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Generalized Gould-Hopper Polynomials

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Abstract

In this paper, we derive generating functions for the generalized Gould-Hopper polynomials in terms of the generalized Lauricella function by using series rearrangement techniques. Further, we derive the summation formulae for that polynomials by using different analytical means on its generating function or by using certain operational techniques. Also, generating functions and summation formulae for the polynomials related to generalized Gould-Hopper polynomials are obtained as applications of main results. In addition, we derive a theorem giving certain families of bilateral generating functions for the generalized Gould-Hopper polynomials. The results obtained here include various families of bilinear and bilateral generating functions, miscellaneous properties and also some special cases for these polynomials.

Keywords: *Generalized Gould-Hopper Polynomials, Lauricella Functions, Summation Formulae, Generating Function.*
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1. Introduction and Preliminaries

Special functions possess a lot of importances in numerous fields of mathematics, physics, engineering and other related disciplines covering different topics such as differential equations, mathematical analysis, functional analysis, mathematical physics, quantum mechanics and so on. Particularly, the family of special polynomials is one of the most useful, widespread and applicable family of special functions. Some of the most considerable polynomials in the theory of special polynomials are generalized Hermite-Kampé de Fériet (or Gould-Hopper) polynomial (see [1]). In the theory of special functions and special polynomials, and forms produced for functions have been studied and developed by several mathematicians cf. [3]-[9], [13] and see also the references cited therein. Generating function is used in a wide variety of research topics, even in modern combinatorics, to find specific properties and formulas for numbers and polynomials. They are used in finding certain properties and formulas for numbers and polynomials in a wide variety of research subjects, indeed, in modern combinatorics. In this study, we will examine the properties of generalized Gould-Hopper polynomials. Summation formulas for these polynomials, bilinear and bilateral generating function relations will be obtained.
First, let's give the notations we will use in our article.
We use the Pochhammer symbol $(\lambda)_n$, defined by ([11])

$$(\lambda)_n = \begin{cases} 1, & if \;\; n=0 \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & if \;\; n \in 1,2,3,\ldots, \end{cases} ,$$

also, we note that

$$(\lambda)_{m+n} = (\lambda)_m (\lambda+m)_n ,$$

$$(n-mk)! = \frac{(-1)^{mk} n!}{(-n)_{mk}}, \;\; 0 \le k \le \left[\frac{n}{m}\right]$$

and

$$(n-M)! = \frac{(-1)^M n!}{(-n)_M}, \;\; 0 \le M \le n,$$

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where M is defined by $M = m_1k_1 + m_2k_2 + ... + m_jk_j$, $m_1, m_2, ..., m_j \in N$; $k_1, k_2, ..., k_j \in N_0 = N \cup \{0\}$. We recall that, the Kampé de Fériet function of two variables is defined by ([11]),

$$F^{p:q;k}_{l:m;n}\left[\begin{array}{l}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{array} x,y\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r \prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}. \tag{1.1}$$

A further generalization of the Kampé de Fériet function (1.1) is the generalized Lauricella function of several variables, which is defined as ([11]):

$$\begin{aligned} & F^{A:B';B'';...;B^{(n)}}_{C:D';D'';...;D^{(n)}}(z_1,z_2,...z_n) \\ = \; & F^{A:B';B'';...;B^{(n)}}_{C:D';D'';...;D^{(n)}}\left(\begin{array}{l}[(a):\theta',\theta'',...,\theta^{(n)}]:[(b'):(\phi')];[(b''):(\phi'')];...;[(b^{(n)}):(\phi^{(n)})];\\ [(c):\psi',\psi'',...,\psi^{(n)}]:[(d'):\delta'];[(d''):\delta''];...;[(d^{(n)}):(\delta^{(n)})];\end{array} z_1,z_2,...,z_n\right)\end{aligned} \tag{1.2}$$

$$\tag{1.3}$$

$$= \sum_{m_1,m_2,...,m_n=0}^{\infty} \Omega(m_1,m_2,...,m_n)\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}...\frac{z_n^{m_n}}{m_n!}, \tag{1.4}$$

where

$$\Omega(m_1,m_2,...,m_n) := \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+m_2\theta''_j+...+m_n\theta_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+m_2\psi''_j+...+m_n\psi_j^{(n)}}} \frac{\prod_{j=1}^{B'}\left(b'_j\right)_{m_1\theta'_j}\prod_{j=1}^{B''}\left(b''_j\right)_{m_2\theta''_j}\cdots\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{m_n\theta_j^{(n)}}}{\prod_{j=1}^{D'}\left(d'_j\right)_{m_1\delta'_j}\prod_{j=1}^{D''}\left(d''_j\right)_{m_2\delta''_j}\cdots\prod_{j=1}^{D^{(n)}}\left(d_j^{(n)}\right)_{m_n\delta_j^{(n)}}}$$

and the coefficients $\theta_j^{(k)}$, $j = 1,2,...,A$; $\phi_j^{(k)}$, $j = 1,2,...,B^{(k)}$; $\psi_j^{(k)}$, $j = 1,2,...,C$; $\delta_j^{(k)}$, $j = 1,2,...,D^{(k)}$; for all $k \in \{1,2,...,n\}$ are real and positive, (a) abbreviates the array of A parameters $a_1, a_2, ..., a_A$, $\left(b^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}, j = 1,2,...,B^{(k)}$; for all $k \in \{1,2,...,n\}$ with similar interpretations for (c) and $\left(d^{(k)}\right)$, $k = 1,2,...,n$; *et cetera*. Note that, when the coefficients in equation (1.4) equal to 1, the generalized Lauricella function (1.4) reduces to a direct multivariable extension of the Kampé de Fériet function (1.1). Taking coefficients equal to 1 in definition (1.4) and for $n = 2$, we have the Kampé de Fériet function of two variables,

$$F^{p:q_1;q_2}_{l:s_1;s_2}(z_1,z_2) = F^{p:q_1;q_2}_{l:s_1;s_2}\left(\begin{array}{l}(a_p):(b'_{q_1});(b''_{q_2});\\(c_1):(d'_{s_1});(d''_{s_2});\end{array} z_1,z_2\right) = \sum_{m_1,m_2=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{m_1+m_2}\prod_{j=1}^{q_1}\left(b'_j\right)_{m_1}\prod_{j=1}^{q_2}\left(b''_j\right)_{m_2}}{\prod_{j=1}^{l}(c_j)_{m_1+m_2}\prod_{j=1}^{s_1}\left(d'_j\right)_{m_1}\prod_{j=1}^{s_2}\left(d''_j\right)_{m_2}}\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}.$$

The Gould-Hopper family of polynomials is defined by the exponential generating function (see [2]),

$$\sum_{n=0}^{\infty} H_n^{(j)}(x,y)\frac{t^n}{n!} = e^{xt+yt^j}, \tag{1.5}$$

where $j \in N$ with $j \geq 2$. In the case $j = 1$, the corresponding generating polynomials are simply expressed by the Newton binomial formula. Upon setting $j = 2$ in (1.5) gives the classical Hermite polynomials $H_n^{(2)}(x,y)$ and the polynomials have been used to define bivariate extensions of some special polynomails (see [3]).
The generalized-Gould-Hopper polynomials (G-GHP) $P_n^{(j,c)}(x,y)$ are defined by the following generating function ([10]):

$$\sum_{n=0}^{\infty} P_n^{(j,c)}(x,y)\frac{t^n}{n!} = c^{xt+yt^j}, \ (c > 1, j \geq 2). \tag{1.6}$$

It is clear that $P_n^{(j,c)}(x,y)$ is explicitly given by ([10]),

$$P_n^{(j,c)}(x,y) = n!\sum_{s=0}^{[n/j]}\frac{x^{n-js}y^s}{(n-js)!s!}(\ln c)^{n+s-js}, \tag{1.7}$$

where the symbol $[n/j]$ denotes the greatest integer less than or equal $[n/j]$. Note that $c = e$ gives $P_n^{(j,e)}(x,y) = H_n^{(j)}(x,y)$ and $c = e$, $j = 2$, gives $P_n^{(2,e)}(x,y) = H_n(x,y)$, where

$$H_n^{(j)}(x,y) = n!\sum_{s=0}^{[n/j]}\frac{x^{n-js}y^s}{(n-js)!s!}, \quad H_n(x,y) = n!\sum_{s=0}^{[n/2]}\frac{x^{n-2s}y^s}{(n-2s)!s!} \tag{1.8}$$

are Hermite-Kampé de Fériet (or Gould-Hopper) polynomials. These polynomials are specified by the generating function

$$\sum_{n=0}^{\infty}H_n^{(j)}(x,y)\frac{t^n}{n!} = e^{xt+yt^j}, \quad \sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!} = e^{xt+yt^2}.$$

Further, we note the following link:

$$H_n^{(j)}(vx,-1) = H_{n,j,v}(x), \tag{1.9}$$

where $H_{n,j,v}(x)$ denotes the generalized Hermite polynomials defined by Lahiri ([14]):

$$\sum_{n=0}^{\infty} H_{n,j,v}(x)\frac{t^n}{n!} = e^{vxt-t^j}.$$

In this paper, we derive the generating functions for the generalized Gould-Hopper polynomials (G-GHP) $P_n^{(j,c)}(x,y)$ in terms of the generalized Lauricella function of two variables $F^{A:B';B''}_{C:D';D''}[.]$ by using series rearrangement techniques. Further, we derive the summation formulae for the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$ by using different analytical means on the generating function of the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$ or by using certain operational techniques.

2. Generating functions for the Generalized Gould-Hopper Polynomials

First, we prove the following generating function for the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$.

Theorem 2.1. *For a suitable bounded sequence $\{f(n)\}_{n=0}^{\infty}$,the following generating function for the generalized-Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$ holds true:*

$$\sum_{n=0}^{\infty} f(n) P_n^{(j,c)}(x,y)\frac{t^n}{n!} = \sum_{n,s=0}^{\infty} f(n+js)\frac{(xt\ln c)^n}{n!}\frac{\left(yt^j\ln c\right)^s}{s!}. \tag{2.1}$$

Proof. Denoting the l.h.s. of equation (2.1) by Δ_1 and using equation (1.7), we find

$$\Delta_1 = \sum_{n=0}^{\infty}\sum_{s=0}^{[n/j]} f(n)\frac{n!x^{n-js}y^s(\ln c)^{n+s-js}t^n}{(n-js)!s!n!}.$$

Replacing n by $n+js$ in the above equation and using (2.2) ([11]):

$$\sum_{n=0}^{\infty}\sum_{k_1,k_2,\ldots,k_r=0}^{M\le n}\phi(k_1,k_2,\ldots,k_r;n) = \sum_{n=0}^{\infty}\sum_{k_1,k_2,\ldots,k_r=0}^{\infty}\phi(k_1,k_2,\ldots,k_r;n+M), \tag{2.2}$$

where M is defined by $M = m_1k_1+m_2k_2+\ldots+m_jk_j$, $m_1,m_2,\ldots,m_j\in N; k_1,k_2,\ldots,k_j\in N_0 = N\cup\{0\}$, we find

$$\Delta_1 = \sum_{n,s=0}^{\infty} f(n+js)\frac{(xt\ln c)^n(yt^j\ln c)^s}{n!s!}.$$

□

Remark 2.2. *Taking*

$$f(n) = \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n\prod_{j=1}^{l}(c_j)_n}$$

in assertion (2.1) of Theorem 2.1 and using definition (1.4) (for $n=2$), we deduce the following consequence of Theorem 2.1.

Corollary 2.3. *The following generating function for the G-GHP $P_n^{(j,c)}(x,y)$ holds true:*

$$\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n\prod_{j=1}^{l}(c_j)_n}P_n^{(j,c)}(x,y)\frac{t^n}{n!} = F^{p:0;0}_{q+l:0;0}\left(\begin{array}{c}[(a)_1^p:1,j] \qquad\qquad :-;-; \\ [(b)_1^q:1,j],\left[(c)_1^l:1,j\right]:-;-;\end{array} xt\ln c,\ yt^j\ln c\right), \tag{2.3}$$

where the notation $(a)_1^p$ is used to represent the product $\prod_{j=1}^{p}a_j$.

Example 2.4. *Taking $p=q=l=1$ in equation (2.3), we get*

$$\sum_{n=0}^{\infty}\frac{(a_1)_n}{(b_1)_n(c_1)_n}P_n^{(j,c)}(x,y)\frac{t^n}{n!} = F^{1:0;0}_{2:0;0}\left(\begin{array}{c}[a_1:1,j]:-;-; \\ [b_1:1,j],[c_1:1,j]:-;-;\end{array} xt\ln c,\ yt^j\ln c\right). \tag{2.4}$$

Further, taking $b_1=c_1=1$ and replacing a_1 by $a+1$ in equation (2.4), we get

$$\sum_{n=0}^{\infty}\binom{a+n}{n}P_n^{(j,c)}(x,y)\frac{t^n}{n!^2} = F^{1:0;0}_{2:0;0}\left(\begin{array}{c}[a+1:1,\ j]:-;-; \\ [1:1,\ j],[1:1,\ j]:-;-;\end{array} xt\ln c,\ yt^j\ln c\right).$$

Remark 2.5. *Taking*

$$f(n) = \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}$$

in assertion (2.1) of Theorem 2.1 and using definition (1.4) (for $n=2$), we deduce the following consequence of Theorem 2.1.

Corollary 2.6. *The following generating function for the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$ holds true:*

$$\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}P_n^{(j,c)}(x,y)\frac{t^n}{n!}=F_{q:0;0}^{p:0;0}\left(\begin{array}{l}[(a)_1^p:1,j]:-;-;\\ [(b)_1^q:1,j]:-;-;\end{array} xt\ln c,\ yt^j\ln c\right). \tag{2.5}$$

Example 2.7. *Taking $p=q=1$ in equation (2.5), we get*

$$\sum_{n=0}^{\infty}\frac{(a_1)_n}{(b_1)_n}P_n^{(j,c)}(x,y)\frac{t^n}{n!}=F_{1:0;0}^{1:0;0}\left(\begin{array}{l}[a_1:1,j]:-;-;\\ [b_1:1,j]:-;-;\end{array} xt\ln c,\ yt^j\ln c\right). \tag{2.6}$$

Next, taking $a_1=b_1=1$ in equation (2.6) and using equations (1.6) in the r.h.s. of the resultant equation, we get

$$\sum_{n=0}^{\infty}P_n^{(j,c)}(x,y)\frac{t^n}{n!}=c^{xt+yt^j}.$$

Remark 2.8. *Taking*

$$f(n)=J_n^{(j)}(w)=\sum_{k=0}^{\infty}\frac{(-1)^k w^k}{k!(n+jk)!},$$

(the generalized Bessel function or the Bessel-Wright function) in Theorem 2.1, we deduce the following consequence of Theorem 2.1.

Corollary 2.9. *The following bilateral generating function for the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$ holds true:*

$$\sum_{n=0}^{\infty}J_n^{(j)}(w)P_n^{(j,c)}(x,y)\frac{t^n}{n!}=F_{1:0;0}^{0:0;0}\left(\begin{array}{l}-:-;\quad -;\\ [1:1,j]:-;-;\end{array} (xt\ln c),\ (yt^j\ln c-w)\right). \tag{2.7}$$

Proof. Denoting the l.h.s. of equation (2.7) by Δ_2 and using definitions

$$J_n^{(j)}(w)=\sum_{k=0}^{\infty}\frac{(-1)^k w^k}{k!(n+jk)!},$$

and (1.7), we find

$$\Delta_2=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{(-1)^k w^k}{k!(n+jk)!}\right)\left(n!\sum_{s=0}^{[n/j]}\frac{x^{n-js}y^s(\ln c)^{n+s-js}}{(n-js)!s!}\right)\frac{t^n}{n!}.$$

Replacing n by $n+js$, we find

$$\Delta_2=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k w^k x^n y^s(\ln c)^{n+s}}{k!(n+js+jk)!n!s!}t^{n+js}.$$

Replacing s by $s-k$ in the above equation and using (2.8) ([11]):

$$\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\phi(s,k)=\sum_{s=0}^{\infty}\sum_{k=0}^{s}\phi(s-k,k), \tag{2.8}$$

we find,

$$\Delta_2=\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^n y^s(\ln c)^{n+s}t^{n+js}}{(n+js)!n!s!}\left(\sum_{k=0}^{s}s!\frac{(-1)^k}{k!(s-k)!}\left(\frac{w}{yt^j\ln c}\right)^k\right). \tag{2.9}$$

Finally, using the expansion ([11])

$$(1-x)^{-\lambda}=\sum_{n=0}^{\infty}(\lambda)_n\frac{x^n}{n!},$$

and definition (1.4) (for $n=2$) in equation (2.9), we get the r.h.s. of assertion (2.7) of Corollary 2.10. □

In the forthcoming section, we establish summation formulae for the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$ by using series rearrangement techniques and also by making use of the operational techniques.

3. Summation Formulue for the Generalized Gould-Hopper Polynomials

First, we prove the following result involving the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$.

Theorem 3.1. *The following summation formula for the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$ holds true:*

$$P_{n+m}^{(j,c)}(w,y)=\sum_{k=0}^{n}\sum_{r=0}^{m}\binom{n}{k}\binom{m}{r}(w-x)^{k+r}(\ln c)^{k+r}P_{n+m-k-r}^{(j,c)}(x,y). \tag{3.1}$$

Proof. Replacing t by $u+t$ in Eq.(1.6) and then using the formula ([11])

$$\sum_{n=0}^{\infty}f(n)\frac{(t+u)^n}{n!}=\sum_{n,m=0}^{\infty}f(n+m)\frac{t^n}{n!}\frac{u^m}{m!}, \tag{3.2}$$

in the resultant equation, we find the following generating function for the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$:

$$\sum_{n=0}^{\infty}P_n^{(j,c)}(x,y)\frac{(t+u)^n}{n!}=c^{x(t+u)+y(t+u)^j},$$

which can be written as

$$\sum_{n,m=0}^{\infty}P_{n+m}^{(j,c)}(x,y)\frac{t^nu^m}{n!m!}\quad=\quad c^{x(t+u)}c^{y(t+u)^j},$$

$$c^{-x(t+u)}\sum_{n,m=0}^{\infty}P_{n+m}^{(j,c)}(x,y)\frac{t^nu^m}{n!m!}\quad=\quad c^{y(t+u)^j}.$$

Replacing x by w in the above equation and equating the resultant equation to the above equation, we find

$$\sum_{n,m=0}^{\infty}P_{n+m}^{(j,c)}(w,y)\frac{t^nu^m}{n!m!}=c^{(w-x)(t+u)}\sum_{n,m=0}^{\infty}P_{n+m}^{(j,c)}(x,y)\frac{t^nu^m}{n!m!},$$

or

$$\sum_{n,m=0}^{\infty}P_{n+m}^{(j,c)}(w,y)\frac{t^nu^m}{n!m!}=\sum_{k=0}^{\infty}\frac{((w-x)(t+u)(\ln c))^k}{k!}\sum_{n,m=0}^{\infty}P_{n+m}^{(j,c)}(x,y)\frac{t^nu^m}{n!m!}$$

which on using formula (3.2) in the first summation on the r.h.s. becomes

$$\sum_{n,m=0}^{\infty}P_{n+m}^{(j,c)}(w,y)\frac{t^nu^m}{n!m!}=\sum_{k,r=0}^{\infty}\frac{(w-x)^{k+r}(\ln c)^{k+r}t^ku^r}{k!r!}\sum_{n,m=0}^{\infty}P_{n+m}^{(j,c)}(x,y)\frac{t^nu^m}{n!m!}. \tag{3.3}$$

Now, replacing n by $n-k$, m by $m-r$ and using (2.8); in the r.h.s. of Eq.(3.3), we find

$$\sum_{n,m=0}^{\infty}P_{n+m}^{(j,c)}(w,y)\frac{t^nu^m}{n!m!}=\sum_{n,m=0}^{\infty}\sum_{k,r=0}^{n,m}\frac{P_{n+m-k-r}^{(j,c)}(x,y)(w-x)^{k+r}(\ln c)^{k+r}}{k!r!}\frac{t^nu^m}{(n-k)!(m-r)!}. \tag{3.4}$$

Finally, one quating the coefficients of like powers of t and u in Eq.(3.4), we get the assertion (3.1) of Theorem 3.1. □

Remark 3.2. *Taking $m=0$ in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.*

Corollary 3.3. *The following summation formula for the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$ holds true:*

$$P_n^{(j,c)}(w,y)=\sum_{k=0}^{n}\binom{n}{k}(w-x)^k(\ln c)^kP_{n-k}^{(j,c)}(x,y). \tag{3.5}$$

Remark 3.4. *Replacing w by w+x in (3.5), we obtain*

$$P_n^{(j,c)}(w+x,y)=\sum_{k=0}^{n}\binom{n}{k}(w)^k(\ln c)^kP_{n-k}^{(j,c)}(x,y). \tag{3.6}$$

Further, we prove the following result involving products of the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$:

Theorem 3.5. *The following summation formula involving products of the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$ holds true:*

$$\frac{\left(\frac{x}{w}\right)^r\left(\frac{X}{W}\right)^s}{r!s!}P_r^{(j,c)}(w,y)P_s^{(j,c)}(W,y)=\sum_{k=0}^{[r/j]}\sum_{l=0}^{[s/j]}\frac{y^{k+l}(\ln c)^{k+l}[(\frac{x}{w})^j-1]^k[(\frac{X}{W})^j-1]^l}{(r-jk)!k!(s-jl)!l!}P_{r-jk}^{(j,c)}(x,y)P_{s-jl}^{(j,c)}(X,y). \tag{3.7}$$

Proof. Consider the product of G-GHP $P_n^{(j,c)}(x,y)$ generating functions (1.6) in the following form:

$$c^{-(xwz-y(-xz)^j+XWZ-y(-XZ)^j)} = \sum_{r,s=0}^{\infty}(-1)^{r+s}P_r^{(j,c)}(w,y)P_s^{(j,c)}(W,y)\frac{(xz)^r(XZ)^s}{r!s!} \tag{3.8}$$

which on replacing t by wz and T by WZ becomes

$$c^{-(xwz+XWZ-y(-wz)^j-y(-WZ)^j)} = \sum_{r,s=0}^{\infty}(-1)^{r+s}P_r^{(j,c)}(x,y)P_s^{(j,c)}(X,y)\frac{(wz)^r(WZ)^s}{r!s!}. \tag{3.9}$$

Next, replacing x by w, w by x, X by W and W by X in (3.9) and equating the resultant equation to (3.9), we find after expanding the exponentials in series

$$\sum_{r,s=0}^{\infty}(-1)^{r+s}P_r^{(j,c)}(w,y)P_s^{(j,c)}(W,y)\frac{(xz)^r(XZ)^s}{r!s!} \tag{3.10}$$

$$= \sum_{k,r=0}^{\infty}(-1)^{r+jk}\frac{y^k(z)^{jk}\left(x^j-w^j\right)^k(\ln c)^k(wz)^r}{r!k!}P_r^{(j,c)}(x,y)\sum_{s,l=0}^{\infty}(-1)^{s+jl}\frac{y^l(Z)^{jl}\left(X^j-W^j\right)^l(\ln c)^l(WZ)^s}{s!l!}P_s^{(j,c)}(X,y). \tag{3.11}$$

(3.12)

Finally, replacing r by $r-mk$, s by $s-ml$ and using equality (3.13) ([11]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/j]}A(k,n-jk), \tag{3.13}$$

in the r.h.s. of Eq. (3.11) and then equating the coefficients of like powers of z and Z, we get formula (3.7). □

Lemma 3.6. *We have the following summation formula for the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$ holds true:*

$$P_n^{(j,c)}(x_1+x_2,y_1+y_2) = \sum_{k=0}^{n}\binom{n}{k}P_{n-k}^{(j,c)}(x_1,y_1)P_k^{(j,c)}(x_2,y_2).$$

Proof. If we take $x \to x_1+x_2$, $y \to y_1+y_2$ in (1.6) and then we use the relation (2.8)

$$\begin{aligned}
\sum_{n=0}^{\infty}P_n^{(j,c)}(x_1+x_2,y_1+y_2)\frac{t^n}{n!} &= c^{(x_1+x_2)t+(y_1+y_2)t^j} \\
&= c^{x_1t+x_2t+y_1t^j+y_2t^j} \\
&= c^{x_1t+y_1t^j}c^{x_2t+y_2t^j} \\
&= \sum_{n=0}^{\infty}P_n^{(j,c)}(x_1,y_1)\frac{t^n}{n!}\sum_{k=0}^{\infty}P_k^{(j,c)}(x_2,y_2)\frac{t^k}{k!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}P_{n-k}^{(j,c)}(x_1,y_1)P_k^{(j,c)}(x_2,y_2)\frac{t^n}{(n-k)!k!}.
\end{aligned}$$

From the coefficients of t^n on the both sides of the last equality, we get the desired result. □

4. Bilinear and Bilateral Generating Functions

In this section, we derive several families of bilinear and bilateral generating functions for the generalized Gould-Hopper polynomials $P_n^{(j,c)}(x,y)$ defined by (1.7) using the similar method considered in (see, [12], [13]).

Theorem 4.1. *For a non-vanishing function $\Omega_\mu(z_1,...,z_r)$ of r complex variables $z_1,...,z_r$ $(r \in \mathbb{N})$ and for $a_k \neq 0$, $\mu, \psi \in \mathbb{C}$, $p \in \mathbb{N}$, let*

$$\Lambda_{\mu,\psi}(z_1,...,z_r,\zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(z_1,...,z_r)\zeta^k$$

and

$$\Theta_{n,p}^{\mu,\psi}(x,y;z_1,...,z_r;\xi) := \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(j,c)}(x,y)\Omega_{\mu+\psi k}(z_1,...,z_r)\frac{\xi^k}{(n-pk)!}.$$

Then, we have

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(x,y;z_1,...,z_r;\frac{\eta}{t^p}\right)t^n = c^{xt+yt^j}\Lambda_{\mu,\psi}(z_1,...,z_r;\eta), \tag{4.1}$$

provided that each member of (4.1) exists.

Proof. For convenience, let T denote the first member of the assertion (4.1) of Theorem 4.1. Then,

$$T = \sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(x,y;z_1,...,z_r;\frac{\eta}{t^p}\right)t^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(j,c)}(x,y)\,\Omega_{\mu+\psi k}(z_1,...,z_r)\frac{\left(\frac{\eta}{t^p}\right)^k}{(n-pk)!}\right)t^n.$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned} T &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k P_n^{(j,c)}(x,y)\,\Omega_{\mu+\psi k}(z_1,...,z_r)\frac{\eta^k}{t^{pk}}\frac{t^{n+pk}}{n!} \\ &= \left(\sum_{n=0}^{\infty} P_n^{(j,c)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty} a_k\Omega_{\mu+\psi k}(z_1,...,z_r)\eta^k\right) \\ &= c^{xt+yt^j}\Lambda_{\mu,\psi}(z_1,...,z_r;\eta), \end{aligned}$$

which completes the proof. □

Theorem 4.2. *For a non-vanishing function $\Omega_\mu(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r \in \mathbb{N})$ and for $a_k \neq 0$, $\mu, \psi \in \mathbb{C}, n, p \in \mathbb{N}$, let*

$$\Lambda_{\mu,\psi}^{n,p}(x_1+x_2,y_1+y_2;z_1,...,z_r;\eta) := \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(j,c)}(x_1+x_2,y_1+y_2)\Omega_{\mu+\psi k}(z_1,...,z_r)\eta^k.$$

Then, we have

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l \binom{n-pl}{k-pl} P_{n-k}^{(j,c)}(x_1,y_1)P_{k-pl}^{(j,c)}(x_2,y_2)\Omega_{\mu+\psi l}(z_1,...,z_r)\eta^l = \Lambda_{\mu,\psi}^{n,p}(x_1+x_2,y_1+y_2;z_1,...,z_r;\eta), \tag{4.2}$$

provided that each member of (4.2) exists.

Proof. Applying the well-known equality

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} A(k,l) = \sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl} A(k+pl,l)$$

and then using Lemma 3.5, we get

$$\begin{aligned} &\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l \binom{n-pl}{k-pl} P_{n-k}^{(j,c)}(x_1,y_1)P_{k-pl}^{(j,c)}(x_2,y_2)\Omega_{\mu+\psi l}(z_1,...,z_r)\eta^l \\ &= \sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl} a_l \binom{n-pl}{k} P_{n-k-pl}^{(j,c)}(x_1,y_1)P_k^{(j,c)}(x_2,y_2)\Omega_{\mu+\psi l}(z_1,...,z_r)\eta^l \\ &= \sum_{l=0}^{[n/p]} a_l\left(\sum_{k=0}^{n-pl}\binom{n-pl}{k} P_{n-k-pl}^{(j,c)}(x_1,y_1)P_k^{(j,c)}(x_2,y_2)\right)\Omega_{\mu+\psi l}(z_1,...,z_r)\eta^l \\ &= \sum_{l=0}^{[n/p]} a_l P_{n-pl}^{(j,c)}(x_1+x_2,y_1+y_2)\Omega_{\mu+\psi l}(z_1,...,z_r)\eta^l \\ &= \Lambda_{\mu,\psi}^{n,p}(x_1+x_2,y_1+y_2;z_1,...,z_r;\eta), \end{aligned}$$

which completes the proof. □

5. Special Cases

As an application of the above theorems, when the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $r \in \mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems.
We first set, $r = m$ and take

$$\Omega_{\mu+\psi k}(y_1,...,y_m) \quad = \quad \Phi^{(\alpha)}_{\mu+\psi k}(y_1,...,y_m) \ (\mu, p \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}),$$

in Theorem 4.1. Recall that, by $\Phi^{(\alpha)}_{\mu+pk}(y_1,...,y_m)$, we denote the multivariable polynomials (see, e.g., [12]) generated by

$$\sum_{n=0}^{\infty}\Phi^{(\alpha)}_{n}(x_1,...,x_r)t^n \quad = \quad (1-x_1t)^{-\alpha}\exp(x_2+...+x_r)t, (|t|<|x_1|^{-1}). \tag{5.1}$$

Then from Theorem 4.1, we obtain the following result which is a class of bilateral generating functions for the multivariable polynomials $\Phi_{\mu+\psi k}(x_1,...,x_r)$ and generalized Gould-Hopper polynomials $P^{(j,c)}_{\mu+\psi k}(x,y)$.

Corollary 5.1. *If*

$$\Lambda_{\mu,\psi}(z_1,...,z_r,\zeta) := \sum_{k=0}^{\infty} a_k\Phi^{(\alpha)}_{\mu+\psi k}(z_1,...,z_r)\zeta^k \quad (a_k \neq 0, \ \mu,\psi\in\mathbb{C}),$$

and

$$\Theta^{\mu,\psi}_{n,p}(x,y;z_1,...,z_r;\xi) := \sum_{k=0}^{[n/p]} a_kP^{(j,c)}_{n-pk}(x,y)\Phi^{(\alpha)}_{\mu+\psi k}(z_1,...,z_r)\frac{\xi^k}{(n-pk)!}.$$

Then, we have

$$\sum_{n=0}^{\infty}\Theta^{\mu,\psi}_{n,p}\left(x,y;z_1,...,z_r;\frac{\eta}{t^p}\right)t^n = c^{xt+yt^j}\Lambda_{\mu,\psi}(z_1,...,z_r;\eta).$$

Remark 5.2. *Using the generating relation (5.1) for the multivariable polynomials $\Phi_{\mu+\psi k}(z_1,...,z_m)$ and getting $a_k=1$, $\mu=0$, $\psi=1$ in Corollary 5.1, we find that*

$$\Lambda_{0,1}(z_1,...,z_m,\zeta) := \sum_{k=0}^{\infty}\Phi^{(\alpha)}_{k}(z_1,...,z_m)\zeta^k$$

and

$$\Theta^{0,1}_{n,p}(x,y;z_1,...,z_m;\xi) := \sum_{k=0}^{[n/p]} {}_kP^{(j,c)}_{n-pk}(x,y)\Phi^{(\alpha)}_{k}(z_1,...,z_m)\frac{\xi^k}{(n-pk)!}.$$

Then, we have

$$\sum_{n=0}^{\infty}\Theta^{0,1}_{n,p}\left(x,y;z_1,...,z_m;\frac{\eta}{t^p}\right)t^n = c^{xt+yt^j}(1-z_1t)^{-\alpha}\exp(z_2+...+z_m)\eta.$$

If we set $r=2$, $z_1=x_3$, $z_2=y_3$,

$$\Omega_{\mu+\psi k}(x_3,y_3) = P^{(j,c)}_{\mu+\psi k}(x_3,y_3),$$

in Theorem 4.2.

Corollary 5.3. *If*

$$\Lambda^{n,p}_{\mu,\psi}(x_1+x_2,y_1+y_2;x_3,y_3;\eta) := \sum_{k=0}^{[n/p]} a_kP^{(j,c)}_{n-pk}(x_1+x_2,y_1+y_2)P^{(j,c)}_{\mu+\psi k}(x_3,y_3)\eta^k, \ (a_k\neq 0, \mu,\psi\in\mathbb{C}, n,p\in\mathbb{N}).$$

Then, we have

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l\begin{pmatrix} n-pl \\ k-pl\end{pmatrix}P^{(j,c)}_{n-k}(x_1,y_1)P^{(j,c)}_{k-pl}(x_2,y_2)P^{(j,c)}_{\mu+\psi l}(x_3,y_3)\eta^l = \Lambda^{n,p}_{\mu,\psi}(x_1+x_2,y_1+y_2;x_3,y_3;\eta).$$

Remark 5.4. *Using Eq. (**??**) and taking $a_l = \begin{pmatrix} n \\ l\end{pmatrix}$, $\mu=0$, $\psi=1$, $p=1$, $\eta^l=\eta^k=1$ in Corollary 5.2, we have*

$$\Lambda^{n,p}_{\mu,\psi}(x_1+x_2,y_1+y_2;z_1,z_2;\eta) := \sum_{k=0}^{[n/p]} a_kP^{(j,c)}_{n-pk}(x_1+x_2,y_1+y_2)P^{(j,c)}_{\mu+\psi k}(x_3,y_3).$$

Then, we have

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}\begin{pmatrix} n \\ l\end{pmatrix}\begin{pmatrix} n-pl \\ k-pl\end{pmatrix}P^{(j,c)}_{n-k}(x_1,y_1)P^{(j,c)}_{k-pl}(x_2,y_2)P^{(j,c)}_{\mu+\psi l}(x_3,y_3) = P^{(j,c)}_{n}(x_1+x_2+x_3,y_1+y_2+y_3).$$

Corollary 5.5. *Taking $j=2,\ c=e$, in Eqs. (3.1), (3.5) and (3.6) and using Eq. (1.8), we get the following summation formulae for the $H_n(x,y)$:*

$$H_{n+m}(w,y)=\sum_{k=}^{n}\sum_{r=0}^{m}\binom{n}{k}\binom{m}{r}(w-x)^{k+r}H_{n+m-k-r}(x,y),$$

$$H_n(w,y)=\sum_{k=0}^{n}\binom{n}{k}(w-x)^kH_{n-k}(x,y),$$

$$H_n(w+x,y)=\sum_{k=0}^{n}\binom{n}{k}w^kH_{n-k}(x,y),$$

respectively. Again, taking $j=2,\ c=e$, in Eq. (3.7) and using Eq. (1.8), we get the following summation formulae involving products of the $H_n(x,y)$:

$$\frac{\left(\frac{x}{w}\right)^r\left(\frac{X}{W}\right)^s}{r!s!}H_r(w,y)H_s(W,y)=\sum_{k=0}^{\left[\frac{r}{2}\right]}\sum_{l=0}^{\left[\frac{s}{2}\right]}\frac{y^{k+l}[(\frac{x}{w})^2-1]^k[(\frac{X}{W})^2-1]^l}{(r-2k)!k!(s-2l)!l!}H_{r-2k}(x,y)H_{s-2l}(X,y).$$

Corollary 5.6. *Taking $y=-1,\ j=2,\ c=e$ and replacing x by vx, w by vw in Eqs. (3.1), (3.5) and (3.6) and using Eq. (1.9), we get the following summation formulae for the generalized Hermite polynomials $H_{n,m,v}(x)$:*

$$H_{n+m,j,v}(w)=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(v)^{n+r}(w-x)^{n+r}H_{n+m-k-r,j,v}(x),$$

$$H_{n,j,v}(w)=\sum_{n=0}^{k}\binom{k}{n}(v)^n(w-x)^nH_{n-k,j,v}(x),$$

$$H_{n,j,v}(w+x)=\sum_{n=0}^{k}\binom{k}{n}(vw)^{k-n}H_{n-k,j,v}(x),$$

respectively. Also, replacing x by vx, w by vw, X by vX, W by vW in Eq. (3.7) and using Eq. (1.9), we get the following summation formula involving products of the generalized Hermite polynomials $H_{n,m,v}(x)$:

$$\frac{\left(\frac{x}{w}\right)^r\left(\frac{X}{W}\right)^s}{r!s!}H_{r,m,v}(w)H_{s,m,v}(W)=\sum_{k=0}^{\left[\frac{r}{m}\right]}\sum_{l=0}^{\left[\frac{s}{m}\right]}\frac{[1-(\frac{x}{w})^m]^k[1-(\frac{X}{W})^m]^l}{(r-mk)!k!(s-ml)!l!}H_{r-mk,m,v}(x)H_{s-ml,m,v}(X).$$

We now discuss some miscellaneous recurrence relations of the G-GHP $P_n^{(j,c)}(x,y)$ given by (1.6). By differentiating each member of the generating function relation (1.7) with respect to x and using

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k),$$

we arrive at the following (differential) recurrence relation for the G-GHP $P_n^{(j,c)}(x,y)$ polynomials:

$$\frac{\partial}{\partial x}P_n^{(j,c)}(x,y)=n\ln c.P_{n-1}^{(j,c)}(x,y).$$

6. Conclusion

In this paper, we have established some generating functions for the generalized Gould Hopper polynomials by using series rearrangement techniques. Also, some summation formulae for that polynomials are derived by using certain operational techniques and by using different analytical means on its generating function. Further, many generating functions and summation formulae for the polynomials related to generalized-Gould Hopper polynomials are obtained as applications of main results. The approach presented in this paper is general and can be extended to establish other properties of special polynomials.

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