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VARIOUS CHARACTERIZATIONS FOR QUATERNIONIC MANNHEIM CURVES IN THREE-DIMENSIONAL EUCLIDEAN SPACE

AYKUT HAS* AND BEYHAN YILMAZ

Abstract. In this article, quaternionic curves in 3-dimensional Euclidean space have examined. Firstly, algebraic properties of quaternions and their basic definitions and theorems are given. Later, some characterizations of the quaternionic Mannheim curves in the 3-dimensional Euclidean space have obtained.

1. Introduction

Quaternions discovered in 1843 during the study of Irish mathematician William Rowan Hamilton to generalize complex numbers into $3-$dimensional space. Real quaternions consist of a combination of two complex numbers. Accordingly, as complex numbers are a subset of quaternions, it is understood that quaternions are a larger number system that includes both real numbers and complex numbers. Quaternions are increasingly used in every field in recent years. Also, quaternions play an important role in various fields such as mechanics, kinematics, and quantum mechanics. On the other hand, these numbers are related to rotational transformation in orthogonal and unit symmetry groups. Quaternions also provide the possibility to represent finite rotations in space, [5, 10, 12].

Since Hamilton, quaternions have been studied by different authors. Every element in real quaternions Q can be expressed as $ae_1 + be_2 + ce_3 + de_4$, where a, b, c and d are real numbers and e_1, e_2, e_3 and e_4 can be taken as the base vector in $4-$dimensional space, [1]. The Serret-Frenet formulas for quaternionic curves in $\mathbb{E}^3$ and $\mathbb{E}^4$ defined by Bharathi and Nagaraj in 1987, [2]. Afterwards, Çöken and Tuna defined these formulas for a quaternionic curve in semi-Euclidean space $\mathbb{E}^4_2$. Also they introduced the definitions quaternionic inclined curves and harmonic curvatures, [3, 4]. In $4-$dimensional Euclidean space the quaternionic Mannheim curves studied by Okuyucu in 2013, [7].

One of the most studied topics in differential geometry is the theory of curves. In the theory of curves, especially geodesics, circles, general helixes, slant helix and

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special curves such as rectifying curves are studied. One of the problems most researched in Euclidean space is the characterization of a regular curve. One of the approaches used in the solution of this problem is to determine the characterizations of these curves by considering the relationship between the Frenet vectors of the two curves. For example, if the principal normal vectors of two curves coincide, this pair of curves is called the Bertrand curve pair. If the tangents of two curves are perpendicular to each other, these curves are called involute-evolute curve pairs. If the normal vector of one of the two curves coincides with the binormal vector of the other, this pair of curves is called the Mannheim curve pair.

Mannheim curves discovered by Mannheim in 1878, [13]. In 1966, with the help of Riccati equations some theorems given regarding Mannheim curves, [8]. In recent years, Liu and Wang have studied the characterization of Mannheim curve pairs in $3-$dimensional Euclidean space, [6]. In addition, studies have been carried out reveal some characteristic features in the Minkowski space, [9, 11].

In this present paper, we firstly prove the theorems related to the Mannheim curve pair in quaternionic form .Then, we obtain some characterizations of spatial quaternionic Mannheim curves with the aid of Serret-Frenet formulas.

2. Preliminaries

In this section, basic definitions and theorems on quaternions, spatial quaternionic curves and quaternionic curves will be given.

Let Q_H be a vector space with characteristic greater than 2 on the H field. In general, the real quaternion form is as follows, where q is a real quaternion.

$$q = ae_1 + be_2 + ce_3 + de_4$$

where a, b, c, d are real numbers and e_i, $(1 \leq i \leq 4)$ are quaternionic units which satisfy the non-commutative multiplication rules

$$e_4 = 1, e_1^2 = e_2^2 = e_3^2 = -1, \quad e_1e_2 = -e_2e_1 = e_3,$$

$$e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2.$$

If we denote $Sq = d$ and $Vq = ae_1 + be_2 + ce_3$, we can rewrite a real quaternion is $q = Sq+Vq$ where Sq and V_q are the scalar part and vectorial part of q, respectively. So, we can show the product of two quaternions as:

$$p \times q = SpSq - \langle Vp, Vq\rangle + SpVq + SqVp + Vp \wedge Vq$$

where $\langle , \rangle$ and $\wedge$ are inner product and cross product in $\mathbb{E}^3$, respectively. The conjugate of q denoted by γq and defined as:

$$\gamma q = -ae_1 - be_2 - ce_3 + de_4$$

which is called the "Hamilton conjugation". This defines bilinear form h as follows

$$h(p, q) = \frac{1}{2}[p \times \gamma q + q \times \gamma q]$$

which is called the quaternion inner product. The norm of q is given by

$$\|q\|^2 = h(q, q) = q \times \gamma q = \gamma q \times q = a^2 + b^2 + c^2 + d^2.$$

If $\|q\|^2 = 1$, then q is called unit quaternion. Also, q is called a spatial quaternion whenever $q + \gamma q = 0$ and called a temporal quaternion whenever $q - \gamma q = 0$.

Theorem 2.1. *Space of spatial quaternions in three dimensional Euclidean space, it is clearly is identified as* $\{p \in Q_H \mid p + \gamma p = 0\}$. *Let* $I = [0,1]$ *indicate the unit spacing in* $\mathbb{R}$. *Let*

$$\begin{aligned} \alpha \quad &: \quad I \subset \mathbb{R} \longrightarrow Q_H \\ s \quad &\longrightarrow \quad \alpha(s) : \sum_{i=1}^{3} \alpha_i(s) e_i \quad (1 \leq i \leq 3) \end{aligned}$$

be a curve with $\{k(s), r(s)\}$ *and* $\{t(s), n(s), b(s)\}$ *denote the Frenet frame of* α. *Then,*

$$\begin{pmatrix} t'(s) \\ n'(s) \\ b'(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & r(s) \\ 0 & -r(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}. \tag{2.1}$$

Theorem 2.2. *Let* $I = [0,1]$ *indicate the unit spacing in* $\mathbb{R}$ *and*

$$\begin{aligned} \alpha \quad &: \quad I \subset \mathbb{R} \longrightarrow Q_H \\ s \quad &\longrightarrow \quad \alpha(s) : \sum_{i=1}^{4} \alpha_i(s) e_i \quad (1 \leq i \leq 4), \end{aligned}$$

be a curve in $\mathbb{E}^4$ *with* $\{K(s), k(s), (r-K)(s)\}$ *and* $\{T(s), N(s), B_1(s), B_2(s)\}$ *denotes the Frenet frame of the curve. Then,*

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B_1'(s) \\ B_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & K(s) & 0 & 0 \\ -K(s) & 0 & k(s) & 0 \\ 0 & -k(s) & 0 & (r-K)(s) \\ 0 & 0 & -(r-K)(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B_1(s) \\ B_2(s) \end{bmatrix}. \tag{2.2}$$

Definition 2.3. Let $\alpha(s)$ and $\beta(s^*)$ be two spatial quaternionic curves in $\mathbb{E}^3$. $\{t(s), n(s), b(s)\}$ and $\{t^*(s^*), n^*(s^*), b^*(s^*)\}$ are the Frenet frames on these curves, respectively. α and β are called spatial quaternionic Mannheim curves if $n(s)$ and $b^*(s^*)$ are linearly dependent, [14].

Definition 2.4. A quaternionic curve $\alpha^{(4)} : I \subset \mathbb{R} \longrightarrow \mathbb{E}^4$ is a quaternionic Mannheim curves if there exist a $\beta^{(4)} : I \subset \mathbb{R} \longrightarrow \mathbb{E}^4$ such that the second Frenet vector at each point of $\alpha^{(4)}$ is included the plane generated by the third Frenet vector and the fourth Frenet vector of $\beta^{(4)}$ at coresponding point under φ where φ is a bijection from $\alpha^{(4)}$ to $\beta^{(4)}$. The curve $\beta^{(4)}$ is called the quaternionic Mannheim partner curve of $\alpha^{(4)}$, [7].

Definition 2.5. Let $\alpha(s)$ and $\beta(s^*)$ be quaternionic curves in $\mathbb{E}^3$ with the Frenet frames $\{t(s), n(s), b(s)\}$ and $\{t^*(s^*), n^*(s^*), b^*(s^*)\}$ according to arc-length parameter s and s^*, respectively. If the pair $\{\alpha, \beta\}$ are a Mannheim pair $n(s)$ and $b^*(s^*)$ are linearly dependent. So, we can write

$$\alpha(s) = \beta(s^*) + \lambda_1 b^*(s^*), \tag{2.3}$$

and

$$\beta(s^*) = \alpha(s) + \lambda_2 n(s^*). \tag{2.4}$$

Corollary 2.6. Let $\alpha(s)$ and $\beta(s^*)$ be quaternionic curves in $\mathbb{E}^3$ with the Frenet frames $\{t(s), n(s), b(s)\}$ and $\{t^*(s^*), n^*(s^*), b^*(s^*)\}$ according to arc-length parameter s and s^*, respectively. If the pair $\{\alpha, \beta\}$ are a Mannheim pair, we can write the following results

$$\begin{aligned} t^* &= \cos\theta t - \sin\theta b, \\ n^* &= \sin\theta t + \cos\theta b, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} t &= \cos\theta t^* + \sin\theta n^*, \\ b &= -\sin\theta t^* + \cos\theta n^*. \end{aligned} \tag{2.6}$$

3. Main Results

Theorem 3.1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a spatial quaternionic Mannheim curves in three-dimensional Euclidean space $\mathbb{E}^3$ with arc-length parameter $s \in [0, 1]$ and $\beta(s^*)$ be the spatial Mannheim curve pair of $\alpha(s)$ with the arc-length parameter s^*. Then the distance between corresponding points are a fixed distance for each $s \in I$.*

Proof. Suppose that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a spatial quaternionic curve. Taking the derivative of equation (2.3) according to s^* and apply Frenet formulas following equations, we have

$$\begin{aligned} \frac{d\alpha}{ds}\frac{ds}{ds^*} &= \beta'(s^*) + \lambda_1' b^*(s^*) + \lambda_1 (b^*)'(s^*), \\ t\frac{ds}{ds^*} &= t^* + \lambda_1' b^* - \lambda_1 r^* n^*. \end{aligned}$$

If both sides of the equality are made the quaternionic inner product of the n, we get

$$\frac{ds}{ds^*} h(t, n) = h((t^* + \lambda_1' b^* - \lambda_1 r^* n^*), n). \tag{3.1}$$

If the left side of the above equation is calculated, we have

$$\begin{aligned} \frac{ds}{ds^*} h(t, n) &= \frac{1}{2}\frac{ds}{ds^*}\left[t \times \gamma n + n \times \gamma t\right] \\ &= 0. \end{aligned} \tag{3.2}$$

On the other hand, right side of equation (3.1) as follows

$$h(t^* + \lambda_1' b^* - \lambda_1 r^* n^*, n) = \frac{1}{2}[(t^* + \lambda_1' b^* - \lambda_1 r^* n^*) \times \gamma n + n \times \gamma (t^* + \lambda_1' b^* - \lambda_1 r^* n^*)]$$

$$\begin{aligned} &= \frac{1}{2}[-(t^* \times n) - \lambda_1'(b^* \times n) + \lambda_1 r^*(n^* \times n) - (n \times t^*) - \lambda_1'(n \times b^*) + \lambda_1 r^*(n \times n^*)] \\ &= \frac{1}{2}[n^* + \lambda_1' + \lambda_1 r^* t^* - n^* + \lambda_1' - \lambda_1 r^* t^*] \\ &= \lambda_1'. \end{aligned} \tag{3.3}$$

From the equations (3.2) and (3.3) we can easily see that

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_1' &= c, \ c \in \mathbb{R}. \end{aligned}$$

$\square$

Theorem 3.2. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a spatial quaternionic Mannheim curves in $\mathbb{E}^3$ with $s \in [0,1]$ and $\beta(s^*)$ be the spatial Mannheim curve pair of $\alpha(s)$ with s^*. Suppose that k, r and k^*, r^* be a curvature and torsions of $\alpha(s)$ and $\beta(s^*)$, respectively. In this case, the following equations are available*

$$\begin{aligned} k &= r^* \sin\theta \frac{ds^*}{ds}, \\ r &= -r^* \cos\theta \frac{ds^*}{ds}. \end{aligned}$$

Proof. Suppose that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a spatial quaternionic curve. By definition of curvature, we have

$$k = h(t', n) \text{ and } r = h(n', b).$$

If the equation (2.3) is used in the first equation given above, we get

$$k = h((\cos\theta t^* + \sin\theta n^*)', n),$$

$$\begin{aligned} k &= h(-\sin\theta \frac{d\theta}{ds} t^* + \cos\theta k^* n^* \frac{ds^*}{ds} + \cos\theta \frac{d\theta}{ds} n^* + \sin\theta(-k^* t^* + r^* b^*)\frac{ds^*}{ds}), b^*) \\ &= \frac{1}{2}[(\sin\theta \frac{d\theta}{ds}(t^* \times b^*) - k^* \cos\theta \frac{ds^*}{ds}(n^* \times b^*) - \cos\theta \frac{d\theta}{ds}(n^* \times b^*) \\ &\quad + k^* \sin\theta \frac{ds^*}{ds}(t^* \times b^*) - r^* \sin\theta \frac{ds^*}{ds}(b^* \times b^*) \\ &\quad + \sin\theta \frac{d\theta}{ds}(b^* \times t^*) - k^* \cos\theta \frac{ds^*}{ds}(b^* \times n^*) - \cos\theta \frac{d\theta}{ds}(b^* \times n^*) \\ &\quad + k^* \sin\theta \frac{ds^*}{ds}(b^* \times t^*) - r^* \sin\theta \frac{ds^*}{ds}(b^* \times b^*)]. \end{aligned}$$

As a result of this equation, we can write

$$k = r^* \sin\theta \frac{ds^*}{ds}.$$

On the other hand, if equation (2.3) is used in the torsion equation, we obtain

$$r = h(-r^* n^* \frac{ds^*}{ds}, (-\sin\theta t^* + \cos\theta n^*)),$$

$$\begin{aligned} r &= \frac{1}{2}[(-r^* n^* \frac{ds^*}{ds}) \times \gamma(-\sin\theta t^* + \cos\theta n^*) + (-\sin\theta t^* + \cos\theta n^*) \times (-r^* n^* \frac{ds^*}{ds})] \\ &= \frac{1}{2}[-r^* \sin\theta \frac{ds^*}{ds}(n^* \times t^*) + r^* \cos\theta \frac{ds^*}{ds}(n^* \times n^*) \\ &\quad - r^* \sin\theta \frac{ds^*}{ds}(t^* \times n^*) + r^* \cos\theta \frac{ds^*}{ds}(n^* \times n^*)]. \end{aligned}$$

If necessary arrangements are made, we can write

$$r = -r^* \cos\theta \frac{ds^*}{ds}.$$

□

Theorem 3.3. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a spatial quaternionic Mannheim curves in $\mathbb{E}^3$ with $s \in [0,1]$ and $\beta(s^*)$ be the spatial Mannheim curve pair of $\alpha(s)$ with s^*. Suppose that k, r and k^*, r^* be a curvature and torsions of $\alpha(s)$ and $\beta(s^*)$, respectively.*

Following equations are available

$$\begin{aligned} k^* &= -\frac{d\theta}{ds}, \\ r^* &= (k\sin\theta - r\cos\theta)\frac{ds}{ds^*}. \end{aligned}$$

Proof. Suppose that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a spatial quaternionic curve. By definition of curvature, we get

$$k^* = h((\cos\theta t - \sin\theta b)', n^*)$$

$$\begin{aligned} k^* &= h((-\sin\theta\frac{d\theta}{ds}t + kn\cos\theta\frac{ds}{ds^*} - \cos\theta\frac{d\theta}{ds}b + rn\sin\theta\frac{ds}{ds^*}), (\sin\theta t + \cos\theta b)) \\ &= \frac{1}{2}[(\sin^2\theta\frac{d\theta}{ds}(t\times t) - k\sin\theta\cos\theta\frac{ds}{ds^*}(n\times t) + \sin\theta\cos\theta\frac{d\theta}{ds}(b\times t) - r\sin^2\theta\frac{ds}{ds^*}(n\times t) \\ &\quad + \sin\theta\cos\theta\frac{d\theta}{ds}(t\times b) - k\cos^2\theta\frac{ds}{ds^*}(n\times b) + \cos^2\theta\frac{d\theta}{ds}(b\times b) - r\sin\theta\cos\theta\frac{ds}{ds^*}(n\times b) \\ &\quad + \sin^2\theta\frac{d\theta}{ds}(t\times t) - k\sin\theta\cos\theta\frac{ds}{ds^*}(t\times n) + \sin\theta\cos\theta\frac{d\theta}{ds}(t\times b) - r\sin^2\theta\frac{ds}{ds^*}(t\times n) \\ &\quad + \sin\theta\cos\theta\frac{d\theta}{ds}(b\times t) - k\cos^2\theta\frac{ds}{ds^*}(b\times n) + \cos^2\theta\frac{d\theta}{ds}(b\times b) - r\sin\theta\cos\theta\frac{ds}{ds^*}(b\times n)]. \end{aligned}$$

As a result of this equation, we can write

$$k^* = -\frac{d\theta}{ds}.$$

On the other hand, using the definition of torsion we obtain

$$\begin{aligned} r^* &= h((\sin\theta t + \cos\theta b)', n) \\ r^* &= h((\cos\theta\frac{d\theta}{ds}t + kn\sin\theta\frac{ds}{ds^*} - \sin\theta\frac{d\theta}{ds}b - rn\cos\theta\frac{ds}{ds^*}), n) \\ &= \frac{1}{2}[-\cos\theta\frac{d\theta}{ds}(t\times n) - k\sin\theta\frac{ds}{ds^*}(n\times n) + \sin\theta\frac{d\theta}{ds}(b\times n) + r\cos\theta\frac{ds}{ds^*}(n\times n) \\ &\quad - \cos\theta\frac{d\theta}{ds}(n\times t) - k\sin\theta\frac{ds}{ds^*}(n\times n) + \sin\theta\frac{d\theta}{ds}(n\times b) + r\cos\theta\frac{ds}{ds^*}(n\times n)]. \end{aligned}$$

If necessary arrangements are made, we can easily see that

$$r = (k\sin\theta - r\cos\theta)\frac{ds}{ds^*}.$$

□

Theorem 3.4. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a spatial quaternionic Mannheim curves in $\mathbb{E}^3$ with $s \in [0,1]$ and $\beta(s^*)$ be the spatial Mannheim curve pair of $\alpha(s)$ with s^*. Suppose that k, r be a curvature and torsions of $\alpha(s)$. Then the following relation exists.*

$$\lambda_2 = \frac{k^2 + r^2}{k}$$

Proof. Suppose that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a spatial quaternionic curve. Taking the derivative of equation (2.4) according to s^* and apply Frenet formulas following equations

$$\begin{aligned}\frac{d\beta^*(s^*)}{ds^*} &= \frac{d\alpha(s)}{ds}\frac{ds}{ds^*} + \lambda_2' n(s) + \lambda \frac{dn(s)}{ds}\frac{ds}{ds^*},\\ t^* &= t\frac{ds}{ds^*} + \lambda_2' n + \lambda_2(-kt + rb)\frac{ds}{ds^*}.\end{aligned}$$

If equation (2.5) is used in the above equation, we get

$$\cos\theta t - \sin\theta b = (1-\lambda_2 k)t\frac{ds}{ds^*} + \lambda_2' n + \lambda_2 rb\frac{ds}{ds^*}. \tag{3.4}$$

If both sides of the equalities are made with the quaternionic inner product of the t, we obtain

$$h(\cos\theta t - \sin\theta b, t) = h((1-\lambda_2 k)t\frac{ds}{ds^*} + \lambda_2' n + \lambda_2 rb\frac{ds}{ds^*}), t).$$

Firstly, if the right side of the above equation is arranged, we can see

$$\begin{aligned}h((1-\lambda_2 k)t\frac{ds}{ds^*} + \lambda_2' n + \lambda_2 rb\frac{ds}{ds^*}), t) &= \frac{1}{2}[-(1-\lambda_2 k)(t\times t)\frac{ds}{ds^*} - \lambda_2'(n\times t) - \lambda_2 r(b\times t)\frac{ds}{ds^*}\\ &\quad -(1-\lambda_2 k)(t\times t)\frac{ds}{ds^*} - \lambda_2'(t\times n) - \lambda_2 r(b\times t)\frac{ds}{ds^*}]\\ &= (1-\lambda_2 k)\frac{ds}{ds^*}. \end{aligned}\tag{3.5}$$

Also, if the left side of the same equation is to be arranged

$$\begin{aligned}h(\cos\theta t - \sin\theta b, t) &= \frac{1}{2}[-\cos\theta(t\times t) + \sin\theta(b\times t) - \cos\theta(t\times t) + \sin\theta(t\times b)]\\ &= \cos\theta.\end{aligned}\tag{3.6}$$

From the equations (3.5) and (3.6), we can write

$$\cos\theta = (1-\lambda_2 k)\frac{ds}{ds^*}. \tag{3.7}$$

On the other hand, if both sides of the equation (3.4) are made with the quaternionic inner product of the b, we get

$$h(\cos\theta t - \sin\theta b, b) = h((1-\lambda_2 k)t\frac{ds}{ds^*} + \lambda_2' n + \lambda_2 rb\frac{ds}{ds^*}), b). \tag{3.8}$$

If the right side of the above equation is arranged,

$$\begin{aligned}h((1-\lambda_2 k)t\frac{ds}{ds^*} + \lambda_2' n + \lambda_2 rb\frac{ds}{ds^*}), b) &= \frac{1}{2}[-(1-\lambda_2 k)(t\times b)\frac{ds}{ds^*} - \lambda_2'(n\times b) - \lambda_2 r(b\times b)\frac{ds}{ds^*}\\ &\quad -(1-\lambda_2 k)(b\times t)\frac{ds}{ds^*} - \lambda_2'(b\times n) - \lambda_2 r(b\times b)\frac{ds}{ds^*}]\\ &= \lambda_2 r\frac{ds}{ds^*}. \end{aligned}\tag{3.9}$$

and if the left side of the same equation is to be arranged, we can see

$$\begin{aligned}h(\cos\theta t - \sin\theta b, b) &= \frac{1}{2}[-\cos\theta(t\times b) + \sin\theta(b\times b) - \cos\theta(b\times t) + \sin\theta(b\times b)]\\ &= -\sin\theta.\end{aligned}\tag{3.10}$$

Using the equations (3.9) and (3.10), we have

$$\sin\theta = -\lambda_2 r\frac{ds}{ds^*}. \tag{3.11}$$

Also, from the equations (3.7) and (3.11) we obtained

$$\cot\theta = \frac{\lambda_2 k - 1}{\lambda_2 r}. \tag{3.12}$$

On the other hand, as seen clearly in Theorem 4,

$$\cot\theta = -\frac{r}{k} \tag{3.13}$$

So, from the equations (3.12) and (3.13), we get

$$\lambda_2 = \frac{k}{k^2 + r^2}.$$

□

Theorem 3.5. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a spatial quaternionic Mannheim curves in $\mathbb{E}^3$ with $s \in [0,1]$ and $\beta(s^*)$ be the spatial Mannheim curve pair of $\alpha(s)$ with s^*. Suppose that k, r and k^*, r^* be a curvature and torsions of $\alpha(s)$ and $\beta(s^*)$, respectively. Following equation is available*

$$r\lambda_1 - r^*\lambda_2 = 0.$$

Proof. Suppose that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a spatial quaternionic curve. Taking the derivative of equation (2.4) according to s and apply Frenet formulas following equations

$$\begin{aligned} \frac{d\beta}{ds^*}\frac{ds^*}{ds} &= \alpha'(s) + \lambda_2' n(s) + \lambda_2 n'(s), \\ t^*\frac{ds^*}{ds} &= (1 - \lambda_2 k)t + \lambda_2' n + \lambda_2 r b. \end{aligned}$$

If we use the equation (2.6) in the last equation, we obtain

$$t^*\frac{ds^*}{ds} = (\cos\theta - \lambda_2 k\cos\theta - \lambda_2 r\sin\theta)t^* + (\sin\theta - \lambda_2 k\sin\theta + \lambda_2 r\cos\theta)n^* + \lambda_2' b^*.$$

On the other hand, if quaternionic inner product with n^* is made on both sides of the equation we have obtained in the last equation above, we get

$$\sin\theta = \lambda_2(k\sin\theta - r\cos\theta).$$

Also, using the result of Theorem 4 in the last equation, we have

$$\begin{aligned} \frac{k}{r^*} &= \lambda_2 r^*, \\ \lambda_2 &= \frac{k}{(r^*)^2}. \end{aligned} \tag{3.14}$$

Taking the derivative of equation (2.3) according to s^*,

$$\begin{aligned} \frac{d\alpha}{ds}\frac{ds}{ds^*} &= \beta'(s^*) + \lambda_1' b^*(s^*) + \lambda_1 (b^*)'(s^*), \\ t\frac{ds}{ds^*} &= t^* + \lambda_1' b^* - \lambda_1 r^* n^*. \end{aligned}$$

If we use the equation (2.5) in the last equation obtained above, we have

$$t\frac{ds}{ds^*} = (\cos\theta - \lambda_1 r^*\sin\theta)t + \lambda_1' n + (-\sin\theta - \lambda_1 r^*\cos\theta)b.$$

Additionally, if both sides in the last equation we obtained above are made with the quaternionic inner product of the b, we get

$$\begin{aligned} \sin\theta &= -\lambda_1 r^* \cos\theta, \\ \sin\theta \frac{ds^*}{ds} &= -\lambda_1 r^* \cos\theta \frac{ds^*}{ds}. \end{aligned}$$

Similarly, if we use the result of Theorem 4 in the last equation we obtained, we can see

$$\begin{aligned} \frac{k}{r^*} &= \lambda_1 r, \\ \lambda_1 &= \frac{k}{r^* r}. \end{aligned} \tag{3.15}$$

Finally, if we use the equations (3.14) and (3.15), we obtain

$$\frac{\lambda_1}{\lambda_2} = \frac{r^*}{r},$$

and then

$$\lambda_1 r - \lambda_2 r^* = 0.$$

□

Theorem 3.6. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a spatial quaternionic Mannheim curves in $\mathbb{E}^3$ with $s \in [0,1]$ and $\beta(s^*)$ be the spatial Mannheim curve pair of $\alpha(s)$ with s^*. Then the following relation exists.*

$$\lambda_1 + \lambda_2 = 0$$

Proof. Suppose that $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a spatial quaternionic curve. Taking the derivative of equation (2.3) according to s^* and apply Frenet formulas following equations

$$\begin{aligned} \frac{d\alpha}{ds}\frac{ds}{ds^*} &= \beta'(s^*) + \lambda_1' b^*(s^*) + \lambda_1 (b^*)'(s^*), \\ t\frac{ds}{ds^*} &= t^* + \lambda_1' b^* - \lambda_1 r^* n^* . \end{aligned}$$

If we use the equation (2.5) in the last equation obtained above, we get

$$t\frac{ds}{ds^*} = (\cos\theta t - \sin\theta b) + \lambda_1' n - \lambda_1 r^* (\sin\theta t + \cos\theta b).$$

If both sides in the last equation obtained above are made with the quaternionic inner product of t, we obtain

$$\frac{ds}{ds^*} = \cos\theta - \lambda_1 r^* \sin\theta.$$

If we multiply both sides of this equation by $\dfrac{ds^*}{ds}$, we can write

$$1 = \cos\theta \frac{ds^*}{ds} - \lambda_1 r^* \sin\theta \frac{ds^*}{ds}$$

and

$$\lambda_1 = -\frac{r^* + r}{kr^*}. \tag{3.16}$$

On the other hand taking the derivative of equation (2.4) according to s and apply Frenet formulas following equations

$$\begin{aligned} \frac{d\beta}{ds^*}\frac{ds^*}{ds} &= \alpha'(s) + \lambda_2' n(s) + \lambda_2 n'(s), \\ t^*\frac{ds^*}{ds} &= (1-\lambda_2 k)t + \lambda_2' n + \lambda_2 r b. \end{aligned}$$

If we use the equation (2.5) in the last equation obtained above, we get

$$(\cos\theta t - \sin\theta b)\frac{ds^*}{ds} = (1-\lambda_2 k)t + \lambda_2' n + \lambda_2 r b.$$

Also, if both sides in the last equation obtained above are made with the quaternionic inner product of the t, we have

$$\cos\theta\frac{ds^*}{ds} = 1 - \lambda_2 k.$$

Using the result of Theorem 4 in the last equation obtained, we can easily see that

$$\lambda_2 = \frac{r^* + r}{kr^*}. \tag{3.17}$$

Finally, if we use the equations (3.16) and (3.17), we obtain

$$\lambda_1 + \lambda_2 = 0.$$

□

4. CONCLUSION

In this paper, some characterizations of the spatial Mannheim curve pair in 3-dimensional Euclidean space have been obtained. While obtaining these characterizations, quaternionic properties are used. It is seen here that the characterizations obtained by Euclidean inner product and the characterizations obtained by quaternionic inner product are identical.

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Department of Mathematics, Kahramanmaraş Sütcü İmam University, Kahramanmaraş, Turkey
Email address: aykuthas@outlook.com,beyhanyilmaz@ksu.edu.tr