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Research Article

Approximation properties related to the Bell polynomials

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ABSTRACT. The authors provide a complete asymptotic expansion for a class of functions in terms of the complete Bell polynomials. In particular, they obtain known asymptotic expansions of some Keller type sequences.

Keywords: Asymptotic expansions, Bell polynomials.

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Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

The first references to the number e were published in 1618 in the table of an appendix of a work on logarithms by John Napier [1, p. xiii]. The discovery of the constant itself is credited to Jacob Bernoulli in 1690 who considered the problem of continuous compounding of interest,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n .$$

Leonhard Euler introduced the letter e as the base for natural logarithms, writing in a letter to Christian Goldbach on 25 November 1731. In 1665, Newton [1, p. 151] discovered

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots .$$

Let us consider the sequence

$$(n+1)\left(1+\frac{1}{n+1}\right)^{n+1} - n\left(1+\frac{1}{n}\right)^n, \qquad n = 1, 2, \ldots . \tag{1.1}$$

The sequence (1.1) is attributed to Felix A. Keller (see, e.g., [2], [3, p. 14], but its origin surely lay in the Euler age). In 1998, H. J. Brothers and J. A. Knox [4, Eq. (8)] gave the following approximation to e,

$$\begin{aligned}
&\frac{(x+1)^{x+1}}{x^x} - \frac{x^x}{(x-1)^{x-1}} \\
=\ &(1+x)\left(1+\frac{1}{x}\right)^x + (1-x)\left(1-\frac{1}{x}\right)^{-x} \\
=\ &e\left(1+\frac{1}{24x^2}+\frac{11}{640x^4}+\frac{5525}{580608x^6}+\mathcal{O}\left(\frac{1}{x^8}\right)\right), \qquad x\to\infty.
\end{aligned}$$

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We recall an excellent result of Alzer and Berg [5, (2.2)]:

$$(x+1)\left(e-\left(1+\frac{1}{x}\right)^x\right)=\frac{e}{2}+\frac{1}{\pi}\int_0^1 \frac{s^s(1-s)^{1-s}\,\sin(\pi\, s)}{x+s}\mathrm{d}s,\qquad x>0. \tag{1.2}$$

Next, using the identity

$$\frac{1}{x+s}=\frac{1}{x+a}\sum_{n=0}^{\infty}\left(\frac{a-s}{x+a}\right)^n,\qquad a\geq 0,\ |a-s|<|x+a|,\ \text{for } s\in[0,1],$$

from (1.2), we deduce that

$$\begin{aligned}(x+1)\left(e-\left(1+\frac{1}{x}\right)^x\right)\\ =\frac{e}{2}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{(x+a)^{n+1}}\int_0^1 s^s(1-s)^{1-s}\,\sin(\pi\, s)\,(a-s)^n\,\mathrm{d}s.\end{aligned} \tag{1.3}$$

For $a=\frac{11}{12}$, (1.3) yields the result of Mortici and Hu [6, (3.1)];
for $a=1$, (1.3) gives an expansion in [7];
for $a=0$, $x>1$, (1.3) becomes

$$(x+1)\left(e-\left(1+\frac{1}{x}\right)^x\right)=\frac{e}{2}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{x^{n+1}}\int_0^1 s^s(1-s)^{n+1-s}\,\sin(\pi\, s)\mathrm{d}s. \tag{1.4}$$

We will review the integrals involved in (1.4) in subsection 3.1. The main result of the paper is the series expansion (2.9) of the function $x\mapsto\left(1+\dfrac{1}{x+a}\right)^{x+b}$ in terms of Bell polynomials. This extends many known results.

1.1. **Complete asymptotic expansion.** Let $(b_n)_{n\geq 0}$ be a sequence of real numbers and $f\colon(0,\infty)\to\mathbb{R}$. Use the symbol $\mathcal{O}$ for Landau's big "O" notation. We recall that $\sum\limits_{n=0}^{\infty}\frac{b_n}{x^n}$ is said to be a complete asymptotic expansion of f as $x\to\infty$, and use the notation

$$f(x)\sim\sum_{n=0}^{\infty}\frac{b_n}{x^n},\qquad \text{as } x\to\infty,$$

if

$$f(x)=\sum_{n=0}^{p}\frac{b_n}{x^n}+\mathcal{O}(x^{-p-1}),\qquad \text{as } x\to\infty$$

for all integers $p\geq 0$.

1.2. **The Bell polynomials.** Let $(x_n)_{n\geq 1}$ be a sequence of numbers. The *complete exponential Bell polynomials* $B_n(x_1,\ldots,x_n)$ (see, e.g., [8, Chapter 2, Section 8], [9, p. 134]) denoted in the sequel by $\mathbf{Bell}_n[x_i]$, are given by the *formal* series identity

$$\exp\left(\sum_{i=1}^{\infty}x_i\frac{t^i}{i!}\right)=\sum_{n=0}^{\infty}\mathbf{Bell}_n[x_i]\,\frac{t^n}{n!}. \tag{1.5}$$

They may be recursively defined as

$$\mathbf{Bell}_0[x_i]:=1,\quad \mathbf{Bell}_{n+1}[x_i]=\sum_{j=0}^{n}\binom{n}{j}\mathbf{Bell}_{n-j}[x_i]\,x_{j+1},\qquad n=0,1,\ldots. \tag{1.6}$$

The following table can be obtained immediately from (1.6).

$$\begin{array}{lcl}
\mathbf{Bell}_0[x_i] &=& 1,\\
\mathbf{Bell}_1[x_i] &=& x_1,\\
\mathbf{Bell}_2[x_i] &=& x_1^2+x_2,\\
\mathbf{Bell}_3[x_i] &=& x_1^3+3x_2x_1+x_3,\\
\mathbf{Bell}_4[x_i] &=& x_1^4+6x_2x_1^2+4x_3x_1+3x_2^2+x_4,\\
\mathbf{Bell}_5[x_i] &=& x_1^5+10x_2x_1^3+10x_3x_1^2+15x_2^2x_1+5x_4x_1+10x_2,x_3+x_5.
\end{array}$$

2. Main Results

It is well known that if $z \mapsto g(z)$ is holomorphic in the disk $|z| < R$, then $z \mapsto \exp(g(z))$ is holomorphic in the disk $|z| < R$. In consequence, the power series expansion of $\exp(g(z))$ has a radius of convergence at least R. So, if the power series $\sum\limits_{i=1}^{\infty} x_i \frac{t^i}{i!}$ has the radius of convergence $R > 0$, then the formal equality (1.5) becomes an equality

$$\exp\left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!}\right) = \sum_{n=0}^{\infty} \mathbf{Bell}_n[x_i]\frac{t^n}{n!}, \quad |t| < R. \tag{2.7}$$

The following is the main result of the paper.

Theorem 2.1. *Let $a, b \in \mathbb{R}$. For*

$$x_{i,a,b} := (-1)^i i!\left(\frac{(a+1)^{i+1}-a^{i+1}}{i+1} - \frac{b\left((a+1)^i - a^i\right)}{i}\right), \quad i = 1, 2, \ldots, \tag{2.8}$$

we obtain the following equality

$$\left(1+\frac{1}{x+a}\right)^{x+b} = e\sum_{n=0}^{\infty}\frac{\mathbf{Bell}_n[x_{i,a,b}]}{n!}\frac{1}{x^n}, \quad |x| > \max(|a|, |a+1|). \tag{2.9}$$

Proof. The following expansion can be obtained by simple calculation,

$$\begin{aligned}
&(x+b)\log\left(1+\frac{1}{x+a}\right)\\
=&(x+b)\log\frac{1+\frac{a+1}{x}}{1+\frac{a}{x}}\\
=&1+\sum_{i=1}^{\infty}(-1)^i i!\left(\frac{(a+1)^{i+1}-a^{i+1}}{i+1} - \frac{b\left((a+1)^i-a^i\right)}{i}\right)\frac{1}{i!x^i},
\end{aligned} \tag{2.10}$$

$|x| > \max(|a|, |a+1|)$. Using (2.7), the proof is complete. □

In particular, we obtain:

Example 2.1. *The following asymptotic expansions hold true:*

$$\begin{aligned}
&\left(1+\frac{1}{x+a}\right)^{x+b}\\
= &\; e\\
&-\frac{e(2a-2b+1)}{2}\frac{1}{x}\\
&+\frac{e\left(36a^2-48ab+36a+12b^2-24b+11\right)}{24}\frac{1}{x^2}\\
&-\frac{e\left(104a^3-168a^2b+156a^2+72ab^2-168ab+94a-8b^3+36b^2-50b+21\right)}{48}\frac{1}{x^3}\\
&+\mathcal{O}(x^{-4}), \qquad x\to\infty,
\end{aligned}$$

$$\left(1+\frac{1}{x+a-\frac{1}{4}}\right)^{x+a+\frac{1}{4}} = e+\mathcal{O}(x^{-2}), \qquad x\to\infty.$$

3. Applications

All known or new results in this section stem from Theorem 2.1.

Corollary 3.1. *From (2.9), we deduce*

$$(3.11)\qquad (x+c)\left(1+\frac{1}{x+a}\right)^{x+b}-ex = e\sum_{k=0}^{\infty}\left(\frac{\mathbf{Bell}_{k+1}[x_{i,a,b}]}{(k+1)!}+c\frac{\mathbf{Bell}_{k}[x_{i,a,b}]}{k!}\right)\cdot\frac{1}{x^k}$$

and, in particular,

$$\begin{aligned}
&(x+c)\left(\frac{1}{x+a}+1\right)^{x+b}-ex\\
&=-\frac{1}{2}e(2a-2b-2c+1)\\
&+e\left(36a^2-48ab-24ac+36a+12b^2+24bc-24b-12c+11\right)\frac{1}{24x}\\
&-e\Big(104a^3-168a^2b-72a^2c+156a^2+72ab^2+96abc-168ab\\
&-72ac+94a-8b^3-24b^2c+36b^2+48bc-50b-22c+21\Big)\frac{1}{48x^2}\\
&+\mathcal{O}(x^{-3}).
\end{aligned}$$

We note that particular cases of (2.9) can be found, e.g., in papers of H. J. Brothers and J. A. Knox [4, 10], C. Mortici and X.-J. Jang [11], C. Mortici and Y. Hu [6].

3.1. **Evaluating the integrals in (1.4).** In this subsection, we obtain the following evaluation of the integrals involved in (1.4).

Proposition 3.1.

$$(3.12)\qquad J_k := \int_0^1 s^s(1-s)^{k-s}\sin(\pi s)\,\mathrm{d}s = (-1)^k\pi\, e\left(\frac{\mathbf{Bell}_{k+1}[x_{i,0,0}]}{(k+1)!}+\frac{\mathbf{Bell}_{k}[x_{i,0,0}]}{k!}\right),$$

$k=1,2,\dots$.

Proof. Taking $a=0$, $b=0$, and $c=1$ in (3.11), we obtain

$$(3.13)\quad (x+1)\left(1+\frac{1}{x}\right)^x - ex = e\sum_{k=0}^{\infty}\left(\frac{\mathbf{Bell}_{k+1}[x_{i,0,0}]}{(k+1)!}+\frac{\mathbf{Bell}_k[x_{i,0,0}]}{k!}\right)\cdot\frac{1}{x^k},\qquad x>1,$$

where

$$(3.14)\qquad x_{0,0,i}=\frac{(-1)^i i!}{i+1},\qquad i=1,2,\ldots.$$

Comparing (1.4) with (3.13), we succeeded in calculating the integrals (3.12). □

For example,

$$\begin{aligned}\int_0^1 s^s(1-s)^{1-s}\sin(\pi s)\,\mathrm{d}s = J_1 &= \frac{e\pi}{24},\\ J_2 &= \frac{e\pi}{48},\\ J_3 &= \frac{73e\pi}{5760},\\ J_4 &= \frac{11e\pi}{1280}.\end{aligned}$$

Note that MATHEMATICA and other assistant software failed to evaluate the integrals (3.12).

3.2. **A generalized Keller function.** Extend now the Keller sequence (1.1) to the function

$$(3.15)\qquad K(a,b,c;x) := (x+c)\left(1+\frac{1}{x+a}\right)^{x+b} - (x+c-1)\left(1+\frac{1}{x+a-1}\right)^{x+b-1} - e,$$

$|x| > \max(|a-1|, |a|, |a+1|) = |a|+1$.
From (3.11), we obtain

$$(3.16)\qquad K(a,b,c;x) = e\sum_{k=2}^{\infty}\frac{1}{x^k}\left(\frac{c\,\mathbf{Bell}_k[x_{i,a,b}]-(c-1)\,\mathbf{Bell}_k[x_{i,a-1,b-1}]}{k!} + \frac{\mathbf{Bell}_{k+1}[x_{i,a,b}]-\mathbf{Bell}_{k+1}[x_{i,a-1,b-1}]}{(k+1)!}\right),$$

$|x| > |a|+1$. We note that, for any parameters a, b, $c \in \mathbb{R}$, the function $K(a,b,c;x)$ is a $\mathcal{O}(x^{-2})$, as $x \to \infty$. For example,

$$\begin{aligned}&K(a,b,c;x)\\ =&\frac{e}{24x^2}\cdot\left(-36a^2+48ab+24ac-36a-12b^2-24bc+24b+12c-11\right)\\ +&\frac{e}{24x^3}\cdot\Big(104a^3-168a^2b-72a^2c+84a^2+72ab^2+96abc-72ab\\ &-24ac+22a-8b^3-24b^2c+12b^2-2b+2c-1\Big)\\ &+\mathcal{O}(x^{-4}),\quad x\to\infty.\end{aligned}$$

In particular, we obtain

$$K(a,a,1;x) = \frac{e-12ea}{24x^2} + \frac{e(4a(6a-1)+1)}{24x^3} + \mathcal{O}(x^{-4}), \qquad x\to\infty,$$

$$K\left(\frac{1}{12},\frac{1}{12},1;x\right) = \frac{5e}{144x^3} + \mathcal{O}(x^{-4}), \qquad x\to\infty,$$

which are cases considered in [7] and [11]. Taking benefit of three free parameters a,b,c, we obtain

$$K\left(-\frac{1}{2},\sqrt{\frac{1}{2}+\frac{1}{\sqrt{6}}},-\frac{1}{6}\sqrt{9+\sqrt{6}};x\right) = \frac{(3+5\sqrt{6})\,e}{720x^4} + \mathcal{O}(x^{-5}).$$

3.3. **On an expansion of Yang.** In [12, Theorem 1], X. Yang obtained the following expansion

$$\left(1+\frac{1}{x}\right)^x = e\left(1-\sum_{k=1}^{\infty}\frac{b_k}{(1+x)^k}\right), \quad x>0, \tag{3.17}$$

where

$$b_1=\frac{1}{2}, \quad b_{k+1}=\frac{1}{k+1}\left(\frac{1}{k+2}-\sum_{i=1}^{k}\frac{b_i}{k+2-i}\right), \quad k=1,2,\ldots.$$

We prove that Yang's formula (3.17) is a particular case of the general Bell-type expansion (2.7) for

$$t=\frac{1}{1+x} \quad \text{and} \quad x_i=-\frac{(i-1)!}{i+1}, \quad i=1,2,\ldots.$$

Indeed, we have

$$\begin{aligned}\left(1+\frac{1}{x}\right)^x e^{-1} &= (1-t)^{1-\frac{1}{t}}\,e^{-1} = \exp\left(\frac{(t-1)\log(1-t)-t}{t}\right)\\ &= \exp\left(-\sum_{i=1}^{\infty}\frac{t^i}{i(i+1)}\right)\\ &= \exp\left(-\sum_{i=1}^{\infty}\frac{(i-1)!}{i+1}\cdot\frac{t^i}{i!}\right) = \sum_{k=0}^{\infty}\frac{\mathbf{Bell}_k[x_i]}{k!}\cdot t^k,\end{aligned}$$

hence

$$\left(1+\frac{1}{x}\right)^x = e\sum_{k=0}^{\infty}\frac{\mathbf{Bell}_k[x_i]}{k!}\cdot\frac{1}{(1+x)^k}, \quad x>0. \tag{3.18}$$

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