\quad + \left(\alpha_1'\sinh\alpha_1\sinh\beta_1 + \cosh\alpha_1\cosh\beta_1\left(\beta_1' - 2\sinh\alpha_1\right)\right) e_2 + \left(\alpha_1'\cosh\alpha_1\right) e_3 \\ &= k_1\varepsilon_2 N. \end{aligned}$$

By using the definition of cross product in $\mathcal{H}_2^3$ we also obtain

$$B_3 = \frac{\cosh^2\alpha_1(\beta_1' - 2\sinh\alpha_1)\varepsilon_2}{k_1}.$$

Now let $B_3 = 0$. From the last equation above, since $\cosh\alpha_1 \neq 0$ then $\beta_1' - 2\sinh\alpha_1 = 0$. Thus we have

$$\nabla_T T = \alpha_1'(\sinh\alpha_1\cosh\beta_1 e_1 + \sinh\alpha_1\sinh\beta_1 e_2 + \cosh\alpha_1 e_3). \tag{23}$$

We can assume that $\alpha_1' \neq 0$ (when $\alpha_1' = 0$ then we have $\nabla_T T = 0$, which implies that γ is a geodesic). Hence we get

$$k_1^2\varepsilon_2 = g(\nabla_T T, \nabla_T T) = \left(\alpha_1'\right)^2(\sinh^2\alpha_1 - \cosh^2\alpha_1) = -\left(\alpha_1'\right)^2. \tag{24}$$

If N is spacelike then $k_1 = 0$, which is a contradiction.

By a similar way, for a timelike curve γ, its tangent vector field can be expressed by

$$T = \sinh\alpha_2\cosh\beta_2 e_1 + \sinh\alpha_2\sinh\beta_2 e_2 + \cosh\alpha_2 e_3, \tag{25}$$

where $\alpha_2 = \alpha_2(s)$ and $\beta_2 = \beta_2(s)$. From (4) we get

$$\begin{aligned}\nabla_T T &= \left(\alpha_2'\cosh\alpha_2\cosh\beta_2 + \sinh\alpha_2\sinh\beta_2(\beta_2' - 2\cosh\alpha_2)\right) e_1 \\ &\quad + \left(\alpha_2'\cosh\alpha_2\sinh\beta_2 + \sinh\alpha_2\cosh\beta_2(\beta_2' - 2\cosh\alpha_2)\right) e_2 \\ &\quad + (\alpha_2'\sinh\alpha_2)e_3 \\ &= k_1\varepsilon_2 N.\end{aligned}$$

Next, we have

$$B_3 = T_1N_2 - T_2N_1 = \frac{\sinh^2\alpha_2(\beta_2' - 2\cosh\alpha_2)}{k_1}\varepsilon_2.$$

Now assume that $B_3 = 0$. If $\sinh\alpha_2 = 0$ then $T = e_3$, that is, γ is a geodesic. So one must have

$$\beta_2' - 2\cosh\alpha_2 = 0.$$

Thus we get

$$\nabla_T T = \alpha_2'(\cosh\alpha_2\cosh\beta_2 e_1 + \cosh\alpha_2\sinh\beta_2 e_2 + \sinh\alpha_2 e_3). \tag{26}$$

Here we can assume that $\alpha_2' \neq 0$ without loss of generality (when $\alpha_2' = 0$ then γ becomes a geodesic again). Then from (26) it follows that

$$k_1^2\varepsilon_2 = g(\nabla_T T, \nabla_T T) = (\alpha_2')^2(\cosh^2\alpha_2 - \sinh^2\alpha_2) = (\alpha_2')^2. \tag{27}$$

If N is timelike then $k_1 = 0$, which is a contradiction again. This completes the proof.

Proposition 3.2: *Let $\gamma: I \to \mathcal{H}_2^3$ be a non-geodesic non-null curve parametrized by arclenght with $B_3 = 0$. Then $k_2^2 = 1$ and γ cannot be biharmonic.*

Proof. Assume that $\gamma: I \to \mathcal{H}_2^3$ is a non-geodesic non-null curve parametrized by arclenght and $\gamma'(s) = T(s)$. If γ is a spacelike curve then from Proposition 3.1 and (24), N must be timelike and $k_1 = \pm\alpha_1' \neq 0$. Using (22), (23), the first Frenet equation and the definition of cross product in $\mathcal{H}_2^3$ it follows that

$$\begin{aligned}N &= \mp(\sinh\alpha_1\cosh\beta_1 e_1 + \sinh\alpha_1\sinh\beta_1 e_2 + \cosh\alpha_1 e_3), \\ B &= T \times N = \pm(\sinh\beta_1 e_1 + \cosh\beta_1 e_2).\end{aligned}$$

From (4) we also have

$$\begin{aligned}\nabla_T N = \mp[&(\alpha_1'\cosh\alpha_1\cosh\beta_1 - \sinh\beta_1)e_1 \\ &+ (\alpha_1'\cosh\alpha_1\sinh\beta_1 - \cosh\beta_1)e_2 + \alpha_1'\sinh\alpha_1 e_3],\end{aligned}$$

which implies that

$$k_2 = g(\nabla_T N, B) = -1.$$

Similarly, if γ is a timelike curve then from Proposition 3.1 and (27), we have N is spacelike and $k_1 = \pm\alpha_2' \neq 0$. Using (26) and the first Frenet equation one obtains

$$N = \pm(\cosh\alpha_2 \cosh\beta_2 e_1 + \cosh\alpha_2 \sinh\beta_2 e_2 + \sinh\alpha_2 e_3),$$
$$B = T \times N = \pm(\sinh\beta_2 e_1 + \cosh\beta_2 e_2).$$

After a straightforward computation we get

$$\begin{aligned}\nabla_T N = \pm[&\left(\alpha_2' \sinh\alpha_2 \cosh\beta_2 + \sinh\beta_2\right) e_1 \\ &+\left(\alpha_2' \sinh\alpha_2 \sinh\beta_2 + \cosh\beta_2\right) e_2 + \alpha_2' \cosh\alpha_2 e_3].\end{aligned}$$

which gives

$$k_2 = g(\nabla_T N, B) = -1.$$

The proof is completed.

Thus we have:

Corollary 3.3: *Let* $\gamma: I \to \mathcal{H}_2^3$ *be a non-geodesic non-null biharmonic helix parametrized by arclenght. Then*

$$\begin{cases} B_3 = \text{constant} \neq 0, \\ k_1^2 \varepsilon_1 \varepsilon_3 + k_2^2 = 1 + 4\varepsilon_3 B_3^2, \\ N_3 = 0. \end{cases} \tag{28}$$

Lemma 3.4: *Let* $\gamma: I \to \mathcal{H}_2^3$ *be a non-geodesic non-null curve parametrized by arclenght. If* $N_3 = 0$ *then*

$$T(s) = \cosh\alpha_0 \cosh\beta(s) e_1 + \cosh\alpha_0 \sinh\beta(s) e_2 + \sinh\alpha_0 e_3 \tag{29}$$

or

$$T(s) = \sinh\vartheta_0 \cosh\rho(s) e_1 + \sinh\vartheta_0 \sinh\rho(s) e_2 + \cosh\vartheta_0 e_3, \tag{30}$$

where α_0, $\vartheta_0 \in R$.

Proof. Let T be the tangent vector field of $\gamma: I \to \mathcal{H}_2^3$ given by $T = T_1 e_1 + T_2 e_2 + T_3 e_3$ and $g(T, T) = \varepsilon_1$. By using (4) we have

$$\begin{aligned}\nabla_T T &= (T_1' - 2T_2 T_3) e_1 + (T_2' - 2T_1 T_3) e_2 + T_3' e_3 \\ &= k_1 \varepsilon_2 \mathrm{N},\end{aligned}$$

which implies that $N_3 = 0$ if and only if $T_3 =$ constant. Then we complete the proof.

Theorem 3.5: *The parametric equations of all non-geodesic spacelike biharmonic curves* γ *of* $\mathcal{H}_2^3$ *are*

$$
\begin{aligned}
x(s) &= \tfrac{1}{a}\cosh\alpha_0\sinh(as+b)+c_1,\\
y(s) &= \tfrac{1}{a}\cosh\alpha_0\cosh(as+b)+c_2,\\
z(s) &= 2\left(\sinh\alpha_0-\tfrac{1}{a}(\cosh\alpha_0)^2\right)s\\
&\quad+\tfrac{2c_1}{a}\cosh\alpha_0\cosh(as+b)-\tfrac{2c_2}{a}\cosh\alpha_0\sinh(as+b)+c_3,
\end{aligned}
\tag{31}
$$

where $a=\sinh\alpha_0\pm\sqrt{5(\sinh\alpha_0)^2+1},\ b,\ c_i\in R,\ (1\le i\le 3)$.

Proof. Assume that $\gamma: I\to\mathcal{H}_2^3$ be a spacelike non-geodesic curve. Then its tangent vector field is given by (29). From Gram-Schmidt procedure we have

$$N(s)=\sinh\beta(s)e_1+\cosh\beta(s)e_2.$$

By taking covariant derivative of the vector field T we get

$$\nabla_T T=\cosh\alpha_0(\beta'-2\sinh\alpha_0)(\sinh\beta\, e_1+\cosh\beta\, e_2)=k_1\varepsilon_2\mathrm{N},$$

where

$$k_1=\left|\cosh\alpha_0(\beta'-2\sinh\alpha_0)\right|. \tag{32}$$

Taking into account the cross product in $\mathcal{H}_2^3$ one obtains

$$
\begin{aligned}
B(s) &= T(s)\times N(s)\\
&= \sinh\alpha_0\cosh\beta(s)e_1+\sinh\alpha_0\sinh\beta(s)e_2+\cosh\alpha_0 e_3.
\end{aligned}
\tag{33}
$$

Moreover,

$$\nabla_T N=\cosh\beta(\beta'-\sinh\alpha_0)e_1+\sinh\beta(\beta'-\sinh\alpha_0)e_2+\cosh\alpha_0 e_3.$$

From the second Frenet equation, it follows that

$$k_2=\sinh\alpha_0(\beta'-2\sinh\alpha_0)-1. \tag{34}$$

Then γ is a spacelike non-geodesic biharmonic curve if and only if

$$\begin{cases}\beta'=\text{constant}\neq 2\sinh\alpha_0,\\ -k_1^2+k_2^2=1-4B_3^2.\end{cases} \tag{35}$$

By substituting (32), (34) and $B_3=\cosh\alpha_0$ in the second equation of (35) we get

$$(\beta')^2-2\beta'(\sinh\alpha_0)-4-4(\sinh\alpha_0)^2=0$$

which gives

$$\beta'=\sinh\alpha_0\pm\sqrt{5(\sinh\alpha_0)^2+1}=a,$$

that is,

$$\beta(s) = as + b, \quad b \in R.$$

To find a differential equation system for the non-geodesic spacelike biharmonic curve $\gamma(s) = (x(s), y(s), z(s))$, by using (3) we first note that

$$\frac{\partial}{\partial x} = e_1 + ye_3, \quad \frac{\partial}{\partial y} = e_2 - xe_3, \quad \frac{\partial}{\partial z} = \frac{1}{2}e_3. \tag{36}$$

Therefore since $T = \frac{d\gamma}{ds}$, we have the following differential equations system

$$\frac{dx}{ds} = \cosh\alpha_0 \cosh(as + b),$$
$$\frac{dy}{ds} = \cosh\alpha_0 \sinh(as + b),$$
$$\frac{dz}{ds} = 2\sinh\alpha_0 + 2\cosh\alpha_0(\sinh(as + b)x(s) - \cosh(as + b)y(s)).$$

Integrating the system gives (31). The proof is completed.

Theorem 3.6: *The parametric equations of all non-geodesic timelike biharmonic curves γ of $\mathcal{H}_2^3$ are*

$$\begin{aligned}
\tilde{x}(s) &= \frac{1}{\tilde{a}}\sinh\vartheta_0 \sinh\left(\tilde{a}s + \tilde{b}\right) + d_1, \\
\tilde{y}(s) &= \frac{1}{\tilde{a}}\sinh\vartheta_0 \cosh\left(\tilde{a}s + \tilde{b}\right) + d_2, \\
\tilde{z}(s) &= 2\left(\cosh\vartheta_0 - \frac{1}{\tilde{a}}(\sinh\vartheta_0)^2\right)s \\
&\quad + \frac{2d_1}{\tilde{a}}\sinh\vartheta_0 \cosh\left(\tilde{a}s + \tilde{b}\right) - \frac{2d_2}{\tilde{a}}\sinh\vartheta_0 \sinh\left(\tilde{a}s + \tilde{b}\right) + d_3,
\end{aligned} \tag{37}$$

where $\tilde{a} = \cosh\vartheta_0 \pm \sqrt{5(\cosh\vartheta_0)^2 - 1}$, $\tilde{b}$, $d_i \in R$, $(1 \le i \le 3)$.

Proof. The tangent vector field of a non-geodesic timelike biharmonic curve $\gamma: I \to \mathcal{H}_2^3$ can be given by (30). From Gram-Schmidt procedure we have

$$N(s) = \sinh\rho(s)e_1 + \cosh\rho(s)e_2,$$

which implies that N is a timelike vector field. If we take the covariant derivative of the tangent vector field T it is easy to see that

$$\begin{aligned}
\nabla_T T &= \sinh\vartheta_0(\rho' - 2\cosh\vartheta_0)(\sinh\rho\, e_1 + \cosh\rho\, e_2) \\
&= k_1 \varepsilon_2 N
\end{aligned}$$

and

$$k_1 = \left|\sinh\vartheta_0(\rho' - 2\cosh\vartheta_0)\right|. \tag{38}$$

Also we have

$$\begin{aligned}
B(s) &= T(s) \times N(s) \\
&= \cosh\vartheta_0 \cosh\rho(s)e_1 + \cosh\vartheta_0 \sinh\rho(s)e_2 + \sinh\vartheta_0 e_3.
\end{aligned} \tag{39}$$

In this case it is obvious that γ is a spacelike vector field. From (4) we get

$$\nabla_T N = \cosh\rho(\rho' - \cosh\vartheta_0)e_1 + \sinh\rho(\rho' - \cosh\vartheta_0)e_2 + \sinh\vartheta_0 e_3.$$

It follows that

$$k_2 = \cosh\vartheta_0(\beta' - 2\cosh\vartheta_0) + 1. \tag{40}$$

Then γ is biharmonic if and only if

$$\begin{cases} \rho' = \text{constant} \neq 2\cosh\vartheta_0, \\ -k_1^2 + k_2^2 = 1 + 4B_3^2. \end{cases} \tag{41}$$

Using (38), (40) and $B_3 = \sinh\vartheta_0$ in the second equation of (41) we get

$$(\rho')^2 - 2\rho'(\cosh\vartheta_0) + 4 - 4(\cosh\vartheta_0)^2 = 0$$

which gives

$$\rho' = \cosh\vartheta_0 \pm \sqrt{5(\cosh\vartheta_0)^2 - 1} = \tilde{a},$$

that is,

$$\rho(s) = \tilde{a}s + \tilde{b}, \quad \tilde{b} \in R.$$

Since $T = \frac{d\gamma}{ds}$, from (36), the differential equations system for the non-geodesic timelike biharmonic curve $\gamma(s) = (\tilde{x}(s), \tilde{y}(s), \tilde{z}(s))$ is the following

$$\frac{d\tilde{x}}{ds} = \sinh\vartheta_0 \cosh(\tilde{a}s + \tilde{b}),$$
$$\frac{d\tilde{y}}{ds} = \sinh\vartheta_0 \cosh(\tilde{a}s + \tilde{b}),$$
$$\frac{d\tilde{z}}{ds} = 2\cosh\vartheta_0 + 2\sinh\vartheta_0\left(\sinh(\tilde{a}s + \tilde{b})\tilde{x}(s) - \cosh(\tilde{a}s + \tilde{b})\tilde{y}(s)\right).$$

If we integrate the above system we obtain (37).

From Theorem 3.5 and Theorem 3.6 we also have

Corollary 3.7: *Let $\gamma: I \to \mathcal{H}_2^3$ be a non-geodesic non-null curve parametrized by arclenght with $N_3 = 0$. Then we have $\varepsilon_1 = -\varepsilon_3$ and N is a timelike vector field, where $\varepsilon_1 = g(T,T)$ and $\varepsilon_3 = g(B,B)$.*

4 Horizontal Biharmonic curves in 3-dimensional Heisenberg group

Let $(x,y) \to H_{(x,y)}$ be a non-integrable two dimensional distribution in $R^3 = R^2_{(x,y)} \times R_z$ defined by $H = \ker w$, where w is a 1-form on R^3. The distribution H is said to be the horizontal distribution. A curve $\gamma \to \gamma(s)$, $\gamma(s) = (x(s), y(s), z(s))$ is called horizontal curve if $\gamma'(s) \in H_{\gamma(s)}$, for all s. By using (36), for a non-null curve γ in 3-dimensional Heisenberg group we can write

$$\gamma'(s) = x'(s)\frac{\partial}{\partial x} + y'(s)\frac{\partial}{\partial y} + z'(s)\frac{\partial}{\partial z} = x'(s)e_1 + y'(s)e_2 + w(\gamma'(s))\frac{\partial}{\partial z}. \tag{42}$$

Then γ is a horizontal curve if

$$\gamma'(s) = x'(s)e_1 + y'(s)e_2, \tag{43}$$

$$w(\gamma'(s)) = z'(s) + 2x'(s)y(s) - 2x(s)y'(s). \tag{44}$$

Theorem 4.1: *The parametric equations of all non-geodesic spacelike horizontal biharmonic curves γ in $\mathcal{H}_2^3$ are*

$$\begin{aligned} x(s) &= \pm\sinh(\pm s + b) + c_1, \\ y(s) &= \pm\cosh(\pm s + b) + c_2, \\ z(s) &= \mp 2s \pm 2c_1\cosh(\pm s + b) \mp 2c_2\sinh(\pm s + b) + c_3, \end{aligned} \tag{45}$$

where $b,\ c_i \in R,\ (1 \le i \le 3)$.

Proof. Let $\gamma: I \to \mathcal{H}_2^3$ be a non-geodesic spacelike horizontal biharmonic curve. Since the tangent vector field of γ can be written as $T = T_1e_1 + T_2e_2 + T_3e_3$ then from (29) and (43) we have

$$T_3 = \sinh\alpha_0 = 0. \tag{46}$$

By using the last equation in (31) we complete the proof.

Theorem 4.2: *There does not exist a non-geodesic timelike horizontal biharmonic curve in $\mathcal{H}_2^3$.*

Proof. Assume that $\gamma: I \to \mathcal{H}_2^3$ is a non-geodesic timelike horizontal biharmonic curve. Then we have $N_3 = 0$ and $T_3 = 0$. Since γ is a timelike curve then Corollary 3.7 implies that N is a timelike and B is a spacelike vector field. Using (4) we have

$$g(\nabla_T T, e_3) = T_3', \quad g(\nabla_T N, e_3) = N_3' - T_2N_1 + T_1N_2, \quad g(\nabla_T B, e_3) = B_3' - T_2B_1 + T_1B_2. \tag{47}$$

On the other hand from the Frenet formulas one can easily see that

$$g(\nabla_T T, e_3) = -k_1N_3, \quad g(\nabla_T N, e_3) = k_1T_3 + k_2B_3, \quad g(\nabla_T B, e_3) = -k_2N_3. \tag{48}$$

It follows from the definition of the cross product in $\mathcal{H}_2^3$, (47) and (48) that

$$k_2 = 1.$$

Substituting the last equation in (19) we get

$$-k_1^2 = 4B_3^2,$$

which is a contradiction. The proof is completed.

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