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Symmetry in Complex Contact Manifolds

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ABSTRACT

We define complex locally $\mathcal{H}$ -symmetric spaces. As an example we prove that complex (κ,μ) -spaces with $\kappa<1$ are locally $\mathcal{H}$ -symmetric.

Keywords:
Complex contact geometry; Symmetry; Local symmetry

INTRODUCTION

Takahashi defined local φ -symmetry for Sasakian manifolds by the curvature condition that

$$g((\nabla_X R)(Y,Z)W,T)=0 \qquad (1)$$

for all horizontal vector fields X,Y,Z,W,T ([12]). There are two generalizations to contact metric manifolds. In [2], contact metric manifolds satisfying the curvature condition (1.1) are called locally φ -symmetric. In [6] another definition is given. A contact metric manifold is called locally φ -symmetric if characteristic reflections are local isometries. This condition leads to infinitely many curvature conditions including the above condition (1.1). Boeckx proved that (κ,μ) -spaces satisfy this condition ([5]). This gives a set of non Sasakian examples.

Symmetry for complex contact metric manifolds is studied by Blair and Mihai in [3], [4]. They defined a complex contact metric manifold to be *GH-locally symmetric* if the reflections in the integral submanifolds of the vertical bundle are isometries. They also proved in [4] that a complex (κ,μ) -space with $\kappa<1$ is GH-locally symmetric.

In this paper, we will use the first generalization of local symmetry and define a complex contact metric manifold to be locally $\mathcal{H}$ -symmetric (in order not to confuse with GH-locally symmetric) if it satisfies the curvature condition (1) and we will give a simple and detailed proof showing that complex (κ,μ) -spaces with $\kappa<1$ satisfy this condition.

PRELIMINARIES

Let M be a complex manifold of dimension $2n+1$. It is called a *complex contact manifold* if it has an open covering $\{\mathcal{O}\}$ of coordinate neighborhoods such that:

1) On each $\mathcal{O}$ there is a holomorphic 1-form ω such that $\omega\wedge(d\omega)^n\neq 0$,

2) On $\mathcal{O}\cap\mathcal{O}'\neq\varnothing$ there is a non-vanishing holomorphic function f such that $\omega'=f\omega$.

The complex contact structure determines a non-integrable subbundle $\mathcal{H}$ by the equation $\omega=0$; $\mathcal{H}$ is called the *complex contact subbundle* or simply the *horizontal subbundle.*

On a complex contact manifold M, there is a Hermitian metric g, local (real) 1 forms u and $v=u\circ J$, local (real) dual vector fields U and $V=-JU$, and (1,1) tensor fields G and $H=GJ$ such that:

1)
$$\begin{aligned}
&G^2=H^2=-I+u\otimes U+v\otimes V,\\
&G^2=H^2=-I+u\otimes U+v\otimes V,\\
&g(U,X)=u(X),\quad g(X,GY)=-g(GX,Y),\\
&GJ=-JG,\quad GU=0,\quad u(U)=1,
\end{aligned}$$

2) On $\mathcal{O}\cap\mathcal{O}'\neq\varnothing$

$$\begin{aligned}
&u'=Au-Bv,\quad v'=Bu+Av,\\
&G'=AG-BH,\quad H'=BG+AH
\end{aligned}$$

where A and B are functions with $A^2+B^2=1$.

As a result of these conditions, the following identities also hold:

3) $HG=-GH=J+u\otimes V-v\otimes U,$
$JH=-HJ=G,\quad g(HX,Y)=g(X,HY),$
$GV=HU=HV=0,\quad uG=vG=uH=vH=0,$
$JV=U,\quad g(U,V)=0,$
$du(X,Y)=g(X,GY)+(\sigma\wedge v)(X,Y),$
$dv(X,Y)=g(X,HY)-(\sigma\wedge u)(X,Y)$

where $\sigma(X)=g(\nabla_X U,V)$, ∇ being the Levi-Civita connection of g (see [1], [7] and [9]).

Here $\omega=f(u-iv)$ where f is a non-vanishing complex-valued function. Also, on the intersections the subbundle generated by U and V is the same as the subbundle generated by U' and V'. Hence we have a global bundle $\mathcal{V}$ orthogonal to $\mathcal{H}$. This bundle is called the *vertical subbundle* and it is typically assumed to be integrable. We refer to a complex contact manifold with the above structure tensors satisfying these conditions as a complex *contact metric manifold*.

In order to split the covariant derivatives of U and V into symmetric and skew-symmetric parts, we define two other local structure tensors:

$$h_U=\frac{1}{2}\mathrm{sym}\mathcal{L}_U G\circ p \text{ and } h_V=\frac{1}{2}\mathrm{sym}\mathcal{L}_V H\circ p$$

where ``sym'' denotes the symmetric part and p denotes the projection $TM\to\mathcal{H}$. These operators satisfy the following properties [2,8]:

$$\begin{aligned}&h_U G=-Gh_U,\quad h_V H=-Hh_V,\\&h_U U=h_U V=h_V U=h_V V=0,\\&\nabla_X U=-GX-Gh_U X+\sigma(X)V,\\&\nabla_X V=-HX-Hh_V X-\sigma(X)U.\end{aligned}$$

In order to define a complex (κ,μ)-space, we consider complex contact metric manifold M with $h_U=h_V=h$. In this case, h anti-commutes with G and H, and hence commutes with J. If the following curvature conditions hold for some constants κ and μ, then M is called a *complex* (κ,μ)*-space* ([11]):

$$\begin{aligned}R(X,Y)U=&\kappa(u(Y)X-u(X)Y)+\mu(u(Y)hX-u(X)hY)\\&+(\kappa-\mu)(v(Y)JX-v(X)JY)\\&+2((\kappa-\mu)g(JX,Y)+(4\kappa-3\mu)u\wedge v(X,Y))V,\end{aligned}\tag{2}$$

$$\begin{aligned}R(X,Y)V=&\kappa(v(Y)X-v(X)Y)+\mu(v(Y)hX-v(X)hY)\\&-(\kappa-\mu)(u(Y)JX-u(X)JY)\\&-2((\kappa-\mu)g(JX,Y)+(4\kappa-3\mu)u\wedge v(X,Y))U,\end{aligned}\tag{3}$$

$$\Omega(X,Y)=(2-\mu)g(JX,Y)+2g(JhX,Y)+2((2-\mu)u\wedge v(X,Y)).\tag{4}$$

Here $\Omega=d\sigma$.

The following theorem is proved in [11].

Theorem 1

Let M be a complex (κ,μ)-space. Then $\kappa\le 1$. If $\kappa=1$, then $h=0$ and M is normal. If $\kappa<1$, then M admits three mutually orthogonal distributions $[0]$, $[\lambda]$ and $[-\lambda]$, defined by the eigenspaces of h, where $\lambda=\sqrt{1-\kappa}$.

Curvature of a complex (κ,μ)-space is completely determined. For details see [11].

Curvature of complex (κ,μ)-spaces

In this section we will write the curvature tensor for a complex (κ,μ)-space. In the expression for the curvature tensor there are several terms. In order to give a simpler expression if we group some terms, we come up with the following tensors which are defined for vector fields X, Y:

$$\begin{aligned}&A(X,Y)=g(X,hY)+(1-\mu/2)g(X,Y),\\&B(X,Y)=g(X,Y)+(2-\mu)/(2\lambda^2)g(X,hY),\\&C(X,Y)=u(X)((\kappa-1+\mu/2)Y+(\mu-1)hY),\\&D(X,Y)=v(X)((\kappa-1-\mu/2)JY-hJY).\end{aligned}$$

Here A,B are $(0,2)$ tensors and C,D are $(1,2)$ tensors.

We also define the following $(0,3)$ tensors:

$$\begin{aligned}f(X,Y,Z)=&g\big(C(X,Y)+D(X,Y)-C(Y,X)-D(Y,X),Z\big)\\&+2g(D(Z,Y),X)-4(2\kappa-1-\mu)v(Z)2u\wedge v(X,Y),\end{aligned}$$

$$\begin{aligned}k(X,Y,Z)=&g\big(C(JX,Y)+D(JX,Y)-C(JY,X)-D(JY,X),Z\big)\\&+2g(D(JZ,Y),X)-4(2\kappa-1-\mu)u(Z)2u\wedge v(X,Y).\end{aligned}$$

Note that when the vector fields are horizontal, the tensors C,D,f and k vanish.

Theorem 2

Let M be a complex (κ,μ)-space with $\kappa<1$. Then, for vector fields X,Y,Z, the curvature tensor is given by

$$
\begin{aligned}
R(X,Y)Z = &(A(Y,Z)+(\kappa-1+\frac{\mu}{2})(u(Y)u(Z)+v(Y)v(Z)))X \\
&-(A(X,Z)+(\kappa-1+\frac{\mu}{2})(u(X)u(Z)+v(X)v(Z)))Y \\
&+(B(Y,Z)+(\mu-1)(u(Y)u(Z)+v(Y)v(Z)))hX \\
&-(B(X,Z)+(\mu-1)(u(X)u(Z)+v(X)v(Z)))hY \\
&-(A(Y,JZ)+(\kappa-1-\frac{\mu}{2})2u\wedge v(Y,Z))JX \\
&+(A(X,JZ)+(\kappa-1-\frac{\mu}{2})2u\wedge v(X,Z))JY \\
&+(2A(X,JY)+(2\kappa-2-\mu)2u\wedge v(X,Y))JZ \\
&-(B(Y,JZ)-2u\wedge v(Y,Z))hJX \\
&+(B(X,JZ)-2u\wedge v(X,Z))hJY \\
&+(2B(X,JY)-4u\wedge v(X,Y))hJZ \\
&+\frac{\mu}{2}(g(Y,GZ)GX-g(X,GZ)GY \\
&+g(Y,HZ)HX-g(X,HZ)HY) \\
&+\frac{2\kappa-\mu}{2\lambda^2}(g(Y,hGZ)hGX-g(X,hGZ)hGY \\
&+g(Y,hHZ)hHX-g(X,hHZ)hHY) \\
&+\mu(g(Y,GX)GZ+g(Y,HX)HZ) \\
&+f(X,Y,Z)U+k(X,Y,Z)V.
\end{aligned}
$$

Proof

First, we write any vector field X uniquely as

$$X = X_{\lambda} + X_{-\lambda} + u(X)U + v(X)V$$

where $X_{\lambda} \in [\lambda]$ and $X_{-\lambda} \in [-\lambda]$. We can write the terms $R(X_{\pm\lambda}, Y_{\pm\lambda})Z_{\pm\lambda}$ using the formulas given in [11]. The terms $R(X,Y)U$, $R(X,Y)V$, $R(U,X)Y$, $R(V,X)Y$, $R(X,U)Y$ and $R(X,V)Y$, can be computed by using the conditions (2) and (3). Then, by using the identities

$$X_{\lambda} = \frac{1}{2}\left(X + \frac{1}{\lambda}hX - u(X)U - v(X)V\right),$$

$$X_{-\lambda} = \frac{1}{2}\left(X - \frac{1}{\lambda}hX - u(X)U - v(X)V\right),$$

we obtain the formula in the theorem. Keep in mind that $hX_{\lambda} = \lambda X_{\lambda}$, $hX_{-\lambda} = -\lambda X_{-\lambda}$ and $hU = hV = 0$. □

When the vector fields are horizontal, the above expression simplifies to

$$
\begin{aligned}
R(X,Y)Z = &A(Y,Z)X - A(X,Z)Y + B(Y,Z)hX - B(X,Z)hY \\
&- A(Y,JZ)JX + A(X,JZ)JY + 2A(X,JY)JZ \\
&- B(Y,JZ)hJX + B(X,JZ)hJY + 2B(X,JY)hJZ \\
&+\frac{\mu}{2}(g(Y,GZ)GX-g(X,GZ)GY \\
&+g(Y,HZ)HX-g(X,HZ)HY) \\
&+\frac{2\kappa-\mu}{2\lambda^2}(g(Y,hGZ)hGX-g(X,hGZ)hGY \\
&+g(Y,hHZ)hHX-g(X,hHZ)hHY) \\
&+\mu(g(Y,GX)GZ+g(Y,HX)HZ.
\end{aligned}
$$

Now we can state and prove our main theorem.

Theorem 3

Let M be a complex (κ,μ)-space with $\kappa < 1$. Then, for horizontal vector fields X,Y,Z and W, we have

$$(\nabla_W R)(X,Y)Z = 0.$$

Proof

For a horizontal fields X,Y,Z and W, we need to compute

$$
\begin{aligned}
(\nabla_W R)(X,Y)Z = &\nabla_W R(X,Y)Z - R(\nabla_W X,Y)Z \\
&- R(X,\nabla_W Y)Z - R(X,Y)\nabla_W Z.
\end{aligned}
$$

First, let us compare the coefficients of X in the 4 terms above. From $\nabla_W R(X,Y)Z$ we have

$$
\begin{aligned}
W(A(Y,Z)) = &g(\nabla_W Y,hZ) + g(Y,\nabla_W hZ) \\
&+(1-\mu/2)(g(\nabla_W Y,Z)+g(Y,\nabla_W Z)).
\end{aligned}
$$

The coefficient of X in $R(X,\nabla_W Y)Z$ is

$$A(\nabla_W Y,Z) = g(\nabla_W Y,hZ) + (1-\mu/2)g(\nabla_W Y,Z),$$

and in $R(X,Y)\nabla_W Z$ is

$$A(Y,\nabla_W Z) = g(Y,h\nabla_W Z) + (1-\mu/2)g(Y,\nabla_W Z).$$

So the coefficient of X in $(\nabla_W R)(X,Y)Z$ is $g(Y,(\nabla_W h)Z)$.

By Lemma 3.5 in [11], for horizontal fields W,Z the covariant derivative of h is given by

$$
\begin{aligned}
(\nabla_W h)Z = &(g(W,hGZ)-(\kappa-1)g(W,GZ))U \\
&+(g(W,hHZ)-(\kappa-1)g(W,HZ))V
\end{aligned}
$$

and hence $g(Y,(\nabla_W h)Z) = 0$.

In $\nabla_W R(X,Y)Z$ we also have the term $A(Y,Z)\nabla_W X$ but that term also appears in $R(\nabla_W X,Y)Z$ and they cancel each other out.

Similarly the coefficient of Y also vanishes and the term $A(X,Z)\nabla_W Y$ in $\nabla_W R(X,Y)Z$ cancels out with its counterpart in $R(X,\nabla_W Y)Z$.

Similar situation happens with the terms hX and hY.

For the terms with JX, JY and JZ, we need $(\nabla_W J)Z$ and $(\nabla_W hJ)Z$. Since W and Z are horizontal, using Lemma 3.1, part (v) in [11] we can write

$$(\nabla_W J)Z = -\mu u(W)HZ + \mu v(W)GZ = 0,$$

and

$$(\nabla_W hJ)Z = (\nabla_W h)JZ + h(\nabla_W J)Z = (\nabla_W h)JZ.$$

Now, if we compute the coefficient of JX in $(\nabla_W R)(X,Y)Z$ we get

$$g(Y,(\nabla_W hJ)Z) + (1-\mu/2)g(Y,(\nabla_W J)Z) = 0.$$

Similarly, the coefficients of JY and JZ vanish also.

Differentiating the term with JX we also get

$$-A(Y,JZ)\nabla_W JX + A(Y,JZ)J\nabla_W X = -A(Y,JZ)(\nabla_W J)X = 0.$$

Similarly for JY and JZ.

Same thing happens with the terms hJX, hJY and hJZ.

By Lemma 3.1, part (v) in [11], for horizontal fields X and W we have

$$(\nabla_W G)X = \sigma(W)HX, (\nabla_W H)X = -\sigma(W)GX.$$

So, by differentiating the term GX we get

$$\begin{matrix}(\mu/2)(g(Y,(\nabla_W G)Z)GX \\ +g(Y,GZ)(\nabla_W G)X)\end{matrix} = \begin{matrix}(\mu/2)\sigma(W)(g(Y,HZ)GX \\ +g(Y,GZ)HX)\end{matrix}.$$

By differentiating the term HX we get

$$\begin{matrix}(\mu/2)(g(Y,(\nabla_W H)Z)HX \\ +g(Y,HZ)(\nabla_W H)X)\end{matrix} = \begin{matrix}-(\mu/2)\sigma(W)(g(Y,GZ)HX \\ +g(Y,HZ)GX)\end{matrix}$$

and they cancel out. Similarly the terms we get from GY and HY, and the terms we get from GZ and HZ cancel each other out.

Same thing happens with the terms hGX and hHX, and with the terms hGY and hHY.

We conclude that, in a complex (κ,μ)-space with $\kappa < 1$, for horizontal vector fields $(\nabla_W R)(X,Y)Z = 0$. □

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