# ON CHARACTERIZATIONS OF SOME SPECIAL CURVES OF SPACELIKE CURVES ACCORDING TO THE TYPE-2 BISHOP FRAME IN MINKOWSKI 3-SPACE

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**Abstract:**

In this paper, first we give a characterization of spacelike inclined curves according to the type-2 Bishop frame in Minkowski 3-space, and then define rectifying curves of spacelike curves according to the type-2 Bishop frame in Minkowski 3-space as their position vectors always lie in the orthogonal complement $\Omega^{\perp}{}_2$ of their vector field $\Omega_2$. Moreover we characterize Bertrand curves in the same space via the new frame. In particular, we study Mannheim partner curves according to type-2 Bishop frame in $E_1^3$ and express such curves in terms of their curvature functions.

**Key Words**: Rectifying curve, Mannheim curve, Bertrand curve, type-2 Bishop frame, inclined curve, Minkowski 3-space.

**Mathematics Subject Classification:** 53A05, 53B25, 53B30

**Özet:**

Biz, bu makalede, öncelikle 3-boyutlu Minkowski uzayında 2. Tip Bishop çatısına göre spacelike inclined eğrilerinin karakterizasyonunu verip, ayrıca konum vektörleri daima, $\Omega_2$ vektör alanının dik tümleyeni olan $\Omega_2{}^{\perp}$ de yatan rektifyan eğrilerini tanımlıyoruz. Buna ek olarak, aynı uzay ve çatı kullanılarak Bertrand eğrileri karakterize edilir. Özellikle, Mannheim eğrileri araştırılır ve eğrilik fonksiyonları cinsinden bu tip eğriler ifade edilir.

## 1 INTRODUCTION

One of the fundamental structures of differential geometry is curves. It is safe to report that the many important results in the theory of the curves in $E^3$ were initiated by G. Monge; and G. Darboux pionnered the moving frame idea.

At the beginning of the twentieth century, A.Einstein's theory opened a door of use of new geometries. One of them, Minkowski space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold, see [19], [20].

One of the interesting problems is the problem of characterization of a regular curve in the theory of curves in the Euclidean and Minkowski spaces, see, [1], [3], [4]. In the solution of the problem, the curvature functions of a regular curve have an effective role. It is known that we can determine the shape and size of a regular curve by using its curvatures and . Another approach to the solution of the problem is to consider the relationship between the corresponding Frenet vectors of two curves, see, [8], [10], [11], [12]. For instance, Bertrand curves and Mannheim curves arise from this relationship, see, [5], [14], [17]. They adapted the geometrical models to relativistic motion of charged particles, see, [6].

Bishop Frame which is also called alternative or parallel frame of the curves was introduced by L.R. Bishop in 1975 by means of parallel vector fields. Recently many researches related to this subject have been made in Euclidean space, see [13],[16], Minkowski spaces see [2], and in dual space, see [15].

Bishop and Serret-Frenet frames have a common vector field, i.e., the tangent vector field of Serret-Frenet frame. The construction of the Bishop frame has some advantages when comparing with the Frenet frame in Euclidean 3-space, see, [13]. That is why he defined this frame that curvature may vanish at some points on the curve. That is, second derivative of the curve may be zero. In this situation, an alternative frame is needed for non continously differentiable curves on which Bishop (parallel transport frame) frame is well defined and constructed in the Euclidean and its ambient spaces.

In this work, using common vector field as the binormal vector of Serret-Frenet frame, we define a rectifying curve of a spacelike curve according to type-2 Bishop frame in Minkowski 3-space as its position vector always lies in the orthogonal complement $\Omega^{\perp}{}_2$ of its principal vector field $\Omega_2$. Moreover we characterize Bertrand curves in the same space. In particular, we study Mannheim partner curves according to type-2 Bishop frame in $E_1^3$ and express such curves in temrs of their curvature functions. Finally we give a characterization regarding to inclined curves according to type-2 Bishop frame in $E_1^3 \cdot$

## 2 PRELIMINARIES

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the Minkowski 3--space $E_1^3$ are briefly presented.

The three dimensional Minkowski space $E_1^3$ is a real vector space $E^3$ endowed with the standard flat Lorentzian metric given by

$$\langle , \rangle_L = -dx_1^2 + dx_2^2 + dx_3^2 ,$$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $E_1^3$. This metric is an indefinite one [19]. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be arbitrary vectors in $E_1^3$, then, [20], the Lorentzian cross product of $u$ and $v$ is defined as

$$u \times v = -\det \begin{bmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

Recall that a vector $v \in E_1^3$ has one of three Lorentzian characters such as: it is a spacelike vector if $\langle v, v \rangle > 0$ or $v = 0$; timelike $\langle v, v \rangle < 0$ and null (lightlike) $\langle v, v \rangle = 0$ for $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $E_1^3$ can locally be spacelike, timelike or null (lightlike) if its velocity vector $\alpha'$ are ,respectively, spacelike, timelike or null (lightlike), for every $s \in I \subset R$ [20].

The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by $\|a\| = \sqrt{|\langle a, a \rangle|}$. The curve $\alpha = \alpha(s)$ is called a unit speed curve if its velocity vector $\alpha'$ is the unit one, i.e., $\|\alpha'\| = 1$. For the vectors $v, w \in E_1^3$, they are said to be orthogonal eachother if and only if $\langle v, w\rangle = 0$. Denote by $\{T, N, B\}$ the moving Serret-Frenet frame along the curve $\alpha = \alpha(s)$ in the space $E_1^3$ [20].

For an arbitrary spacelike curve $\alpha = \alpha(s)$ in $E_1^3$, the Serret-Frenet formulae are given as follows

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \gamma\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1}$$

where $\gamma = \mp 1$, and the functions $\kappa$ and $\tau$ are, respectively, the first and second (torsion) curvature and also

$$T(s) = \alpha'(s), N(s) = \frac{T'(s)}{\kappa(s)}, B(s) = T(s) \times N(s) \text{ and } \tau(s) = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2(s)}.$$

- If $\gamma = -1$, then $\alpha(s)$ is a spacelike curve with spacelike principal normal $N$ and timelike binormal $B$, and its Serret-Frenet invariants are given as

$$\kappa(s) = \sqrt{\langle T'(s), T'(s)\rangle} \text{ and } \tau(s) = -\langle N'(s), B(s)\rangle.$$

- If $\gamma = 1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal $N$ and spacelike binormal $B$, also we obtain its Serret-Frenet invariants as

$$\kappa(s) = \sqrt{-\langle T'(s), T'(s)\rangle} \text{ and } \tau(s) = \langle N'(s), B(s)\rangle.$$

The Lorentzian sphere $S_1^2$ of radius $r > 0$ and with the center in the origin of the space $E_1^3$ is defined, [19], by

$$S_1^2(r) = \{p = (p_1, p_2, p_3) \in E_1^3 : \langle p, p\rangle = r^2\}.$$

**Theorem 2.1,** [2]**,** Let $\alpha = \alpha(s)$ be a spacelike unit speed curve with a spacelike principal normal. If $\{\Omega_1, \Omega_2, B\}$ is an adapted frame, then we have

$$\begin{bmatrix} \Omega_1^{'} \\ \Omega_2^{'} \\ B^{'} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & -\xi_2 \\ -\xi_1 & -\xi_2 & 0 \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix}. \tag{2}$$

**Theorem 2.2** Let $\{T, N, B\}$ and $\{\Omega_1, \Omega_2, B\}$ be Frenet and Bishop frames, respectively. There exists a relation between them as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sinh\theta(s) & \cosh\theta(s) & 0 \\ \cosh\theta(s) & \sinh\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{3}$$

where $\theta$ is the angle between the vectors $N$ and $\Omega_1$ [2].

Using (2), we have

$$B^{'} = \tau N = -\xi_1\Omega_1 - \xi_2\Omega_2,$$

and taking the norm of both sides, we find

$$\tau = \sqrt{\left|\xi_2^2 - \xi_1^2\right|},$$

and

$$\sqrt{\left|(\frac{\xi_1}{\tau})^2 - (\frac{\xi_2}{\tau})^2\right|} = 1.$$

By (5), we express

$$\xi_1 = \tau(s)\cosh\theta(s), \xi_2 = \tau(s)\sinh\theta(s). \tag{4}$$

The frame $\{\Omega_1, \Omega_2, B\}$ is properly oriented, and $\tau$ and $\theta(s) = \int_0^s \kappa(s)ds$ are polar coordinates for the curve $\alpha = \alpha(s)$. We shall call the set $\{\Omega_1, \Omega_2, B, \xi_1, \xi_2\}$ as type-2 Bishop invariants of the curve $\alpha = \alpha(s)$ in $E_1^3$.

**Definition 2.1.** Let $\alpha = \alpha(s)$ be a spacelike curve in $E_1^3$ and $V_1$ be the first Frenet vector field of $\alpha$. If

$$\langle V_1, X \rangle = \cosh\theta, \text{ (constant)}, \tag{5}$$

for a constant unit vector field $X \in \chi(E_1^3)$, then $\alpha$ is called a general helix (inclined curve) in $E_1^3$ [18].

**Definition 2.2.** Let $\{\alpha, \alpha^*\}$ be Bertrand curves in $E_1^3$, these curves are said to be Bertrand curves such that they satisfy the following relation:

$$\alpha = \alpha^* + \lambda N^*,$$

where $N^*$ is the principal normal vector of the curve $\alpha^*$ and $\lambda \in R$.

**Definition 2.3.** Let $\{\alpha, \alpha^*\}$ be Mannheim curves in $E_1^3$, these curves are said to be Mannheim curves such that they satisfy the following relation:

$$\alpha = \alpha^* + \lambda B^*,$$

where $N^*$ is the principal normal vector of the curve $\alpha^*$ and $\lambda \in R$.

# 3 ON CHARACTERİZATİONS OF SOME SPECIAL CURVES OF SPACELIKE CURVES ACCORDING TO THE TYPE-2 BISHOP FRAME IN $E_1^3$

In this section, we study characterizations of some special curves such as inclined, rectifying, Bertrand curves.

## 3.1 A characterization for inclined curves according to the type-2 Bishop frame in $E_1^3$

**Definition 3.1.** Let $\alpha \subset E_1^3$ be a curve in $E_1^3$, the function

$$H(s) = \frac{\xi_2(s)}{\xi_1(s)} \tag{6}$$

is called harmonic curvature function of the curve $\alpha$ provided that $\xi_1 \neq 0$ and $\xi_2 \neq 0$ according to the type-2 Bishop frame in $E_1^3$.

**Theorem 3.1.** If the curve $\alpha \subset E_1^3$ is an inclined curve, then the harmonic curvature $H$ is constant.

**Proof.** From (5), we write that

$$\frac{\xi_2(s)}{\xi_1(s)} = \tanh\varphi(s), \tag{7}$$

and differentiating (7) with respect to $s$, we find

$$\frac{d\varphi}{ds} = \frac{(\frac{\xi_2}{\xi_1})'}{1-(\frac{\xi_2}{\xi_1})^2}, \tag{8}$$

and using $H(s) = \frac{\xi_2(s)}{\xi_1(s)}$ in (6), we get

$$\frac{d\varphi}{ds} = \frac{H'}{1-H^2}, \tag{9}$$

and integrating (9), we reach out

$$\varphi = \int \frac{H'}{1-H^2} ds.$$

The solution of this integral is found as follows:

$$\varphi = \frac{1}{2}\ln(\frac{H-1}{H+1}) + c,$$

where $c \in R$. If the curve $\alpha$ is an inclined curve which satisfies (7), then $\varphi$ is a constant. As a result, we obtain that

$$H = \frac{1+e^{2\varphi}}{1-e^{2\varphi}} = constant.$$

Hence, the proof is completed.

### 3.2 Rectifying curves according to the type-2 Bishop frame in $E_1^3$

In the Euclidean space, rectifying curves were introduced by B.Y. Chen in [3] as space curves whose position vectors always lie in its rectifying plane spanned by the tangent and the binormal vector fields of the curve, i.e., $T$ and $B$, respectively. Accordingly, the position vector with respect to a chosen origin of a rectifying curve $\alpha$ in $E^3$ satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s),$$

where $\lambda(s)$ and $\mu(s)$ are arbitrary differentiable functions in arc-length parameter $s \in I \subset E$ [1]. In this section, we define rectifying curves according to the type-2 Bishop frame in Minkowski 3-space in analogy to its Euclidean case. Position vector of the rectifying curves always lies in the orthogonal complement $\Omega_2^{\perp}$ of its principal normal vector field $\Omega_2$. Consequently, the complement $\Omega_2^{\perp}$ is given by

$$\alpha(s) = \lambda(s)\Omega_1(s) + \mu(s)B(s) \tag{10}$$

for the differentiable functions $\lambda(s)$ and $\mu(s)$ in arc-length parameter $s$.

Next, we characterize rectifying curves according to their curvature functions $\xi_1$, $\xi_2$ and then give the necessary and sufficient conditions for an arbitrary curve to be a rectifying curve in $E_1^3$. Moreover, we obtain an explicit equation of rectifying curve in $E_1^3$.

**Theorem 3.2.** There is a rectifying curve of a spacelike curve according to the type-2 Bishop frame in $E_1^3$ if and only if its position vector is as:

$$X(s) = \alpha(s) + \left(\int \xi_1(s)\mu(s)ds\right)\Omega_1(s) - \left(\int \xi_1(s)\lambda(s)ds\right)B(s).$$

**Proof.** Assume that there is a rectifying curve of a spacelike curve according to the type-2 Bishop frame in $E_1^3$, so the equation (10) is satisfied. Differentiating (10) with respect to $s$, and using Bishop equations (2) in it, we reach

$$T(s) = (\lambda'(s) - \xi_1(s)\mu(s))\Omega_1(s) - \xi_2(s)\mu(s)\Omega_2(s) + (\xi_1(s)\lambda(s) + \mu'(s))B(s),$$

and it is followed by

$$\begin{cases} \lambda'(s) - \xi_1(s)\mu(s) = 0, \\ \xi_2(s)\mu(s) = 0, \\ \xi_1(s)\lambda(s) + \mu'(s) = 0, \end{cases} \tag{11}$$

so we easily get

$$\begin{cases} \lambda^2(s) + \mu^2(s) = c, \\ \lambda(s) = \int \xi_1(s)\mu(s)ds, \\ \mu(s) = -\int \xi_1(s)\lambda(s)ds, \end{cases} \tag{12}$$

where $c \in R$. Thus, the functions $\lambda(s)$ and $\mu(s)$ are expressed in terms of the curvature functions $\xi_1(s)$ and $\xi_2(s)$. Moreover, using the last equation in (11), and the relation (12), we easily find that the curvatures $\xi_1(s)$ and $\xi_2(s)$ satisfy the equation

$$(-\int \xi_1(s)\mu(s)ds)\xi_1(s) + (\int \xi_1(s)\lambda(s)ds)\xi_2(s) = \mp(\sqrt{c-\lambda^2})'. \tag{13}$$

Conversely, assume that the curvatures $\xi_1(s)$ and $\xi_2(s)$ satisfy (13), then the position vector $X$ of rectifying curve is given by

$$X(s) = \alpha(s) + (\int \xi_1(s)\mu(s)ds)\Omega_1(s) - (\int \xi_1(s)\lambda(s)ds)B(s). \tag{14}$$

### 3.3 Bertrand curves of the spacelike curves according to the type-2 Bishop frame in $E_1^3$

**Definition 3.2.** Let α and $\alpha^*$ be regular curves according to the type-2 Bishop frame in $E_1^3$, these curves are said to be Bertrand curves such that they satisfy the following relation:

$$\alpha^* = \alpha + l\Omega_2, \tag{15}$$

where $\Omega_2$ is the vector and $l \in R$

**Theorem 3.3.** A spacelike curve according to the type-2 Bishop frame in $E_1^3$ does not admit a Bertrand mate curve.

**Proof.** Taking the norm of (15), and then differentiating it, we have

$$\frac{d}{ds}(l^2) = \frac{d}{ds}(\alpha^* - \alpha) = 2(\alpha^* - \alpha)(\alpha^{*'} - \alpha'). \tag{16}$$

Using the following expressions

$$\alpha^{*'} = T^{*'} = \sinh\bar{\theta}\Omega_1^* - \cosh\bar{\theta}\Omega_2^*,$$
$$\alpha' = T = \sinh\theta\Omega_1 - \cosh\theta\Omega_2, \tag{17}$$

which are obtained from in (16) through using (2), we get

$$\frac{d}{ds}(l^2) = 2(\alpha^* - \alpha)(\sinh\bar{\theta}\Omega_1^* - \cosh\bar{\theta}\Omega_2^* - \sinh\theta\Omega_1 + \cosh\theta\Omega_2), \tag{18}$$

and rearranging (18), we find

$$\frac{d}{ds}(l^2) = -\lambda(\cosh\bar{\theta} - \cosh\theta). \tag{19}$$

If we take $\bar{\theta} = \theta$ in (19), then $\theta$ is constant, and also $\kappa^* = \kappa$.

Let's continue to study by differentiating $T^*.T$ with respect to the parameter $s$, then we have

$$\frac{d}{ds}(T^*.T) = T^{*'}.T + T^*.T',$$

and using Frenet derivative formulae we get

$$\frac{d}{ds}(T^*.T) = \kappa^*[\cosh\bar{\theta}\sinh\theta\Omega_1^*\Omega_1 - \cosh\theta\sinh\bar{\theta}] + \kappa\sinh\bar{\theta}\cosh\theta\Omega_1^*\Omega_1 - \cosh\bar{\theta}\sinh\theta]. \tag{20}$$

Using trigonometric transformation formulae, and rearranging (20), we find

$$\frac{d}{ds}(T^*.T) = \kappa\sinh 2\theta(\cosh\varphi - 1), \tag{21}$$

since

$$\Omega_1^* = \cosh\varphi\Omega_1 + \sinh\varphi B,$$

where $\varphi$ is the angle between the vectors $\Omega_1$ and $B$. In (21), if $\kappa = 0$, we get $\theta = 0$ or $\varphi = \frac{\pi}{2}$, or if $\kappa \neq 0$, we obtain $\theta = 0$ or $\varphi = \frac{\pi}{2}$. In this case, we can write that

$$\Omega_1^* = B.$$

Conversely, let's take

$$T^*.T = 0 \tag{22}$$

and (15) into consideration, then differentiating (15) gives

$$\frac{d\alpha^*}{ds^*}\frac{ds^*}{ds}T = T + l(-\xi_2 B)T = 0, \tag{23}$$

or

$$T^*\frac{ds^*}{ds} = T + l'\Omega_2 - l\xi_2 B$$

and inner product of this expression with $T$ using (23) in (22), we end up with

$$\frac{ds}{ds^*} = 0.$$

This is a contradiction. Also, if $\varphi \neq \frac{\pi}{2}, \theta = 0$ and $\kappa = 0$, then

$$\Omega_1^* = -\Omega_2 \text{ and } \Omega_2^* = \Omega_1. \tag{24}$$

Finally, the result (24) contradicts with the following relation

$$\Omega_2^* = \Omega_2$$

which is a defintion of Bertrand curve.

## 4 MANNHEIM CURVES OF THE SPACELIKE CURVES ACCORDING TO THE TYPE-2 BISHOP FRAME IN $E_1^3$

It is well known in differential geometry that If there exists a corresponding relationship between the space curves $\alpha$ and $\alpha^*$ such that the principal normal lines of $\alpha$ coincide with the binormal lines of $\alpha^*$ at the corresponding points of the curves, then $\alpha$ is called as a Mannheim curve, and $\alpha^*$ is called as a Mannheim partner curve of $\alpha$. The pair of $\{\alpha, \alpha^*\}$ is said to be a Mannheim pair [17].

**Definition 4.1.** Let $\alpha$ and $\alpha^*$ be regular curves according to the type-2 Bishop frame in $E_1^3$, these curves are said to be Mannheim curves such that they satisfy the following relation:

$$\alpha = \alpha^* + \lambda B^*, \tag{25}$$

where $B^*$ is the binormal vector and $\lambda \in R$.

**Theorem 4.1.** Let $\alpha$ and $\alpha^*$ be spacelike curves according to the type-2 Bishop frame in $E_1^3$. The curve $\alpha^*$ is a Mannheim curve such that $\{\alpha, \alpha^*\}$ is a Mannheim pair.

**Proof.** For curves to be Mannheim pair according to the type-2 Bishop frame in $E_1^3$, the vectors $\Omega_2$ and $B^*$ have to be linearly dependent. Thus, from (25), we easily find that

$$\alpha^* = \alpha - \lambda\Omega_2. \tag{26}$$

**Theorem 4.2.** The distance between the corresponding points of Mannheim partner curves according to the type-2 Bishop frame in $E_1^3$ is constant.

**Proof.** Let $\{\alpha, \alpha^*\}$ be a Mannheim pair, then differentiating (25) and using (2), we obtain

$$\Omega_1 \frac{ds}{ds^*} = (1-\lambda\xi_1^*)\Omega_1^* - \lambda\xi_2^*\Omega_2^* + \lambda' B^*. \tag{27}$$

Since $\Omega_2$ and $B^*$ are linearly dependent, using

$$\langle \Omega_2, B^* \rangle = 0,$$

we get

$$\lambda' = 0.$$

Thus, $\lambda$ is a non-zero constant. On the other hand, the distance function between these points is

$$d(\alpha(s), \alpha^*(s^*)) = \|\alpha(s) - \alpha^*(s^*)\| = \|\lambda B^*\| = |\lambda|.$$

**Theorem 4.3.** Let $\{\alpha, \alpha^*\}$ be a Mannheim pair according to the type-2 Bishop frame in $E_1^3$, then there is a relation between the Bishop curvatures of the curvatures of the curves $\alpha$ and $\alpha^*$ as follows:

$$\xi_2 = \sqrt{\frac{\xi_1^{*2} - \xi_2^{*2}}{(\lambda\xi_2^*)^2 - (1-\lambda\xi_1^*)^2}}. \tag{28}$$

**Proof.** Let $\{\alpha, \ \alpha^*\}$ be a Mannheim pair according to the type-2 Bishop frame in $E_1^3$, then considering non-zero constant $\lambda$ in (27), we have

$$\Omega_1 \frac{ds}{ds^*} = (1-\lambda\xi_1^*)\Omega_1^* - \lambda\xi_2^*\Omega_2^*, \tag{29}$$

and also we know that the following relation holds

$$\Omega_1 = \cos\varphi\Omega_1^* + \sin\varphi\Omega_2^*, \tag{30}$$

where $\varphi$ is the angle between the vectors $\Omega_1$ and $\Omega_1^*$ at the corresponding points of the curves $\alpha$ and $\alpha^*$. Using (30) in (29), we get

$$\cos\varphi = (1-\lambda\xi_1^*)\frac{ds^*}{ds}, \quad \sin\varphi = -\lambda\xi_2^*\frac{ds^*}{ds}. \tag{31}$$

Differentiating (26) gives

$$\Omega_1^* = \Omega_1 \frac{ds}{ds^*} + \lambda\xi_2 B\frac{ds}{ds^*}, \tag{32}$$

and also we know that there is the following relation

$$B = \cosh\varphi\Omega_1^* + \sinh\varphi\Omega_2^*, \tag{33}$$

using (30), (32), and (33), we find

$$\Omega_1^* = \left(\frac{-\sinh\varphi}{\sin\varphi\cosh\varphi - \cos\varphi\sinh\varphi}\right)\Omega_1 + \left(\frac{\sin\varphi}{\sin\varphi\cosh\varphi - \cos\varphi\sinh\varphi}\right)B. \tag{34}$$

Also considering (34) together with (32), we get

$$\frac{ds}{ds^*}=\frac{\sinh\varphi}{\cos\varphi\sinh\varphi-\sin\varphi\cosh\varphi}, \tag{35}$$

$$\lambda\xi_2=-\frac{\sin\varphi}{\sinh\varphi}, \tag{36}$$

and

$$\lambda\xi_2\frac{ds}{ds^*}=\frac{\sin\varphi}{\sin\varphi\cosh\varphi-\cos\varphi\sinh\varphi}. \tag{37}$$

Using (31) and (36) in (37), we reach

$$\sinh\varphi=\frac{\xi_2^*}{\xi_2}\frac{ds^*}{ds},\quad \cosh\varphi=\frac{\xi_1^*}{\xi_2}\frac{ds^*}{ds}, \tag{38}$$

and

$$\frac{ds^*}{ds}=\sqrt{\frac{1}{(1-\lambda\xi_1^*)^2+\lambda^2\xi_2^{*2}}}. \tag{39}$$

From (38) and (39), we have the equation (28).

**Corollary 4.1.** There is the relation as

$$\frac{\xi_1^*}{\xi_2^*}=\frac{1}{\mu} \tag{40}$$

and also the curvatures of the curve $\alpha^*$ are obtained

$$\xi_1^*=\frac{\sigma}{\lambda(\sigma-\mu)},\quad \xi_2^*=\frac{\sigma\mu}{\lambda(\sigma-\mu)}, \tag{41}$$

where $\sigma=\tan\varphi, \mu=\tanh\varphi,$ and $\lambda\in R.$

**Proof.** From (31), we have

$$\tanh\varphi\xi_1^*-\xi_2^*=0. \tag{42}$$

Putting $\tanh\varphi=\mu,$ (42) turns into

$$\mu\xi_1^* - \xi_2^* = 0. \tag{43}$$

From (31), we find

$$\cot\varphi = -\frac{1}{\lambda\xi_2^*} + \frac{\xi_1^*}{\xi_2^*}, \tag{44}$$

substituting $\cot\varphi = \frac{1}{\sigma}$, we have

$$\xi_2^* = \frac{\sigma\mu}{\lambda(\sigma-\mu)}.$$

The curvature $\xi_1^*$ is also found in the same way.

**Corollary 4.2.** Let $\{\alpha, \ \alpha^*\}$ be a Mannheim pair according to the type-2 Bishop frame in $E_1^3$, then the curvature $\xi_1$ of the curve $\alpha$ as follows:

$$\xi_1 = -\frac{\sin\varphi}{\cosh\varphi}\frac{d\varphi}{ds^*}. \tag{45}$$

**Proof.** From (30), we find that

$$\left\langle \Omega_1, \Omega_1^* \right\rangle = \cos\varphi. \tag{46}$$

Differentiating (46) with respect to $s^*$, we find

$$\left\langle \xi_1 B, \Omega_1^* \right\rangle + \left\langle \Omega_1, \xi_1^* B^* \right\rangle = -\sin\varphi \frac{d\varphi}{ds^*}. \tag{47}$$

Using (33) in (47), we reach the result (45).

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