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Time fractional problem via inner product including weighted function

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Abstract

In this research, we discuss the construction of analytic solution of homogenous initial boundary value problem including PDEs of fractional order. Since homogenous initial boundary value problem involves Caputo fractional order derivative, it has classical initial and boundary conditions. By means of separation of variables method and the inner product defined on $L^2[0, l]$, the solution is constructed in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem including fractional derivative in Caputo sense used in this study. We defined a new inner product with a weighted function to get coefficients in the Fourier series. Illustrative example presents the applicability and influence of separation of variables method on fractional mathematical problems.

Keywords: *Caputo fractional derivative, dirichlet boundary conditions, separation of variables, spectral method, weighted ınner product*

Ağırlıklı iç çarpım ile zaman kesirli problem

Öz

Bu çalışmada, kesirli mertebeden kısmi diferansiyel denklemler içeren homojen başlangıç sınır değer probleminin analitik çözümünü araştırıyoruz. Homojen başlangıç sınır değeri problemi Caputo kesirli mertebe türevini içerdiğinden klasik başlangıç ve sınır koşullarına sahiptir. Değişkenlerine ayırma yöntemi ve $L^2[0, l]$ de tanımlanan ağırlıklı iç çarpım ile çözüm, bu çalışmada kullanılan Caputo anlamında kesirli türevi içeren bir Sturm-Liouville özdeğer probleminin özfonksiyonlarına göre bir Fourier

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serisi şeklinde oluşturulmuştur. Fourier serisindeki katsayıları elde etmek için ağırlıklı fonksiyona sahip yeni bir iç çarpım tanımlanmıştır. Çözülen örnek, değişkenlerine ayırma yönteminin kesirli matematik problemleri üzerindeki uygulanabilirliğini ve etkisini göstermektedir.

Anahtar kelimeler: *Caputo kesirli türev, dirichlet sınır koşulları, değişkenlerine ayırma, spektral method, ağırlıklı iç çarpım*

1. Introduction

As PDEs of fractional order plays an influential role in mathematical models of processes in various branches of science such as applied mathematics, physics chemistry etc., the interest of this topic increases enormously. Because of non-local property of fractional derivative, the model with fractional derivative for physical problems turns out to be the best choice to analyze the behaviour of the complex non linear processes. That is why, it attracts increasing number of researchers. The derivatives in the sense of Caputo are one of the most common one since mathematical models with Caputo derivatives give better results than the analysis of ones including other fractional derivatives. This conclusion is supported by various papers [1-16]. In addition, the derivative of a constant function in Caputo sense is zero which does not hold by many fractional derivatives. The solution of fractional differential equations in the sense of Caputo derivative is obtained in terms of Mittag-Leffler function or its derivations, as a result, the Mittag-Leffler function (MLF) plays a vital role in the solutions of fractional differential equations. It takes the place of exponential function which is a significant function to form the solution of integer order differential equations [17-25].

The main goal of this study is to establish the analytic solution of following time fractional differential equations with Dirichlet boundary and initial conditions.

$$D_t^{\alpha}u(x,t) = u_{xx}(x,t) + Bu_x(x,t) - Cu(x,t), \quad (1)$$
$$u(0,t) = u(l,t) = 0, \quad (2)$$
$$u(x,0) = f(x)e^{-\frac{B}{2}x}, \quad (3)$$

where $0 < \alpha < 1, 0 \leq x \leq l,\ 0 \leq t \leq T,\ B, C \in \mathbb{R}$.

This problem models diffusion processes in terms of fractional derivative. The outcomes of this model is much more better than the outcomes of the ones including integer order derivatives, since fractional derivative is a non-local operator whereas integer order derivative is a local operator. The novelty of this research is that the solution of this fractional problem is constructed by means of seperation of variables with weighted inner product defined on $L^2[0,l]$ as:

$$< u,v \geq \frac{2}{l}\int_0^l u(x)v(x)e^{Bx}dx,$$

where the functions $u(x)$ and $v(x)$ belong to the function space $L^2[0,l]$.

2. Preliminary results

In this subsection, fundamental definitions are recalled.
The q^{th} order fractional derivative of $u(t)$ in Caputo sense is defined as

$$D^q u(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} u^{(n)}(s) ds, \qquad t \in [t_0, t_0 + T], \tag{4}$$

where $u^{(n)}(t) = \frac{d^n u}{dt^n}, n-1 < q < n$. Notice that Caputo fractional derivative coincides with ordinary derivative when the order of the derivative is integer.
The q^{th} order Caputo fractional derivative for $0 < q < 1$ is established as follows:

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s) ds, \qquad t \in [t_0, t_0 + T]. \tag{5}$$

The two-parameters MLF which is taken into account in eigenvalue problem, is given by

$$E_{\alpha,\beta}(\lambda(t-t_0)^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^{\alpha})^k}{\Gamma(\alpha k+\beta)}, \alpha, \beta > 0, \tag{6}$$

including constant λ Especially, for $t_0 = 0, \alpha = \beta = q$ we have

$$E_{\alpha,\beta}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+q)}, q > 0. \tag{7}$$

MLF coincides with exponential function i.e., $E_{1,1}(\lambda t) = e^{\lambda t}$ for $q = 1$. For details see [26,27].

3. Main results

The solution of the problem (1)-(3) is established by employing the separation of variables method which leads to the following form:

$$u(x,t;\alpha) = X(x)\, T(t;\alpha), \tag{8}$$

where $0 \leq x \leq l,\ 0 \leq t \leq T$.
After substitution of (8) into (1) and arrangement, we have

$$\frac{D_t^{\alpha}(T(t;\alpha))}{T(t;\alpha)} + C = \frac{X''(x) + BX'(x)}{X(x)} = -\lambda. \tag{9}$$

The related Sturm-Liouville problem is obtained as follows:

$$X''(x) + BX'(x) + \lambda X(x) = 0, \tag{10}$$
$$X(0) = X(l) = 0. \tag{11}$$

The solution of this problem is determined in terms of the exponential function in the following form:

$$X(x) = e^{rx}. \tag{12}$$

Hence the characteristic equation is computed in the following form:

$$r^2 + Br + \lambda = 0. \tag{13}$$

Case 1. If $B^2 - 4\lambda > 0$, the equation (13) have two distinct real roots r_1, r_2. Hence the solution of Sturm-Liouville problem (10)-(11) becomes

$$X(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}. \tag{14}$$

The first boundary condition yields

$$X(0) = 0 = c_1 + c_2, \tag{15}$$

which leads to

$$c_1 = -c_2. \tag{16}$$

Similarly second boundary condition leads to

$$X(l) = c_1(e^{r_1 l} - e^{r_2 l}) = 0 \Longrightarrow c_1 = 0, c_2 = 0, \tag{17}$$

which implies that there is not any solution for $B^2 - 4\lambda > 0$.
Case 2. If $B^2 - 4\lambda = 0$, the equation (13) have two coincident roots r_1, r_2 such that $r_1 = r_2$. Hence the solution of Sturm-Liouville problem (10)-(11) becomes

$$X(x) = c_1 e^{r_1 x} + c_2 x e^{r_2 x}. \tag{18}$$

By making use of the first boundary condition, we have

$$X(0) = c_1 = 0. \tag{19}$$

Similarly second boundary condition leads to

$$X(l) = c_2 l e^{r_1 l} \Longrightarrow c_2 = 0, \tag{20}$$

which implies that there is no solution for B^2-4$\lambda = 0$.
Case 3. If $B^2 - 4\lambda < 0$, the roots of characteristic equation are complex. Hence the solution of Sturm-Liouville problem (10)-(11) becomes

$$X(x) = e^{-\frac{B}{2}x}\left(k_1 \cos\left(\frac{\sqrt{4\lambda - B^2}}{2}x\right) + ik_2 \sin\left(\frac{\sqrt{4\lambda - B^2}}{2}x\right)\right). \tag{21}$$

By making use of the first boundary condition we have

$$X(0) = k_1 = 0. \tag{22}$$

Similarly last boundary condition leads to

$$X(l) = e^{-\frac{B}{2}l} i k_2 \sin\left(\frac{\sqrt{4\lambda - B^2}}{2} l\right) = 0, \tag{23}$$

which implies that

$$\sin\left(\frac{\sqrt{4\lambda - B^2}}{2} l\right) = 0, \tag{24}$$

which yields the following eigenvalues

$$\lambda_n = \frac{4w_n^2 + B^2 l^2}{4l^2} = \frac{4w_n^2 + (Bl)^2}{(2l)^2}, n = 0,1,2,3, \dots, \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \tag{25}$$

where $w_n = n\pi, (n = 0,1,2,3, \dots)$ satisfy the equation $\sin(w_n) = \sin\left(\frac{\sqrt{4\lambda_n - B^2}}{2} l\right) = 0$.
As a result, the solution is obtained as follows:

$$X_n(x) = \sin\left(w_n\left(\frac{x}{l}\right)\right) e^{-\frac{B}{2}x}, n = 1,2,3, \dots \tag{26}$$

The second equation in (9) for eigenvalue λ_n yields the fractional differential equation below:

$$\frac{D_t^\alpha(T(t;\alpha))}{T(t;\alpha)} = -(C + \lambda), \tag{27}$$

which yields the following solutions

$$T_n(t;\alpha) = E_{\alpha,1}\left(-\left(C + \frac{4w_n^2 + (Bl)^2}{(2l)^2}\right) t^\alpha\right), n = 0,1,2,3, \dots \tag{28}$$

The solution for every eigenvalue λ_n is constructed as

$$u_n(x,t;\alpha) = E_{\alpha,1}\left(-\left(C + \frac{4w_n^2 + (Bl)^2}{(2l)^2}\right) t^\alpha\right) \sin\left(w_n\left(\frac{x}{l}\right)\right) e^{-\frac{B}{2}x}, n = 0,1,2,3, \dots, \tag{29}$$

which leads to the following general solution

$$u(x,t;\alpha) = \sum_{n=1}^{\infty} A_n e^{-\frac{B}{2}x} \sin\left(w_n\left(\frac{x}{l}\right)\right) E_{\alpha,1}\left(-\left(C + \frac{4w_n^2 + (Bl)^2}{(2l)^2}\right) t^\alpha\right). \tag{30}$$

Note that it satisfies boundary condition as well as fractional differential equation.
The coefficients of general solution are established by taking the following initial condition into account:

$$u(x,0) = f(x) e^{-\frac{B}{2}x} = \sum_{n=1}^{\infty} A_n e^{-\frac{B}{2}x} \sin\left(w_n\left(\frac{x}{l}\right)\right). \tag{31}$$

The coefficients A_n for $n = 1,2,3, \dots$ are determined by the help of inner product with weighted function defined on $L^2[0, l]$ $as < u, v \geq \frac{2}{l}\int_0^l u(x)v(x)e^{Bx}dx$:

$$A_n = \frac{2}{l} < e^{-\frac{B}{2}x}\sin\left(w_n\left(\frac{x}{l}\right)\right), f(x)e^{-\frac{B}{2}x} >= \frac{2}{l}\int_0^l \sin\left(w_n\left(\frac{x}{l}\right)\right)\, e^{-\frac{B}{2}x} f(x)e^{-\frac{B}{2}x}e^{Bx}dx = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right) f(x)dx. \quad (32)$$

4. Illustrative example

Consider the following initial boundary value problem with Dirichlet boundary conditions:

$$u_t = u_{xx} + u_x - u,$$
$$u(0,t) = 0,\ u(2,t) = 0,$$
$$u(x,0) = -\sin(\pi x)\, e^{-\frac{1}{2}x}, \quad (33)$$

the solution of which is accomplished as follows:

$$u(x,t) = -e^{-\frac{1}{2}x}\sin(\pi x)\, e^{-\left(\pi^2+\frac{5}{4}\right)t}, \quad (34)$$

where $0 \leq x \leq 2, 0 \leq t \leq T$.
Now let us consider the following fractional initial boundary value problem with Dirichlet boundary conditions:

$$D_t^\alpha u(x,t) = u_{xx}(x,t) + u_x(x,t) - u(x,t), \quad (35)$$
$$u(0,t) = u(2,t) = 0, \quad (36)$$
$$u(x,0) = \sin(\pi x)\, e^{-\frac{1}{2}x}, \quad (37)$$

where $0 < \alpha < 1, 0 \leq x \leq 2,\ 0 \leq t \leq T$.
It is clear from Eq. (30) that the series form of the solution is accomplished as follows:

$$u(x,t;\alpha) = \sum_{n=1}^{\infty} A_n e^{-\frac{1}{2}x}\sin\left(w_n\left(\frac{x}{2}\right)\right) E_{\alpha,1}\left(-\left(1+\frac{w_n^2+1}{4}\right)t^\alpha\right), \quad (38)$$

where $w_n = n$.
Substituting $t = 0$ into the general solution (38) and utilizing the initial condition (37), we have

$$-\sin(\pi x)\, e^{-\frac{1}{2}x} = \sum_{n=1}^{\infty} A_n e^{-\frac{1}{2}x}\sin\left(w_n\left(\frac{x}{2}\right)\right). \quad (39)$$

The coefficients A_n for $n = 0,1,2,3, \dots$ are determined by the help of the inner product as follows:

$$A_n =< \sin\left(w_n\left(\frac{x}{2}\right)\right) e^{-\frac{1}{2}x}, -\sin(\pi x)\, e^{-\frac{1}{2}x} >$$
$$= \int_0^2 \sin\left(w_n\left(\frac{x}{2}\right)\right) e^{-\frac{1}{2}x}(-\sin(\pi x)) e^{-\frac{1}{2}x}\, e^x dx$$

For $n \neq 2, A_n = 0. n = 2$ we get

$$A_2 = -\int_0^2 \sin^2(\pi x)\, dx = -\int_0^2 \left(\frac{1}{2} - \frac{\cos(2\pi x)}{2}\right) dx = \left(\frac{x}{2} - \frac{\sin(2\pi x)}{4}\right)\Big|_{x=0}^{x=2} = -1. \tag{40}$$

Thus

$$u(x,t;\alpha) = -e^{-\frac{1}{2}x} \sin(\pi x)\, E_{\alpha,1}\left(-\left(\pi^2 + \frac{5}{4}\right) t^\alpha\right). \tag{41}$$

The accuracy of the method can be observed that as the order α of fractional derivative tends to 1, the solution (41) tends to the solution (34). The graphics of solutions for Example and Problem (33) in 2D are given in Figure 1.

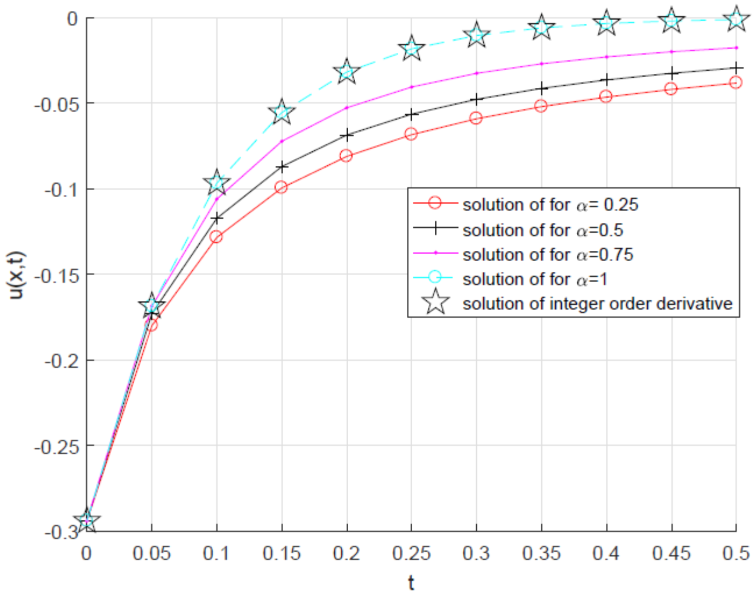

Figure.1. The graphics of solutions for Example in 2D at x=0.25 for different α.

As it is clear from the figure 1 that as α gets closer to 1, the solution of fractional problem gets closer to the solution of integer order problem. It is also obvious that when the time variable t is very small, the solutions of the fractional problem with any order α are close to each other and as time goes on discrepancy among the solutions becomes more obvious.

5. Conclusion

The main motivation of obtaining the series solution of time fractional initial boundary value problem including Dirichlet boundary conditions is accomplished by utilizing separation of variables method in terms of the solutions of related Sturm-Liouville

eigenvalue problem. It is shown that this method works effectively with newly defined inner product. At the end of the this research, we reach the conclusion that the suitable weighted inner product allows us to construct the solution of any fractional differential equations without any difficulty.
Based on the analytic solution, we reach the conclusion that diffusion processes decays exponential with time until initial condition is reached. As α tends to 0, the rate of decaying increases. This implies that in the mathematical model for diffusion of the matter which has small diffusion rate the value of α must be close to 0. This model can account for various diffusion processes of various methods. For the future works similar problems with different boundary conditions are taken into consideration.

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