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Numerical Simulation for Singularly Perturbed Problem with Two Nonlocal Boundary Conditions

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Abstract. In this paper, numerical solution for singularly perturbed problem with nonlocal boundary conditions is obtained. Finite difference method is used to discretize this problem on the Bakhvalov-Shishkin mesh. The some properties of exact solution are analyzed. The error is obtained first-order in the discrete maximum norm. Finally, an example is solved to show the advantages of the finite difference method.

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1. Introduction

We study the following singularly perturbed boundary value problem with two nonlocal boundary conditions:

$$\varepsilon^2 u''(x) + \varepsilon b(x) u'(x) - g(x, u(x)) = 0, \quad 0 < x < \ell, \tag{1.1}$$

$$u(0) - \int_{\ell_0}^{\ell_1} k_0(x)\, u(x)\, dx - A = 0, \tag{1.2}$$

$$u(\ell) - \int_{\ell_0}^{\ell_1} k_1(x)\, u(x)\, dx - B = 0, \quad 0 \leq \ell_0 < \ell_1 \leq \ell, \tag{1.3}$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, A and B are given constants, the functions $b(x) \geqslant 0$ and $g(x, u)$ are sufficiently smooth on $[0, \ell]$ and $[0, \ell] \times \mathbb{R}$, respectively, and $k_0(x)$ and $k_1(x)$ are continuous functions on $[\ell_0, \ell_1]$. We note here that

$$0 < \beta \leq \frac{\partial g}{\partial u} \leq \beta^* < \infty.$$

Singularly perturbation problems occur very frequently in fluid mechanics, fluid dynamics, quantum mechanics, elasticity, aerodynamics, meteorology, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography [13, 14, 22–25, 27].

The highest order derivative terms of such differential equations are multiplied by a small parameter ε. Therefore, these differential equations are called singularly perturbed differential equations. Due to the parameter ε, a sudden and

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rapid change of thin transition layers occurs in the solution (around the points 0 and ℓ). In all domain except this thin transition layers, the solution is regular and slow. For this reason, suitable numerical methods have been used to solve such problems. For example, the finite elements method and the finite differences method [2, 3, 7, 9–11, 15]. In the last two decades, a great deal of research has been made on numerical methods for solving singularly perturbed problems, see [5, 12–14, 17, 20–25, 27, 29] and references herein. Existence and uniqueness of the solutions of such problems are researched in [4, 6, 8]. When we look at similar problems in the literature, we see that the singularly perturbed differential equation with two integral boundary conditions has been investigated with as the boundary function method and the theory of differential inequalities were applied to obtain the uniformly valid asymptotic solution of the problem in [22]. Another study is that the method of generalized quasilinearization for the boundary value problem with two integral boundary conditions has been used in [23]. We can see various studies on singularly perturbed problems with nonlocal boundary condition in the references [1–3, 9–11, 16, 18, 19, 26, 28, 30–32].

Studies above, related to different types of singularly perturbed equations were concerned only with integral condition. On the other hand, our problem is concerned with two parameter singularly perturbed problem together with the two integral boundary conditions. We have also derived algorithms to obtain Bakhvalov-Shishkin mesh to the problem for the first time. The Bakhvalov-Shishkin mesh gives a stronger error bound for $\varepsilon \le CN^{-1}$. This is a modification of the Shishkin mesh described by Bakhvalov. But the original Bakhvalov mesh requires the solution of a nonlinear equation to determine the transition point where the mesh switches from coarse to fine. Instead, the transition points are as in the Shishkin mesh [18]. There are many studies on the Bakhvalov-Shishkin mesh see references there in [33, 34].

In this s, we examine some properties of the exact solution of (1.1)-(1.3) in Section 2. Finite difference scheme on a non-uniform Bakhvalov-Shishkin type mesh for problem (1.1)-(1.3) are obtained in Section 3. Convergence properties of the scheme are analyzed in Section 4. An example is presented in Section 5.

Assumption 1: In this study, C means a positive constant independent of independent of ε and the mesh parameter. For any continuous function, $\vartheta(x)$, defined in the associated interval, maximum norm has the form $\|\vartheta\|_\infty = \max_{[0,\ell]} |\vartheta(x)|$.

Assumption 2: Throughout the study, $\varepsilon \le CN^{-1}$ will be assumed as in practice, where N is a positive even integer.

2. Preliminariy for Some Properties of the Exact Solution

In this part, we present some properties of the exact solution that are necessary in the application of the method used in the study.

Definition 2.1 ([10, 11]). Let L, L_0 and L_1 be the finite difference operators in (2.3)-(2.5) and $v \in C^2[0, \ell]$. If $L_0(v) \ge 0$, $L_1(v) \ge 0$ and $L(v) \le 0$ for all $0 < x < \ell$, then $v(x) \ge 0$ for all $0 \le x \le \ell$.

Definition 2.2 ([10, 11]). Let l, l_0 and l_1 be the finite difference operators in (3.6)-(3.8). If v is any mesh function defined on $\bar{\omega}_N$ such that $l_0(v) \ge 0$, $l_1(v) \ge 0$ and $l(v) \le 0$ for all $i = 1, 2, ..., N$; then $v_i \ge 0$ for all $i = 0, 1, ..., N$.

Lemma 2.3. *Let $u(x)$ be the solution of the problem (1.1)-(1.3), $b(x) \in C^1[0, \ell]$, $\gamma = \int_{\ell_0}^{\ell_1} (|k_0(x)| + |k_1(x)|)\, dx < 1$, $\partial g/\partial u - \varepsilon b'(x) \ge \beta_*$ and $|\partial g/\partial x| \leqslant C$ for $x \in [0, \ell]$, then the following estimates hold,*

$$\|u\|_\infty \leqslant C_0, \tag{2.1}$$

$$\left|u'(x)\right| \leqslant C\left\{1 + \frac{1}{\varepsilon}\left(\exp(-\frac{\mu_0 x}{\varepsilon}) + \exp(-\frac{\mu_1(\ell - x)}{\varepsilon})\right)\right\}, 0 \leqslant x \leqslant \ell, \tag{2.2}$$

where

$$C_0 = (1-\gamma)^{-1}\left(|A| + |B| + \beta^{-1}\|G\|_\infty\right),$$
$$G(x) = g(x, 0), \quad \|u\|_\infty = \max_{[0,\ell]} |u(x)|,$$
$$\mu_0 = \frac{1}{2}\left(\sqrt{b^2(0) + 4\beta_*} + c(0)\right),$$
$$\mu_1 = \frac{1}{2}\left(\sqrt{b^2(\ell) + 4\beta_*} - c(\ell)\right).$$

Proof. Let us begin the proof of (2.1) by rewriting the equation (1.1)-(1.3) in the following form:

$$\varepsilon^2 u''(x) + \varepsilon b(x)u'(x) - c(x)u(x) = G(x), \quad 0 < x < \ell, \tag{2.3}$$

$$u(0) - \int_{\ell_0}^{\ell_1} k_0(x)u(x)\,dx = A, \tag{2.4}$$

$$u(\ell) - \int_{\ell_0}^{\ell_1} k_1(x)u(x)\,dx = B, \tag{2.5}$$

where

$$c(x) = \frac{\partial g}{\partial u}(x, \xi u(x)), \quad 0 < \xi < 1.$$

Using the maximum principle in (2.3)-(2.5), we obtain the inequality as

$$|u(x)| \le |u(0)| + |u(\ell)| + \beta^{-1}\|G\|_\infty, \quad x \in [0, \ell], \tag{2.6}$$

where $u(0)$ and $u(\ell)$ are given as

$$|u(0)| \le |A| + \int_{\ell_0}^{\ell_1} |k_0(x)|\,|u(x)|\,dx, \tag{2.7}$$

and

$$|u(\ell)| \le |B| + \int_{\ell_0}^{\ell_1} |k_1(x)|\,|u(x)|\,dx. \tag{2.8}$$

Inserting (2.7) and (2.8) in (2.6), we have

$$|u(x)| \le C_0.$$

So, this completes the proof of (2.1).

Now, let us use the problem (2.3)-(2.5) for proving the accuracy of (2.2) as follows.

Therefore, we first write the equation (2.3) as

$$Lv = \Phi(x),$$
$$v(0) = u'(0), \quad v(\ell) = u'(\ell),$$

where

$$u'(x) = v(x), \quad \Phi(x) = G'(x) - \varepsilon b'(x)u'(x) + c'(x)u(x).$$

Now, we estimate $v(x)$ by splitting it into $v(x) = v_1(x) + v_2(x)$ and consider each term seperately, where $v_1(x)$ and $v_2(x)$ are the solutions of the following problems, respectively,

$$Lv_1 = \Phi(x), \quad v_1(0) = 0, \quad v_1(\ell) = 0, \tag{2.9}$$

$$Lv_2 = 0, \quad v_2(0) = u'(0), \quad v_2(\ell) = u'(\ell). \tag{2.10}$$

For the solution of (2.9) we obtain

$$v_1(x) \le \beta^{-1}\|\Phi\|_C,$$

so that by Lemma 2.1

$$|\Phi(x)| \le C,$$

thus

$$|v_1(x)| \le C.$$

In order to estimate the solution of the problem (2.10), we take as.

$$v_2(x) = w_0(x) + w_1(x) + R_\varepsilon(x),$$

where $w_0(x)$, $w_1(x)$ and $R_\varepsilon(x)$ are the solutions of the following problems, respectively:

$$\varepsilon^2 w_0''(x) + \varepsilon b(x)w_0'(x) - c(x)w_0(x) = 0, \tag{2.11}$$

$$w_0(0) = u'(0), \quad w_0(\ell) = 0, \tag{2.12}$$

$$\varepsilon^2 w_1''(x) + \varepsilon b(x)w_1'(x) - c(x)w_1(x) = 0, \tag{2.13}$$

$$w_1(0) = 0, \quad w_1(\ell) = u'(\ell), \tag{2.14}$$

$$LR_\varepsilon = \Psi_\varepsilon(x), \tag{2.15}$$

$$R_\varepsilon(0) = R_\varepsilon(\ell) = 0, \tag{2.16}$$

with
$\Psi_\varepsilon(x) = [b(0) - b(x)]\varepsilon w_0'(x) + [c(x) - c(0)]w_0(x) + [b(\ell) - b(x)]\varepsilon w_0'(x) + [c(x) - c(\ell)]w_0(x) \leq C.$

The solutions of the problems (2.11),(2.12) and (2.13),(2.14) can be easily obtained as □

$$w_0(x) = \frac{u'(0)sinh(\frac{\mu_0(\ell-x)}{2\varepsilon})exp(\frac{-b(0)x}{2\varepsilon})}{\varepsilon sinh(\frac{\mu_0\ell}{2\varepsilon})} \leq \frac{Cexp(\frac{-\mu_0 x}{2\varepsilon})}{\varepsilon},$$

and

$$w_1(x) = \frac{u'(\ell)sinh(\frac{\mu_1 x}{2\varepsilon})exp(\frac{b(\ell)(\ell-x)}{2\varepsilon})}{\varepsilon sinh(\frac{\mu_1\ell}{2\varepsilon})} \leq \frac{Cexp(\frac{-\mu_1(\ell-x)}{2\varepsilon})}{\varepsilon},$$

where

$$\mu_0 = \left(\sqrt{b^2(0) + 4c(0)}\right), \quad \mu_1 = \left(\sqrt{b^2(\ell) + 4c(0)}\right).$$

The solution of problem (2.15)-(2.16) is estimated with the help of inequality (2.1) as follows
$R_\varepsilon(x) \leq C.$

Using the above bounds in the following inequality, we have the following inequality

$$|v_2(x)| \leq |w_0(x)| + |w_1(x)| + |R_\varepsilon(x)| \leq C\left\{1 + \frac{1}{\varepsilon}\left(\exp(-\frac{\mu_0 x}{\varepsilon}) + \exp(-\frac{\mu_1(\ell-x)}{\varepsilon})\right)\right\}.$$

Finally, we find the evaluation as

$$u'(x) = v(x) = v_1(x) + v_2(x) \leq C\left\{1 + \frac{1}{\varepsilon}\left(\exp(-\frac{\mu_0 x}{\varepsilon}) + \exp(-\frac{\mu_1(\ell-x)}{\varepsilon})\right)\right\}.$$

This gives the proof of (2.2).

3. Application of the Finite Difference Method

In this part, we construct the problem (1.1)-(1.3) using the finite difference method on Bakhvalov-Shishkin type mesh.

Bakhvalov-Shishkin Mesh is defined as follow:

The interval $[0,\ell]$ is divided into the three subintervals $[0,\sigma_1]$, $[\sigma_1, \ell-\sigma_2]$ and $[\ell-\sigma_2,\ell]$. Here σ_1 and σ_2 are introduced as the transition points and are written as follows:

$$\sigma_1 = \min\left\{\frac{1}{4}, \mu_0^{-1}\varepsilon\ln N\right\}, \quad \sigma_2 = \min\left\{\frac{1}{4}, \mu_1^{-1}\varepsilon\ln N\right\}.$$

The mesh points $\bar{\omega}_N = \{x_i\}_{i=0}^N$ are introduced through a set of the equalities:

$$x_i = \begin{cases} -\mu_0^{-1}\varepsilon\ln\left[1 - 4(1-N^{-1})\frac{i}{N}\right], & i = 0,...,\frac{N}{4}, \\ \sigma_1 + (i - \frac{N}{4})h, \ h = \frac{2(1-\sigma_2-\sigma_1)}{N}, & i = \frac{N}{4}+1,...,\frac{3N}{4}, \\ 1 + \mu_1^{-1}\varepsilon\ln\left[1 - 4(1-N^{-1})(1-\frac{i}{N})\right], & i = \frac{3N}{4},...,N. \end{cases}$$

The difference scheme will be constructed on the following nonuniform mesh

$$\omega_N = \{0 < x_1 < x_2 < ... < x_{N-1} < \ell\},$$

and

$$\bar{\omega}_N = \omega_N \cup \{x_0 = 0, x_N = \ell\}.$$

Before describing the numerical method, we introduce some notations for the mesh functions. We set $v_i = v(x_i)$ for any $v(x)$ on $\bar{\omega}_N$. For any mesh function v_i defined on $\bar{\omega}_N$ we use the following finite difference operators:

$$v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i}, \quad v_{x,i} = \frac{v_{i+1} - v_i}{h_{i+1}}, \quad v_{\overset{0}{x},i} = \frac{v_{x,i} + v_{\bar{x},i}}{2},$$

$$v_{\widehat{x},i}=\frac{v_{i+1}-v_i}{\hbar_i},\quad v_{\bar{x}\widehat{x},i}=\frac{v_{x,i}-v_{\bar{x},i}}{\hbar_i},\quad \hbar_i=\frac{h_i+h_{i+1}}{2},$$
$$\|v\|_\infty\equiv\|v\|_{\infty,\bar{\omega}_N}:=\max_{0\leqslant i\leqslant N}|v_i|,\quad h_i=x_i-x_{i-1}, i=1,2,...,N.$$

The discretization for (1.1) begins with the identity

$$\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}Lu(x)\varphi_i(x)dx=0,\ 1\leqslant i\leqslant N-1, \tag{3.1}$$

where the basis functions $\{\varphi_i(x)\}_{i=1}^{N-1}$ are of the form

$$\varphi_i(x)=\begin{cases}\varphi_i^{(1)}(x), & x_{i-1}<x<x_i,\\ \varphi_i^{(2)}(x), & x_i<x<x_{i+1},\\ 0, & \text{otherwise.}\end{cases}$$

The functions $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$ are defined by

$$\varphi_i^{(1)}(x)\ =\ \frac{e^{\frac{b_i(x-x_{i-1})}{\varepsilon}}-1}{e^{\frac{ih_i}{\varepsilon}}-1},\qquad \varphi_i^{(2)}(x)=\frac{1-e^{-\frac{b_i(x_{i+1}-x)}{\varepsilon}}}{1-e^{-\frac{b_ih_{i+1}}{\varepsilon}}}.$$

The coefficient χ_i in (3.1) is given as follow:

$$\chi_i=\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}\varphi_i(x)dx=\Big\{\ \hbar_i^{-1}\left(\frac{h_i}{1-e^{\frac{a_ih_i}{\varepsilon}}}+\frac{h_{i+1}}{1-e^{-\frac{a_ih_{i+1}}{\varepsilon}}}\right).$$

We easily rearrange (3.1) and write

$$-\varepsilon^2\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}\varphi_i'(x)u'(x)\,dx+\varepsilon b_i\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}\varphi_i(x)u'(x)\,dx,$$
$$-f(x_i,u_i)+R_i=0,\quad i=1,2,...,N-1, \tag{3.2}$$

by letting

$$\begin{aligned}R_i\ &=\ \varepsilon\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}[b(x)-b(x_i)]\,\varphi_i(x)u'(x)\,dx,\\ &\quad-\chi_i^{-1}\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}dx\varphi_i(x)\int_{x_{i-1}}^{x_{i+1}}\frac{d}{dx}f(\xi,u(\xi))K_{0,i}^*(x,\xi)\,d\xi,\end{aligned} \tag{3.3}$$

where

$$K_{0,i}^*(x,\xi)=T_0(x-\xi)-T_0(x_i-\xi),\ T_0(\lambda)=1,\ \lambda\geqslant0;\ T_0(\lambda)=0,\ \lambda<0.$$

Using the interpolating quadrature rules (2.1) and (2.2) from [18] with weight functions $\varphi_i(x)$ on subintervals (x_{i-1},x_{i+1}) for (3.2), and we obtain

$$\varepsilon^2\left\{\chi_i^{-1}\left(1+0.5\varepsilon^{-1}\hbar_ib_i(\chi_{2,i}-\chi_{1,i})\right)\right\}u_{\bar{x}\widehat{x},i}+\varepsilon b_iu_{\overset{0}{x},i}-f(x_i,u_i)+R_i=0,$$

where

$$\chi_{1,i}=\hbar_i^{-1}\int_{x_{i-1}}^{x_i}\varphi_i^{(1)}(x)dx=\Big\{\ \hbar_i^{-1}\left(\frac{\varepsilon}{b_i}+\frac{h_i}{1-e^{\frac{b_ih_i}{\varepsilon}}}\right),$$

$$\chi_{2,i}=\hbar_i^{-1}\int_{x_i}^{x_{i+1}}\varphi_i^{(2)}(x)dx=\Big\{\ \hbar_i^{-1}\left(\frac{h_{i+1}}{1-e^{\frac{b_ih_{i+1}}{\varepsilon}}}-\frac{\varepsilon}{b_i}\right).$$

The following difference approach is obtained from the above equations

$$\varepsilon\theta_iu_{\bar{x}\widehat{x},i}+\varepsilon b_iu_{\overset{0}{x},i}-f(x_i,u_i)+R_i=0,\ 1\leqslant i\leqslant N-1,$$

where

$$\theta_i = \chi_i^{-1} \left(1 + 0.5\varepsilon^{-1} \hbar_i b_i \left(\chi_{2,i} - \chi_{1,i}\right)\right). \tag{3.4}$$

After some calculations of (3.4), we deduce that

$$\theta_i = \left\{ \frac{a_i \hbar_i}{2\varepsilon} \left(\frac{h_{i+1}\left(e^{\frac{b_i h_i}{\varepsilon}} - 1\right) + h_i \left(1 - e^{-\frac{b_i h_{i+1}}{\varepsilon}}\right)}{h_{i+1}\left(e^{\frac{b_i h_i}{\varepsilon}} - 1\right) - h_i \left(1 - e^{-\frac{b_i h_{i+1}}{\varepsilon}}\right)} \right) \right. .$$

Now let us determine approximation for (1.2) and (1.3). Let x_{N_0} and x_{N_1} be the mesh points nearest to ℓ_0 and ℓ_1, respectively.

$$\begin{aligned} \int_{\ell_0}^{\ell_1} k_0(x)\, u(x)\, dx &= \int_{\ell_0}^{x_{N_0}} k_0(x)\, u(x)\, dx + \int_{x_{N_0}}^{x_{N_1}} k_0(x)\, u(x)\, dx \\ &\quad + \int_{x_{N_1}}^{\ell_1} k_0(x)\, u(x)\, dx, \end{aligned}$$

and

$$\begin{aligned} \int_{x_{N_0}}^{x_{N_1}} k_0(x)\, u(x)\, dx &= \sum_{i=N_0}^{N_1} \left(\int_{x_{i-1}}^{x_i} k_0(x)\, dx \right) u(x_i) + \bar{r}_i \\ &= M_0(u) + \bar{r}_i, \end{aligned}$$

where

$$M_0(u) = \sum_{i=N_0}^{N_1} \left(\int_{x_{i-1}}^{x_i} k_0(x)\, dx \right) u(x_i), \tag{3.5}$$

$$\bar{r}_i = \sum_{i=N_0}^{N_1} \int_{x_{i-1}}^{x_i} k_0(x)\, dx \int_{x_{i-1}}^{x_i} u'(\xi) \left(T_0(x-\xi) - 1\right) d\xi,$$

We find the following approximation for (1.2):

$$u_0 - M_0(u) = A + r_0, \tag{3.6}$$

where

$$r_0 = \int_{\ell_0}^{x_{N_0}} k_0(x)\, u(x)\, dx + \int_{x_{N_1}}^{\ell_1} k_0(x)\, u(x)\, dx + \bar{r}_i. \tag{3.7}$$

For (1.3), we find the following approximation similar to (1.2):

$$u_\ell - M_1(u) = B + r_1, \tag{3.8}$$

where

$$M_1(u) = \sum_{i=N_0}^{N_1} \left(\int_{x_{i-1}}^{x_i} k_1(x)\, dx \right) u(x_i), \tag{3.9}$$

$$r_1 = \int_{\ell_0}^{x_{N_0}} k_1(x)\, u(x)\, dx + \int_{x_{N_1}}^{\ell_1} k_1(x)\, u(x)\, dx + \tilde{r}, \tag{3.10}$$

$$\tilde{r} = \sum_{i=N_0}^{N_1} \int_{x_{i-1}}^{x_i} dx k_1(x) \int_{x_{i-1}}^{x_i} u'(\xi) \left(T_0(x-\xi) - 1\right) d\xi.$$

If we neglect the error functions R_i, r_0 and r_1 in (3.3), (3.7) and (3.10), then we deduce the following finite difference scheme for the problem (1.1)-(1.3):

$$\varepsilon^2 \theta_i y_{\bar{x}\widehat{x},i} + \varepsilon b_i y_{\overset{0}{x},i} - g(x_i, y_i) = 0, \; 1 \leqslant i \leqslant N-1, \tag{3.11}$$

$$y_0 - M_0(y) = A, \tag{3.12}$$

$$y_N - M_1(y) = B, \tag{3.13}$$

where θ_i, $M_0(y)$ and $M_1(y)$ are given by (3.4), (3.5) and (3.9), respectively.

4. Analysis of Stability

In this section, we will investigate the stability of the method by applying Lemma 4.1. Here we denote the estimate of error functions R_i, r_0 and r_1 with Lemma 4.2.

We define error function z_i as $z_i = y_i - u_i$, which is the solution of the discrete problem (3.11)-(3.13).

$$\varepsilon^2\theta_i z_{\bar{x}\widehat{x},i} + \varepsilon b_i z_{\overset{0}{x},i} - [g(x_i, y_i) - g(x_i, u_i)] = R_i, \ \ 1 < i < N, \tag{4.1}$$

$$z_0 - M_0(z) = r_0, \tag{4.2}$$

$$z_N - M_1(z) = r_1. \tag{4.3}$$

Lemma 4.1. *If z_i is the solution of (4.1)-(4.3), then the following estimate holds:*

$$\|z\|_{\infty,\bar{\omega}_N} \le C\left(\beta^{-1}\|R\|_{\infty,\omega_N} + |r_0| + |r_1|\right).$$

Proof. We rewrite the problem (4.1)-(4.3) in the form

$$lz_i := \varepsilon^2\theta_i z_{\bar{x}\widehat{x},i} + \varepsilon b_i z_{\overset{0}{x},i} - c_i z_i = R_i, \ \ 1 < i < N, \tag{4.4}$$

$$l_0 z := z_0 - M_0(z) = r_0, \tag{4.5}$$

$$l_1 z := z_N - M_1(z) = r_1, \tag{4.6}$$

where

$$c_i = \frac{\partial g}{\partial u}(x_i, u_i + \eta z_i), \ \ 0 < \eta < 1,$$

and $u_i + \eta z_i$ is the intermediate point.

Using the maximum principle in (4.4), it is easy to obtain that

$$\|z\|_{\infty,\bar{\omega}_N} \le |z_0| + |z_N| + \beta^{-1}\|R\|_{\infty,\omega_N}. \tag{4.7}$$

Using the boundary conditions (4.5) and (4.6), we find that

$$|z_0| \le |r_0| + \sum_{i=N_0}^{N_1}\left(\int_{x_{i-1}}^{x_i}|k_0(x)|\,dx\right)|z_i|, \tag{4.8}$$

$$|z_N| \le |r_1| + \sum_{i=N_0}^{N_1}\left(\int_{x_{i-1}}^{x_i}|k_1(x)|\,dx\right)|z_i|. \tag{4.9}$$

By inserting the inequalities (4.8) and (4.9) in (4.7), we obtain

$$\begin{aligned}
\|z\|_{\infty,\bar{\omega}_N} \le\ & \beta^{-1}\|R\|_{\infty,\omega_N} + |r_0| + \sum_{i=N_0}^{N_1}\left(\int_{x_{i-1}}^{x_i}|k_0(x)|\,dx\right)|z_i| \\
& + |r_1| + \sum_{i=N_0}^{N_1}\left(\int_{x_{i-1}}^{x_i}|k_1(x)|\,dx\right)|z_i| \\
\le\ & \beta^{-1}\|R\|_{\infty,\omega_N} + |r_0| + |r_1| + \\
& + \|z\|_{\infty,\bar{\omega}_N}\left(\int_{\ell_0}^{\ell_1}|k_0(x)|\,dx + \int_{\ell_0}^{\ell_1}|k_1(x)|\,dx\right).
\end{aligned}$$

From here, we have

$$\|z\|_{\infty,\bar{\omega}_N} \le (1-\lambda)^{-1}\left(\beta^{-1}\|R\|_{\infty,\omega_N} + |r_0| + |r_1|\right),$$

where $\lambda = \sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i}(|k_0(x)| + |k_1(x)|)\,dx < 1$. So, the proof of Lemma 4.1 is completed. □

Lemma 4.2. *Under the assumptions of Section 1 and Lemma 2.1, the following estimates are valid for the error functions R_i, r_0 and r_1 :*

$$\|R\|_{\infty,\omega_N} \le CN^{-1},$$

$$|r_0| \le CN^{-1}, \tag{4.10}$$

$$|r_1| \le CN^{-1}. \tag{4.11}$$

Proof. From the expression (3.3) for R_i in the Bakhvalov-Shishkin mesh we have the following evaluation

$$|R_i| \leq C\left\{h_i + h_{i+1} + \int\limits_{x_{i-1}}^{x_{i+1}} \left(1 + \left|u'(\xi)\right|\right) d\xi\right\}, \quad 1 \leq i \leq N.$$

This inequality, together with (2.2), enables us to write as follows:

$$|R_i| \leq C\left\{h_i + h_{i+1} + \frac{1}{\varepsilon}\int\limits_{x_{i-1}}^{x_{i+1}} \left(e^{-\frac{\mu_0 x}{\varepsilon}} + e^{-\frac{\mu_1(\ell - x)}{\varepsilon}}\right) dx\right\}. \tag{4.12}$$

1) The remainder term R_i^1 is evaluated as. For $x_i \in [0, \sigma_1]$, $\sigma_1 \leq \frac{1}{4}$:

$$x_{i-1} = -\mu_0^{-1}\varepsilon \ln\left[1 - 4(1 - N^{-1})\frac{i-1}{N}\right], \ i = 1, ..., \frac{N}{4},$$

$$h_i = -\mu_0^{-1}\varepsilon \ln\left[1 - 4(1 - N^{-1})\frac{i}{N}\right] + \mu_0^{-1}\varepsilon \ln\left[1 - 4(1 - N^{-1})\frac{i-1}{N}\right], \tag{4.13}$$

$$h_{i+1} = -\mu_0^{-1}\varepsilon \ln\left[1 - 4(1 - N^{-1})\frac{i+1}{N}\right] + \mu_0^{-1}\varepsilon \ln\left[1 - 4(1 - N^{-1})\frac{i}{N}\right]. \tag{4.14}$$

Applying the mean value theorem in (4.13)-(4.14), we obtain that

$$h_i = \mu_0^{-1}\varepsilon \frac{4(1 - N^{-1})N^{-1}}{1 - 4i_1(1 - N^{-1})N^{-1}} \leq CN^{-1}, h_{i+1} \leq CN^{-1}. \tag{4.15}$$

From (4.12) and (4.15), we have

$$\begin{aligned} |R_i| &\leq C\left\{CN^{-1} + CN^{-1} + \mu_0^{-1}\left[e^{\frac{-\mu_0 x_{i-1}}{\varepsilon}} - e^{\frac{-\mu_0 x_{i+1}}{\varepsilon}}\right] - \mu_1^{-1}\left[e^{\frac{-\mu_1(1-x_{i+1})}{\varepsilon}} - e^{\frac{-\mu_1(1-x_{i-1})}{\varepsilon}}\right]\right\} \\ &\leq CN^{-1}, \ i = 1, ..., \frac{N}{4} - 1, \end{aligned}$$

where

$$\begin{aligned} e^{-\frac{\mu_0(x_{i-1})}{\varepsilon}} - e^{-\frac{\mu_0(x_{i+1})}{\varepsilon}} &\leq \frac{1}{N} e^{\frac{-\mu_0(i-1-\frac{N}{4})h}{\varepsilon}}\left(1 - e^{\frac{-2\mu_0 h}{\varepsilon}}\right) \\ &\leq CN^{-1}, \end{aligned}$$

and similarly

$$e^{-\frac{\mu_1(1-x_{i+1})}{\varepsilon}} - e^{-\frac{\mu_1(1-x_{i-1})}{\varepsilon}} \leq CN^{-1}.$$

From (4.12), we come to conclusion as

$$|R_i| \leq CN^{-1}, \ i = 1, ..., \frac{N}{4}.$$

2) The remainder term R_i is evaluated as. For $x_i \in [\sigma_1, 1 - \sigma_2]$:

$$x_i = \sigma_1 + (i - \frac{N}{4})h, \quad i = \frac{N}{4} + 1, ..., \frac{3N}{4}, \tag{4.16}$$

where

$$h = \frac{2(1 - \sigma_2 - \sigma_1)}{N}. \tag{4.17}$$

From (4.12), (4.16) and (4.17), we have

$$|R_i| \leq CN^{-1}, \ i = \frac{N}{4} + 1, ..., \frac{3N}{4},$$

where

$$\begin{aligned} e^{-\frac{\mu_0(x_{i-1})}{\varepsilon}} - e^{-\frac{\mu_0(x_{i+1})}{\varepsilon}} &\leq \frac{1}{N} e^{\frac{-\mu_0(i-1-\frac{N}{4})h}{\varepsilon}}\left(1 - e^{\frac{-2\mu_0 h}{\varepsilon}}\right) \\ &\leq CN^{-1}, \end{aligned}$$

and similarly

$$e^{-\frac{\mu_1(1-x_{i+1})}{\varepsilon}}-e^{-\frac{\mu_1(1-x_{i-1})}{\varepsilon}}\leq CN^{-1}.$$

3) The remainder term R_i is evaluated as. For $x_i\in[1-\sigma_2,1]$:

$$x_{i-1}=1+\mu_1^{-1}\varepsilon\ln\left[1-4(1-N^{-1})(1-\frac{i-1}{N})\right],\ \ i=\frac{3N}{4},...,N,$$

$$h_i=\mu_1^{-1}\varepsilon\left\{\ln[1-4(1-N^{-1})(1-\frac{i}{N})]-\ln[1-4(1-N^{-1})(1-\frac{i-1}{N})]\right\}. \tag{4.18}$$

By applying the mean value theorem in (4.18), we obtain

$$h_i\leq CN^{-1}. \tag{4.19}$$

Using the inequality (4.19), we have

$$\hbar_i\leq CN^{-1}. \tag{4.20}$$

Thus, from (4.12), (4.19) and (4.20), we can write

$$|R_i|\ \leq\ CN^{-1},\ \ i=\frac{3N}{4},...,N,$$

where

$$\begin{aligned}e^{-\frac{\mu_1(x_{i-1})}{\varepsilon}}-e^{-\frac{\mu_1(x_{i+1})}{\varepsilon}} &= e^{-\mu_1(1+\mu_1^{-1}\varepsilon\ln\left[1-4(1-N^{-1})(1-\frac{i-1}{N})\right])}\\ &- e^{-\mu_1(1+\mu_1^{-1}\varepsilon\ln\left[1-4(1-N^{-1})(1-\frac{i+1}{N}\right])}\leq CN^{-1}.\end{aligned}$$

Now, we evaluate (3.7) for the proof of (4.10) as

$$\begin{aligned}|r_0| &\leq \sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i}dx\,|k_0(x)|\int_{x_{i-1}}^{x_i}\left|u'(\xi)\right||T_0(x-\xi)-1|\,d\xi\\ &\quad+\int_{\ell_0}^{x_{N_0}}|k_0(x)|\,|u(x)|\,dx+\int_{x_{N_1}}^{\ell_1}|k_0(x)|\,|u(x)|\,dx\\ &\leq h\max_{[x_{i-1},\,x_i]}|k_0(x)|\sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i}\left|u'(\xi)\right||T_0(x-\xi)-1|\,d\xi+O(h)\\ &\leq 2h\max_{[x_{i-1},\,x_i]}|k_0(x)|\int_0^{\ell}\left|u'(x)\right|dx+O(h)\\ &\leq Ch.\end{aligned} \tag{4.21}$$

When $[x_{N_0},\ x_{N_1}]$ is inside the interval $[\sigma_1,\ \ell-\sigma_2]$, we obtain from the inequality (4.21)

$$|r_0|\leq CN^{-1}.$$

When $[x_{N_0},\ x_{N_1}]$, by the inequality (4.21), we have

$$|r_0|\leq Ch\leq\frac{C\varepsilon\ln N}{N}\leq CN^{-1}.$$

The same estimate is done for the interval $[\ell-\sigma_2,\ell]$ in a similar way. The proof of (4.11) from (3.10) is given as

$$\begin{aligned}
|r_1| &\le \sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i} dx\,|k_1(x)|\int_{x_{i-1}}^{x_i}\left|u'(\xi)\right||T_0(x-\xi)-1|\,d\xi \\
&\quad+\int_{\ell_0}^{x_{N_0}}|k_1(x)|\,|u(x)|\,dx+\int_{x_{N_1}}^{\ell_1}|k_1(x)|\,|u(x)|\,dx \\
&\le h\max_{[x_{i-1},\,x_i]}|k_1(x)|\sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i}\left|u'(\xi)\right||T_0(x-\xi)-1|\,d\xi+O(h) \\
&\le 2h\max_{[x_{i-1},\,x_i]}|k_1(x)|\int_0^{\ell}\left|u'(x)\right|dx+O(h) \\
&\le Ch. \qquad (4.22)
\end{aligned}$$

If $[x_{N_0},\ x_{N_1}]$ is inside of the intervals $[0,\sigma_1]$, $[\sigma_1,\ell-\sigma_2]$ and $[\ell-\sigma_2,\ell]$, respectively, we have from the inequality (4.22)

$$|r_1|\le CN^{-1}.$$

All these complete the proof of Lemma 4.2. □

Finally, from Lemma 4.1 and 4.2, the following significant theorem gives us convergence result of the proposed method.

Theorem 4.3. *Suppose that $b(x), f(x)\in C^1[0,\ell]$. Let u be the solution of (1.1)-(1.3) and y be the solution of (3.11)-(3.13). Then, the following $\varepsilon-$ uniform estimate satisfies*

$$\|y-u\|_{\infty,\bar{\omega}_N}\le CN^{-1}.$$

5. Numerical Illustration

Here we will test the difference scheme on a problem. We solve the nonlinear problem (3.11)-(3.13) using the following quasilinearization technique [10]:

$$\begin{aligned}
\varepsilon^2\theta_i y^{(n)}_{\bar{x}\widehat{x},i}+\varepsilon b_i y^{(n)}_{\overset{0}{x},i}-f\left(x_i,y_i^{(n-1)}\right)-\frac{\partial f}{\partial y}\left(x_i,y_i^{(n-1)}\right)\left(y_i^{(n)}-y_i^{(n-1)}\right)&=0,\\
y_0^{(n)}&=\sum_{i=N_0}^{N_1}h_i k_{0,i}y_i^{(n-1)}+A,\\
y_N^{(n)}&=\sum_{i=N_0}^{N_1}h_i k_{1,i}y_i^{(n-1)}+B,
\end{aligned}$$

for $n\ge 1$ and $y_i^{(0)}$ given for $1\le i\le N$.

Example 1. Our test problem is as follows:

$$\varepsilon^2u''+\varepsilon(1+x)u'-2u+\arctan(x+u)=0,\ 0<x<1,$$

$$u(0)=\int_{0.5}^{1}\cos(\pi x)\,u(x)\,dx+2,\quad u(1)=\int_{0.5}^{1}\sin(\pi x)u(x)\,dx+3.$$

The exact solution of this example is not available. Hence we compare approximate solutions in double mesh as.

$$e_\varepsilon^N=\max_i\left|y_i^{\varepsilon,N}-\tilde{y}_{2i}^{\varepsilon,2N}\right|.$$

Convergence rates are defined as:

$$P_\varepsilon^N=\frac{\ln\left(e_\varepsilon^N/e_\varepsilon^{2N}\right)}{\ln 2}.$$

The $\varepsilon-$uniform errors e^N are estimated the following form:

$$e^N = \max_{\varepsilon} e_{\varepsilon}^N.$$

The rates of uniform convergence P_{ε}^N for different values of ε and N are presented in Table 1. These increase monotonically towards one. It is understood from the results of the numerical experiment is consistent with the theoretical results. In Figure 1, as N values increase, the graph gets closer to the coordinate axes in the boundary layer regions around $x = 0$ and $x = 1$.

Table 1. Approximate maximum errors e_{ε}^N and the rates of convergence p_{ε}^N on $\bar{\omega}_N$ for values of ε and N of Example 1.

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
2^{-10}	0.228121	0.116023	0.058333	0.029137	0.014051	0.007009
	0.97	0.99	1.00	1.00	1.00	
2^{-12}	0.230496	0.116093	0.058346	0.029115	0.014008	0.007001
	0.98	0.99	1.00	1.05	0.99	
2^{-14}	0.230540	0.116110	0.058349	0.029112	0.014559	0.007102
	0.97	0.99	1.00	1.07	1.08	
2^{-16}	0.230549	0.116116	0.058351	0.029111	0.014531	0.007102
	0.98	0.99	1.00	1.00	1.03	
2^{-18}	0.230548	0.116125	0.058331	0.029112	0.014499	0.007106
	0.98	0.99	1.00	1.00	1.02	
2^{-20}	0.230559	0.116150	0.058287	0.029140	0.014359	0.007001
	0.98	0.99	1.00	1.02	1.03	
e^N	0.230559	0.116150	0.058351	0.029140	0.014559	0.007009
p^N	0.98	0.99	1.00	1.02	1.08	

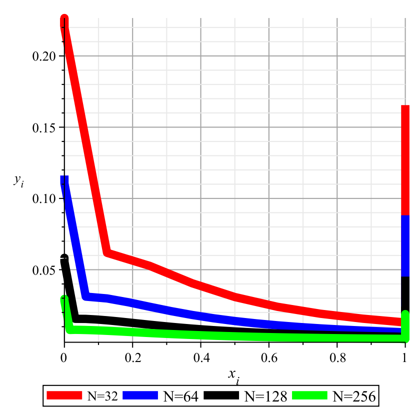

Figure 1. Error distrubition for $\varepsilon = 2^{-14}$ and N values.

6. Conclusion

We have studied the finite difference method on the non-uniform mesh for solving singularly perturbed semilinear boundary value problem with two integral boundary conditions. We have applied the present method on a test problem. As a result, the method has ε-uniform convergence with respect to the perturbation parameter ε in the discrete maximum norm. The obtained numerical results have been tabulated in terms of maximum absolute errors and rates of convergence (see Table 1). Approximate solution curves have been shown in Figure 1, in terms of ε and for increasing

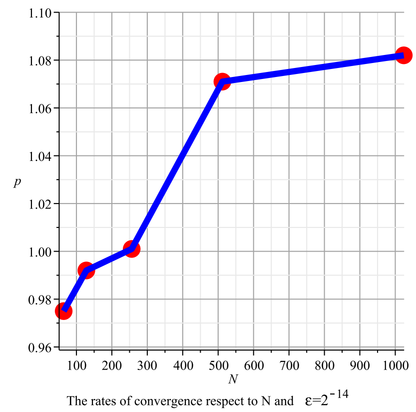

Figure 2. The rates of convergence P_ε^N for $\varepsilon = 2^{-14}$, $N = 64, 128, 256, 512, 1024$.

and N. As N value increases, the convergence values get closer to one, which also proves how perfect the method is in Table 1. The scheme can be effectively applied also in the case when the original problem has a solution with certain singularities.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

Authors Contribution Statement

The authors wrote, read and approved the final version of the manuscript.

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