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Research Article

Durrmeyer type operators on a simplex

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ABSTRACT. The paper contains the definition and certain approximation properties of a sequence of Durrmeyer type operators on a simplex, which preserve affine functions and make a link between the multidimensional "genuine" Durrmeyer operators and the multidimensional Bernstein operators.

Keywords: Multidimensional linear positive operators, Durrmeyer type operators, Bernstein operators on a simplex, limit operators, estimates with moduli of continuity.

2020 Mathematics Subject Classification: 41A36, 41A25.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and consideration.

1. INTRODUCTION

Durrmeyer operators introduced in [10] and independently by Lupaş [17], were one of the most fecund source of inspiration in approximation by positive linear operators. They were be known especially after the paper by Derriennic [7]. In References, we give only a very partial review of contributions in this field. The extension to Jacobi weight was considered by the author in [20], see also [21], [5]. The limit of Durrmeyer operators with Jacobi weight yields the so named "genuine" Durrmeyer operators considered firstly by Chen [6], Goodman and Sharma [14], see also [19], [26], [11]. The eigen-structure of this operators was studied in [22]. For other modifications of Bernstein-Durrmeyer operators mention [18], [27], [16], [3], [1], [15].

In this paper, we are especially interested in the following modification. In [24], there was constructed a family of operators depending on a parameter ρ, with property that they preserve linear functions, which make a link between the genuine-Durrmeyer operators and the Bernstein operators in the following mode:

$$\mathbb{U}_n^{\rho}(f)(x) = (1-x)^n f(0) + x^n f(1) + \sum_{k=0}^{n} \frac{\int_0^1 f(t) t^{k\rho-1}(1-t)^{(n-k)\rho-1} dt}{B(k\rho, (n-k)\rho)} p_{n,k}(x),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $(0 \le k \le n)$, $f \in C[0,1]$, $x \in [0,1]$. For $\rho = 1$, these operators coincide with genuine–Durrmeyer operators and on the other hand $\lim_{\rho\to\infty} U_n^{\rho} = B_n$, where B_n are the Bernstein operators. These operators are studied more completely in Gonska and the author in [12]. The eigen-structure of operators U_n^{ρ} was given in Gonska, Raşa, Stănilă [13].

The extension of Durrmeyer operators on a simplex is very natural. Mention that the first Durrmeyer operators on a simplex were considered by Derriennic [8]. The multidimensional Durrmeyer operators with Jacobi weight were considered by Ditzian [9] and the equivalent of

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the "genuine"-Durrmeyer operators on a simplex are given by Waldron [28]. The genuine Durrmeyer operators on a simplex preserves affine functions. The generalization of the Durrmeyer operators on a simplex with regard to a arbitrary measure was made by Berdysheva and Jetter [4], see also [25].

The aim of this paper is to extend operators U_n^ρ on a simplex, obtaining the family of operators $\mathbb{U}_n^\rho$, which preserve affine functions.

We also construct an additional class of operators $\mathbb{M}_n^{\rho,\mathbf{a}}$, depending on a scalar parameter ρ and on a vector parameter $\mathbf{a}$ and we prove that operators $\mathbb{U}_n^\rho$ are the limit of operators $\mathbb{M}_n^{\rho,\mathbf{a}}$ when $\mathbf{a} \to (-1,\dots,-1)$. This class of operators $\mathbb{M}_n^{\rho,\mathbf{a}}$ allows to obtain more simply certain properties of operators $\mathbb{U}_n^\rho$.

2. Preliminaries and Definitions

Let $p \in \mathbb{N}$. For any vector $\mathbf{x} = (x_1,\dots,x_p)$, denote $|\mathbf{x}| = x_1+\dots+x_p$. For any $p \in \mathbb{R}$, consider the standard simplex in $\mathbb{R}^p$.

$$\Delta_p = \{(x_1,\dots,x_p)|\ x_i \geq 0,\ \ |\mathbf{x}| \leq 1\}.$$

If $g \in C(\Delta_p)$, $p \in \mathbb{N}$, denote by $\int_{\Delta_p} g$ the volume integral of g on Δ_p.

Fix $m \in \mathbb{N}$. Denote $\mathbf{e}_k = (0,\dots,0,1,0,\dots,0) \in \mathbb{R}^m$,$(1 \leq k \leq m)$, where the digit 1 appears at the k-th place. Denote also $\mathbf{e}_0 = (0,\dots,0) \in \mathbb{R}^m$.

Let $\mathbf{x} = (x_1,\dots,x_m) \in \mathbb{R}^m$. Denote the Euclidean norm of $\mathbf{x}$ by $\|\mathbf{x}\| = \sqrt{x_1^2+\dots+x_m^2}$, and the L_1 norm of $\mathbf{x}$ by $\|\mathbf{x}\|_1 = |x_1|+\dots+|x_m|$. If $f \in C(\Delta_m)$, denote $\|f\| = \max_{\mathbf{x}\in\Delta_m} |f(\mathbf{x})|$.

For vectors $\mathbf{v}_0,\dots,\mathbf{v}_p \in \mathbb{R}^m$, denote

$$\Delta_{[\mathbf{v}_0,\dots,\mathbf{v}_p]} = \left\{ \sum_{i=0}^p t_i\mathbf{v}_i \middle| t_0,\dots,t_p \geq 0, t_0+\dots t_p = 1 \right\},$$

the simplex with vertices $\mathbf{v}_0,\dots,\mathbf{v}_p$. Numbers $t_0,\dots,t_p$ are the barycenter coordinates of a point in $\Delta_{[\mathbf{v}_0,\dots,\mathbf{v}_r]}$. Note that $\Delta_m = \Delta_{[\mathbf{e}_0,\dots,\mathbf{e}_m]}$.

Fix also a number $n \in \mathbb{N}$. Put

$$\Lambda = \{\mathbf{k} = (k_0,\dots,k_m)|\mathbf{k} \geq 0,\ |\mathbf{k}| = n\}.$$

For $\mathbf{k} \in \Lambda$, denote $\operatorname{supp} \mathbf{k} := \{i \in \{0,1,\dots,m\}|k_i > 0\}$. If $\operatorname{supp} \mathbf{k} = \{i_0 < \dots < i_p\}$, define $D_\mathbf{k} = \Delta_{[\mathbf{e}_{i_0},\dots,\mathbf{e}_{i_p}]}$.

If $g \in C(D_\mathbf{k})$, denote by $\int_{D_k} g\,d\sigma$ the integral of g on D_k. In the case when $D_\mathbf{k} = \Delta_m$, $\int_{D_k} g\,d\sigma = \int_{\Delta_m} g$. If $g \in C(\Delta_m)$, then the restriction of g to $D_\mathbf{k}$ is denoted also by g.

For $\mathbf{k} \in \Lambda$, with $\operatorname{supp} \mathbf{k} = \{i_0 < \dots < i_p\}$ consider function $\theta_\mathbf{k} : \Delta_p \to D_\mathbf{k}$ defined by

$$\theta_\mathbf{k}(x_{i_1},\dots,x_{i_p}) = \sum_{s=1}^p x_{i_s}\mathbf{e}_{i_s} + \left(1 - \sum_{s=1}^p x_{i_s}\right)\mathbf{e}_{i_0},\ (x_{i_1},\dots,x_{i_p}) \in \Delta_p. \tag{2.1}$$

Lemma 2.1. *Let* $\mathbf{k} \in \Lambda$, *with* $\operatorname{supp} \mathbf{k} = \{i_0 < \dots < i_p\}$.

i) If $i_0 = 0$, *then*

$$\int_{D_\mathbf{k}} g d\sigma = \int_{\Delta_p} g \circ \theta_\mathbf{k},\ g \in C(D_\mathbf{k}); \tag{2.2}$$

ii) If $i_0 > 0$, *then*

$$\int_{D_\mathbf{k}} g d\sigma = \sqrt{p+1} \int_{\Delta_p} g \circ \theta_\mathbf{k},\ g \in C(D_\mathbf{k}). \tag{2.3}$$

Proof. Let prove ii). We have $\theta(\Delta_p) = D_{\mathbf{k}}$. We can write $\theta_{\mathbf{k}}(x_{i_1}, \dots, x_{i_p}) = \mathbf{e}_{i_0} + \sum_{s=1}^{p} x_{i_s}(\mathbf{e}_{is} - \mathbf{e}_{i_0})$. Then $\frac{\partial \theta_{\mathbf{k}}}{\partial x_{i_s}} = \mathbf{e}_{i_s} - \mathbf{e}_{i_0}$. Hence

$$\det[\partial\theta_{\mathbf{k}} \cdot (\partial\theta_{\mathbf{k}})^T] := \det\left[\left\langle \frac{\partial \theta_{\mathbf{k}}}{\partial x_{i_s}}, \frac{\partial \theta_{\mathbf{k}}}{\partial x_{i_t}} \right\rangle\right]_{1 \le s,t \le p} = \det\begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2 \end{pmatrix} = p+1.$$

Then,

$$\int_{D_{\mathbf{k}}} g\, d\sigma = \int_{\Delta_p} (g \circ \theta_{\mathbf{k}}) \sqrt{\det[\partial\theta_{\mathbf{k}} \cdot (\partial\theta_{\mathbf{k}})^T]} = \sqrt{p+1} \int_{\Delta_p} g \circ \theta_{\mathbf{k}}.$$

Using the same method, point i) is immediate. □

Let $\mathbf{k} = (k_0, \dots, k_m) \in \Lambda$. For $\mathbf{x} = (x_1, \dots, x_m) \in \Delta_m$, denote

$$p_{n,\mathbf{k}}(\mathbf{x}) = \binom{n}{k_0\ k_1\ \dots k_m} (1 - |\mathbf{x}|)^{k_0} (x_1)^{k_1} \dots (x_m)^{k_m},$$

where

$$\binom{n}{k_0\ k_1\ \dots k_m} = \frac{n!}{k_0! k_1! \dots k_m!}.$$

The Bernstein operators on the simplex Δ_m are given by

$$\mathbb{B}_n(f)(\mathbf{x}) = \sum_{\mathbf{k} \in \Lambda} f\left(\frac{\mathbf{k}}{n}\right) p_{n,\mathbf{k}}(\mathbf{x}),\ f \in C(\Delta_m),\ \mathbf{x} \in \Delta_m. \tag{2.4}$$

Fix a number $\rho > 0$. For $\mathbf{k} \in \Lambda$ such that supp $\mathbf{k} = \{i_0, \dots, i_p\}$ consider function $Q_{\mathbf{k}}^{\rho} : D_{\mathbf{k}} \to \mathbb{R}$ defined by

$$Q_{\mathbf{k}}^{\rho}\left(\sum_{s=0}^{p} t_s \mathbf{e}_{i_s}\right) = \prod_{s=0}^{p} t_s^{k_{i_s}\rho - 1},\ \sum_{s=0}^{p} t_s \mathbf{e}_{i_s} \in D_{\mathbf{k}}. \tag{2.5}$$

For $\beta = (\beta_0, \dots, \beta_p)$, $b_0, \dots, b_p > 0$, consider multidimensional beta function

$$B(\beta) = \frac{\Gamma(\beta_0) \dots \Gamma(\beta_p)}{\Gamma(|\beta|)},$$

where Γ is gamma function. If $p = 0$, then $B(\beta) = 1$.
Let $\mathbf{k} \in \Lambda$, supp $\mathbf{k} = \{i_0 < \dots < i_p\}$. From relation (2.5) and relations (2.2) and (2.3), it follows that

$$\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho} d\sigma = B(k_{i_0}\rho, \dots, k_{i_p}\rho),\ \text{if } i_0 = 0; \tag{2.6}$$

$$\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho} d\sigma = \sqrt{p+1} B(k_{i_0}\rho, \dots, k_{i_p}\rho),\ \text{if } i_0 > 0. \tag{2.7}$$

Definition 2.1. Operators $\mathbb{U}_n^{\rho} : C(\Delta_m) \to C(\Delta_m)$ are defined by

$$\mathbb{U}_n^{\rho}(f)(\mathbf{x}) = \sum_{\mathbf{k} \in \Lambda} F_{n,\mathbf{k}}^{\rho}(f) p_{n,\mathbf{k}}(\mathbf{x}),\ f \in C(\Delta_m),\ \mathbf{x} \in \Delta_m, \tag{2.8}$$

where

$$F_{n,\mathbf{k}}^{\rho}(f) = \frac{\int_{D_{\mathbf{k}}} f Q_{\mathbf{k}}^{\rho} d\sigma}{\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho} d\sigma},\ \mathbf{k} \in \Lambda,\ f \in C(\Delta_m). \tag{2.9}$$

Remark 2.1. For $\rho = 1$, operators $\mathbb{U}_n^{\rho}$ coincide with operators constructed by Waldron [28].

Definition 2.2. For a vector $\mathbf{a} = (a_0, \ldots, a_m)$, with $a_i > -1$, $(0 \le i \le m)$, $\rho \ge 1$ and $n \in \mathbb{N}$ define

(2.10) $$\mathbb{M}_n^{\rho,\mathbf{a}}(f, \mathbf{x}) = \sum_{\mathbf{k}\in\Lambda} F_{n,\mathbf{k}}^{\rho,\mathbf{a}}(f) p_{n,\mathbf{k}}(\mathbf{x}),\ f \in C(\Delta_m),\ \mathbf{x} \in \Delta_m,$$

where

(2.11) $$F_{n,\mathbf{k}}^{\rho,\mathbf{a}}(f) = \frac{\int_{\Delta_m} f P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}},\ f \in C(\Delta_m),\ \mathbf{k} \in \Lambda$$

and

$$P_{\mathbf{k}}^{\rho,\mathbf{a}}(\mathbf{x}) = \prod_{s=0}^{m} x_s^{k_s\rho + a_s},\ \mathbf{x} = (x_1, \ldots, x_m) \in \Delta_m,\ x_0 = 1 - |\mathbf{x}|.$$

3. Link properties

Theorem 3.1. *For any $\rho \ge 1$, $n \in \mathbb{N}$ and $f \in C(\Delta_m)$, we have*

(3.12) $$\lim_{\mathbf{a}\to-\mathbf{1}} \mathbb{M}_n^{\rho,\mathbf{a}}(f)(\mathbf{x}) = \mathbb{U}_n^{\rho}(f)(\mathbf{x}),\ \textit{uniformly for } \mathbf{x} \in \Delta_m,$$

where $-\mathbf{1} = (-1, \ldots, -1) \in \mathbb{N}^{m+1}$.

Proof. It is sufficient to show that

(3.13) $$\lim_{\mathbf{a}\to-\mathbf{1}} F_{n,\mathbf{k}}^{\rho,\mathbf{a}}(f) = F_{n,\mathbf{k}}^{\rho}(f),\ \mathbf{k} \in \Lambda,\ f \in C(\Delta_m).$$

If $\operatorname{supp}\ \mathbf{k} = \{0, 1, \ldots, m\}$, then it is possible to pass to limit $\mathbf{a} \to -\mathbf{1}$ by simple replacement $\rho = -1$, because

$$\lim_{\mathbf{a}\to-\mathbf{1}} F_{n,\mathbf{k}}^{\rho,\mathbf{a}}(f) = \frac{\int_{\Delta_m} (f Q_{\mathbf{k}}^{\rho}) \circ \theta_{\mathbf{k}}}{\int_{\Delta_m} Q_{\mathbf{k}}^{\rho} \circ \theta_{\mathbf{k}}} = \frac{\int_{D_{\mathbf{k}}} f Q_{\mathbf{k}}^{\rho} d\sigma}{\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho} d\sigma}$$

and these integrals exist.
In the sequel, we consider that $\operatorname{supp}\ \mathbf{k} = \{i_0 < \ldots < i_p\} \subset \{0, 1, \ldots, m\}$, with $p < m$. Also, we denote $\{i_{p+1}, \ldots, i_m\} := \{0, 1, \ldots, m\} \setminus \operatorname{supp}\ \mathbf{k}$.
If $p = 0$, then $D_{\mathbf{k}} = \{\mathbf{e}_{i_0}\}$ and $\pi_{\mathbf{k}}(\mathbf{x}) = \mathbf{e}_{i_0}$, $\mathbf{x} \in \Delta_m$. Then $\frac{\int_{\Delta_m} (f\circ\pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} = f(\mathbf{e}_{i_0})$ and on the other hand it follows $F_{n,\mathbf{k}}^{\rho}(f) = \frac{\int_{D_{\mathbf{k}}} (f Q_{\mathbf{k}}^{\rho})\circ\theta_{\mathbf{k}}}{\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho}\circ\theta_{\mathbf{k}}} = f(\mathbf{e}_{i_0})$ and (3.13) is clear. We consider now that $p \ge 1$. We have to consider two cases.
Case 1. $0 \notin \operatorname{supp}\ \mathbf{k}$. Then, $0 \in \{i_{p+1}, \ldots, i_m\}$. Consider function $\pi_{\mathbf{k}} : \Delta_m \to D_{\mathbf{k}}$, given by

$$\pi_{\mathbf{k}}(\mathbf{x}) = \sum_{s=1}^{p} x_{i_s} \mathbf{e}_{i_s} + \left(1 - \sum_{s=1}^{p} x_{i_s}\right) \mathbf{e}_{i_0},\ \mathbf{x} \in \Delta_m.$$

Hence $D_{\mathbf{k}} = \Delta_{[\mathbf{e}_{i_0}, \ldots, \mathbf{e}_{i_p}]} = \pi(\Delta_m)$. We decompose

(3.14) $$F_{n,\mathbf{k}}^{\rho,\mathbf{a}}(f) = \frac{\int_{\Delta_m} (f \circ \pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} + \frac{\int_{\Delta_m} (f - f \circ \pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}}.$$

We show that

(3.15) $$\lim_{\mathbf{a}\to-\mathbf{1}} \frac{\int_{\Delta_m} (f \circ \pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} = F_{n,\mathbf{k}}^{\rho}(f).$$

We can write

$$\int_{\Delta_m} (f \circ \pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}$$
$$= \int_0^1 dx_{i_1} \int_0^{1-x_{i_1}} dx_{i_2} \dots \int_0^{1-\sum_{s=1}^{p-1} x_{i_s}} f(\pi_{\mathbf{k}}(\mathbf{x})) \prod_{s=1}^{p} x_{i_s}^{k_{i_s}\rho + a_{i_s}} V_{\mathbf{k}}(x_{i_1}, \dots, x_{i_p}) dx_{i_p},$$

where

$$V_{\mathbf{k}}(x_{i_1}, \dots, x_{i_p}) = \int_0^{1-\sum_{s=1}^{p} x_{i_s}} dx_{i_{p+1}} \dots \int_0^{1-\sum_{s=1}^{m-1} x_{i_s}} x_{i_0}^{k_{i_0}\rho + a_{i_0}} \prod_{s=p+1}^{m} x_{i_s}^{a_{i_s}} dx_{i_m},$$

where $x_{i_0} = 1 - \sum_{s=1}^{m} x_{i_s}$. Denote $u = 1 - \sum_{s=1}^{p} x_{i_s}$. Using the change of variables $x_{i_s} = u y_{i_s}$, $p+1 \le s \le m$ one obtains $x_{i_0} = u\left(1 - \sum_{s=p+1}^{m} y_{i_s}\right)$ and then

$$\begin{aligned} V_{\mathbf{k}}(x_{i_1}, \dots, x_{i_p}) &= u^{m-p+\sum_{s=p+1}^{m} a_{i_s} + a_{i_0} + \rho k_{i_0}} B(\rho k_{i_0} + a_{i_0} + 1, a_{i_{p+1}} + 1, \dots a_{i_m} + 1) \\ &= u^{m-p+\sum_{s=p+1}^{m} a_{i_s} + a_{i_0} + \rho k_{i_0}} \frac{\Gamma(k_{i_0}\rho + a_{i_0} + 1) \prod_{s=p+1}^{m} \Gamma(a_{i_s} + 1)}{\Gamma\left(a_{i_0} + \sum_{s=p+1}^{m} a_{i_s} + \rho k_{i_0} + m - p + 1\right)}. \end{aligned}$$

We have

$$\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}} = \frac{\prod_{s=0}^{p} \Gamma(k_{i_s}\rho + a_{i_s} + 1) \prod_{s=p+1}^{m} \Gamma(a_{i_s} + 1)}{\Gamma\left(|a| + n\rho + m + 1\right)}. \tag{3.16}$$

By combining the relations above, we get

$$\frac{\int_{\Delta_m} (f \circ \pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} = \int_0^1 dx_{i_1} \int_0^{1-x_{i_1}} dx_{i_2} \dots \int_0^{1-\sum_{s=1}^{p-1} x_{i_s}} f(\pi_{\mathbf{k}}(\mathbf{x})) T_{\mathbf{k}}^{\mathbf{a}}(x_{i_1}, \dots, x_{i_p}) dx_{i_p}, \tag{3.17}$$

where

$$\begin{aligned} T_{\mathbf{k}}^{\mathbf{a}}(x_{i_1}, \dots, x_{i_p}) &= \frac{\Gamma(|\mathbf{a}| + n\rho + m + 1)}{\Gamma\left(a_{i_0} + \sum_{s=p+1}^{m} a_{i_s} + \rho k_{i_0} + m - p + 1\right) \prod_{s=1}^{p} \Gamma(k_{i_s}\rho + a_{i_s} + 1)} \\ &\times \prod_{s=1}^{p} x_{i_s}^{k_{i_s}\rho + a_{i_s}} \left(1 - \sum_{s=1}^{p} x_{i_s}\right)^{m-p+\sum_{s=p+1}^{m} a_{i_s} + k_{i_0}\rho + a_{i_0}}. \end{aligned} \tag{3.18}$$

It is possible to pass to limit $\mathbf{a} \to -\mathbf{1}$ in (3.18) and it follows

$$\lim_{\mathbf{a} \to -\mathbf{1}} T_{\mathbf{k}}^{\mathbf{a}}(x_{i_1}, \dots, x_{i_p}) = \frac{\Gamma(n\rho)}{\prod_{s=0}^{p} \Gamma(k_{i_s}\rho)} \prod_{s=1}^{p} x_{i_s}^{k_{i_s}\rho - 1} \left(1 - \sum_{s=1}^{p} x_{i_s}\right)^{\rho k_{i_0} - 1}. \tag{3.19}$$

By taking into account relations (3.17), (3.19), (2.5), (2.7), (2.3) and then (2.9), we have successively

$$\begin{aligned}\lim_{\mathbf{a}\to-\mathbf{1}} \frac{\int_{\Delta_m}(f\circ\pi_{\mathbf{k}})P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}} &= \frac{\int_{\Delta_p}(f\cdot Q_{\mathbf{k}}^{\rho})\circ\theta_{\mathbf{k}}}{B(k_{i_0}\rho,\ldots,k_{i_p}\rho)}\\ &= \frac{\sqrt{p+1}\int_{\Delta_p}(f\cdot Q_{\mathbf{k}}^{\rho})\circ\theta_{\mathbf{k}}}{\sqrt{p+1}B(k_{i_0}\rho,\ldots,k_{i_p}\rho)}\\ &= \frac{\int_{D_{\mathbf{k}}} f\cdot Q_{\mathbf{k}}^{\rho}\,d\sigma}{\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho}\,d\sigma}\\ &= F_{n,\mathbf{k}}^{\rho}(f).\end{aligned}$$

So that relation (3.15) was proved. Now, we show that

$$\lim_{\mathbf{a}\to-\mathbf{1}} \frac{\int_{\Delta_m}(f-f\circ\pi_{\mathbf{k}})P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}=0. \tag{3.20}$$

Consider on $\mathbb{R}^m$ the norm $\|\mathbf{x}\|_1$, defined in the beginning. Let $\varepsilon>0$. There exist $0<\delta<1$, such that if $\mathbf{x}\in\Delta_m$, $\|\mathbf{x}-\pi_{\mathbf{k}}(\mathbf{x})\|<\delta$, then $|f(\mathbf{x})-f(\pi_{\mathbf{k}}(\mathbf{x}))|<\varepsilon$. Decompose $\Delta_m=A\cup B$, where $A=\{\mathbf{x}\in\Delta_m|\ \|\mathbf{x}-\pi_{\mathbf{k}}(x)\|_1<\delta\}$ and $B=\Delta_m\setminus A$. Then

$$\left|\frac{\int_A(f-f\circ\pi_{\mathbf{k}})P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}\right|\le\varepsilon. \tag{3.21}$$

Let $\mathbf{x}\in B$. We have

$$\mathbf{x}-\pi_{\mathbf{k}}(\mathbf{x})=\sum_{j=1}^m x_j\mathbf{e}_j-\sum_{s=1}^p x_{i_s}\mathbf{e}_{i_s}-\left(1-\sum_{s=1}^p x_{i_s}\right)\mathbf{e}_{i_0}=\sum_{s=p+1}^m x_{i_s}\mathbf{e}_{i_s}+\left(-1+\sum_{s=1}^p x_{i_s}\right)\mathbf{e}_{i_0}.$$

Therefore

$$\|\mathbf{x}-\pi_{\mathbf{k}}(\mathbf{x})\|_1=\sum_{s=p+1}^m x_{i_s}\mathbf{e}_{i_s}+1-\sum_{s=1}^p x_{i_s}=x_0+2\sum_{s=p+1}^m x_{i_s}.$$

Since $x_0\in\{i_{p+1},\ldots,i_m\}$, it results $\|\mathbf{x}-\pi_{\mathbf{k}}(\mathbf{x})\|_1\le 3\sum_{s=p+1}^m x_{i_s}$. It follows that there is at least an index $j\in\{i_{p+1},\ldots,i_m\}$, such that $x_i\ge\frac{\delta}{3m}$. Define

$$B_j=\left\{\mathbf{x}\in\Delta_m|x_j\ge\frac{\delta}{3m}\right\},\ j\in\{i_{p+1},\ldots,i_m\}.$$

From above, it follows that $B\subset\bigcup_{j\in\{0,i_{p+1},\ldots,i_m\}}B_j$. Therefore

$$\left|\frac{\int_B(f-f\circ\pi_{\mathbf{k}})P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}\right|\le\frac{2\|f\|\int_B P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}\le 2\|f\|\sum_{j\in\{i_{p+1},\ldots,i_m\}}\frac{\int_{B_j}P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}.$$

We show that

$$\lim_{\mathbf{a}\to-\mathbf{1}}\frac{\int_{B_j}P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m}P_{\mathbf{k}}^{\rho,\mathbf{a}}}=0,\ j\in\{i_{p+1},\ldots,i_m\}. \tag{3.22}$$

We can consider that $a_j<0$, $(0\le j\le m)$. Let $j=i_r$, with $p+1\le r\le m$. In integral $\int_{B_j}P_{\mathbf{k}}^{\rho,\mathbf{a}}$ if we make the change of variables: $x_j=\frac{\delta}{3m}+\left(1-\frac{\delta}{3m}\right)y_j$ and $x_\ell=\left(1-\frac{\delta}{3m}\right)y_\ell$, for

$\ell \in \{1, \ldots, m\} \setminus \{j\}$. Also $x_0 = \left(1 - \frac{\delta}{3m}\right) y_0$, where $y_0 = 1 - (y_1 + \ldots + y_m)$. Then, we have the equivalence $(x_1, \ldots, x_m) \in B_j \Longleftrightarrow (y_1, \ldots, y_m) \in \Delta_m$. We get

$$\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}} = \left(1 - \frac{\delta}{3m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \int_{\Delta_m} \prod_{s=0}^{p} y_{i_s}^{k_{i_s}\rho+a_{i_s}} \prod_{s=p+1,\, s\neq r}^{m} y_{i_s}^{a_{i_s}} \left(\frac{\delta}{3m} + \left(1 - \frac{\delta}{3m}\right) y_{i_r}\right)^{a_{i_r}}.$$

Since $a_{i_r} < 0$ and $\frac{\delta}{3m} < 1$, we obtain $\left(\frac{\delta}{3m} + \left(1 - \frac{\delta}{3m}\right) y_{i_r}\right)^{a_{i_r}} \leq \left(\frac{\delta}{3m}\right)^{a_{i_r}} \leq \left(\frac{\delta}{3m}\right)^{-1} = \frac{3m}{\delta}$. Consequently

$$\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}} \leq \frac{3m}{\delta}\left(1 - \frac{\delta}{3m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \frac{\prod_{s=0}^{p} \Gamma(k_{i_s}\rho + a_{i_s} + 1) \prod_{s=p+1,\, s\neq r}^{m} \Gamma(a_{i_s} + 1)\Gamma(1)}{\Gamma(n\rho + |a| - a_{i_r} + m + 1)}.$$

By taking into account relation (3.16), it results:

$$\frac{\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} \leq \frac{3m}{\delta}\left(1 - \frac{\delta}{3m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \frac{\Gamma(n\rho + |\mathbf{a}| + m + 1)}{\Gamma(n\rho + |\mathbf{a}| - a_{i_r} + m + 1)\Gamma(a_{i_r} + 1)}.$$

But

$$\lim_{\mathbf{a}\to-\mathbf{1}} \left(1 - \frac{\delta}{3m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \frac{\Gamma(n\rho + |\mathbf{a}| + m + 1)}{\Gamma(n\rho + |\mathbf{a}| - a_{i_r} + m + 1)} = \left(1 - \frac{\delta}{3m}\right)^{n\rho} \frac{\Gamma(n\rho)}{\Gamma(n\rho + 1)}$$

and $\lim_{\mathbf{a}\to-\mathbf{1}} \Gamma(a_{i_r} + 1) = \infty$. Then, one obtains relation (3.22).
Case 2. $i_0 = 0$. Then $\operatorname{supp} \mathbf{k} = \{0 = i_0 < i_1 < \ldots < i_p\}$, where $0 \leq p \leq m - 1$. Define the function $\pi_{\mathbf{k}} : \Delta_m \to D_{\mathbf{k}}$, by

$$\pi_{\mathbf{k}}(\mathbf{x}) = \sum_{s=1}^{p} x_{i_s}\mathbf{e}_{i_s}, \quad \mathbf{x} = (x_1, \ldots, x_m) \in \Delta_m.$$

The method of the proof is similar as in Case 1. Consider the decomposition of the form given in (3.14). First, we show the corresponding relation (3.15). For $(x_{i_1}, \ldots, x_{i_p}) \in \Delta_p$, we denote

$$U(x_{i_1}, \ldots, x_{i_p}) = \left\{ (x_{i_{p+1}}, \ldots, x_{i_m}) \middle| x_{i_s} \geq 0,\ (p+1 \leq s \leq m),\ \sum_{s=p+1}^{m} x_{i_s} \leq 1 - \sum_{s=1}^{p} x_{i_s} \right\}.$$

Then, we can write

$$\int_{\Delta_m} (f \circ \pi_{\mathbf{k}}) \cdot P_{\mathbf{k}}^{\rho,\mathbf{a}} = \int_{\Delta_p} f\left(\sum_{s=1}^{p} x_{i_s}\mathbf{e}_{i_s}\right) \prod_{s=1}^{p} x_{i_s}^{k_{i_s}\rho+a_{i_s}} dx_{i_1} \ldots dx_{i_p} \times \int_{U(x_{i_1},\ldots,x_{i_p})} \prod_{s=p+1}^{m} x_{i_s}^{a_{i_s}} x_0^{k_0\rho+a_0} dx_{i_{p+1}} \ldots dx_{i_m}.$$

Denote $u = 1 - x_{i_1} - \ldots - x_{i_p}$. Using the change of variables $x_\ell = uy_{i_\ell}$, $p+1 \le \ell \le m$ in the interior integral, we obtain

$$\int_{U(x_{i_1},\ldots,x_{i_p})} \prod_{s=p+1}^{m} x_{i_s}^{a_{i_s}} x_0^{k_0\rho+a_0} dx_{i_{p+1}} \ldots dx_{i_m}$$
$$= u^{m-p+a_0+a_{i_{p+1}}+\ldots+a_{i_m}+k_0\rho} B(k_0\rho + a_0 + 1, a_{i_{p+1}} + 1, \ldots, a_{i_m} + 1).$$

By taking also into account relation (3.16), we obtain

$$\frac{\int_{\Delta_m} (f \circ \pi) \cdot P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}}$$
$$= \int_{\Delta_p} f\left(\sum_{s=1}^{p} x_{i_s}\mathbf{e}_{i_s}\right) \prod_{s=1}^{p} x_{i_s}^{k_{i_s}\rho+a_{i_s}} \left(1 - \sum_{s=1}^{p} x_{i_s}\right)^{m-p+a_0+\sum_{s=p+1}^{m} a_{i_s}+k_0\rho} dx_{i_1}\ldots dx_{i_p}$$
$$\times \frac{\Gamma(n\rho + |\mathbf{a}| + m + 1)}{\Gamma(k_0\rho + a_0 + a_{i_{p+1}} + \ldots + a_{i_m} + m - p + 1)\prod_{s=1}^{p}\Gamma(k_{i_s}\rho + a_{i_s} + 1)}.$$

Using (2.1), (2.6), (2.2) and (2.9), it follows

$$\lim_{\mathbf{a}\to\mathbf{1}} \frac{\int_{\Delta_m} (f \circ \pi_{\mathbf{k}}) \cdot P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} = \frac{\int_{\Delta_p} (f \cdot Q_{\mathbf{k}}^{\rho}) \circ \theta_{\mathbf{k}}}{B(k_{i_0}\rho, \ldots, k_{i_p}\rho)} = \frac{\int_{D_{\mathbf{k}}} f \cdot Q_{\mathbf{k}}^{\rho}}{\int_{D_{\mathbf{k}}} Q_{\mathbf{k}}^{\rho}} = F_{n,\mathbf{k}}^{\rho}(f).$$

So that relation (3.15) is proved.

In order to prove the corresponding relation (3.20), let $\varepsilon > 0$ arbitrarily chosen. There is $0 < \delta < 1$, such that inequality $\|\mathbf{x} - \pi_{\mathbf{k}}(\mathbf{x})\|_1 < \delta$, $\mathbf{x} \in \Delta_m$ implies $|f(\mathbf{x}) - f(\pi_{\mathbf{k}}(x))| < \varepsilon$. Consider the sets $A = \{\mathbf{x} \in \Delta_m | \ \|\mathbf{x} - \pi_{\mathbf{k}}(\mathbf{x})\|_1 < \delta\}$ and $B = \Delta_m \setminus A$. We have

$$\left|\frac{\int_{\Delta_m} (f - f \circ \pi_{\mathbf{k}}) P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}}\right| \le \varepsilon + 2\|f\| \frac{\int_B P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}}.$$

If $\mathbf{x} \in B$, there is $j \in \{p+1, \ldots, m\}$ such that $x_j \ge \frac{\delta}{m}$. Indeed, otherwise we have $\|\mathbf{x} - \pi_{\mathbf{k}}(\mathbf{x})\|_1 = x_{i_{p+1}} + \ldots x_{i_m} < (m-p)\frac{\delta}{m} \le \delta$, which is a contradiction. Define

$$B_j := \{\mathbf{x} = (x_1, \ldots, x_m) \in \Delta_m | x_j \ge \frac{\delta}{m}\},\ j \in \{i_{p+1}, \ldots, i_m\}.$$

Therefore $B \subset \bigcup_{j=p+1}^{m} B_j$, which implies

$$\frac{\int_B P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} \le \sum_{j=p+1}^{m} \frac{\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}}.$$

Fix $r \in \{p+1, \ldots, m\}$ and $j = i_r$. With the change of variables $x_{i_r} = \frac{\delta}{m} + \left(1 - \frac{\delta}{m}\right) y_{i_r}$ and $x_\ell = \left(1 - \frac{\delta}{m}\right) y_\ell$, $\ell \in \{1, \ldots, m\} \setminus \{r\}$; the condition $\mathbf{x} \in B_j$ is equivalent to $(y_1, \ldots, y_m) \in \Delta_m$. We obtain $x_0 = \left(1 - \frac{\delta}{m}\right) y_0$ and then

$$\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}}$$
$$= \left(1 - \frac{\delta}{m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_s}} \int_{\Delta_m} \prod_{\ell=0,\ \ell\ne i_r}^{m} y^{k_\ell\rho+a_\ell} \left(\frac{\delta}{m} + \left(1 - \frac{\delta}{m}\right) y_{i_r}\right)^{a_{i_r}} dy_1 \ldots dy_m.$$

We have $\left(\frac{\delta}{m} + \left(1 - \frac{\delta}{m}\right) y_{i_r}\right)^{a_{i_r}} \le \left(\frac{\delta}{m}\right)^{a_{i_r}} < \left(\frac{\delta}{m}\right)^{-1} = \frac{m}{\delta}$. Then

$$\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}} \leq \frac{m}{\delta}\left(1-\frac{\delta}{m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \times \frac{\prod_{s=0}^{p}\Gamma(k_{i_s}\rho+a_{i_s}+1)\prod_{s=p+1,\ s\neq r}^{m}\Gamma(a_{i_s}+1)\Gamma(1)}{\Gamma(n\rho+|\mathbf{a}|-a_{i_r}+m+1)}.$$

Using also relation (3.16), it follows

$$\frac{\int_{B_j} P_{\mathbf{k}}^{\rho,\mathbf{a}}}{\int_{\Delta_m} P_{\mathbf{k}}^{\rho,\mathbf{a}}} \leq \frac{m}{\delta}\left(1-\frac{\delta}{m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \frac{\Gamma(n\rho+|\mathbf{a}|+m+1)}{\Gamma(n\rho+|\mathbf{a}|-a_j+m+1)\Gamma(a_j+1)}.$$

But

$$\lim_{\mathbf{a}\to\mathbf{1}}\left(1-\frac{\delta}{m}\right)^{m+n\rho+|\mathbf{a}|-a_{i_r}} \frac{\Gamma(n\rho+|\mathbf{a}|+m+1)}{\Gamma(n\rho+|\mathbf{a}|-a_j+m+1)} = \left(1-\frac{\delta}{m}\right)^{n\rho}\frac{\Gamma(n\rho)}{\Gamma(n\rho+1)}$$

and $\lim_{\mathbf{a}\to\mathbf{1}}\Gamma(a_j+1)=\infty$. Then, the corresponding relations (3.22) are true. Now, it is simple to deduce that (3.20) is valid. □

Remark 3.2. In the case $\rho=1$, Theorem 3.1 was proved in [28] but using a method which is not applicable here. In unidimensional case, Theorem 3.1 was proved in [12].

Theorem 3.2. *For any $f\in C(\Delta_m)$, we have*

(3.23) $$\lim_{\rho\to\infty}\mathbb{U}_n^{\rho}(f)(\mathbf{x})=\mathbb{B}_n(f)(\mathbf{x}),\ \textit{uniformly for } \mathbf{x}\in\Delta_m.$$

Proof. It is sufficient to show that

(3.24) $$\lim_{\rho\to\infty}F_{n,\mathbf{k}}^{\rho}(f)=f\left(\frac{k_1}{n},\ldots,\frac{k_m}{n}\right),\ \mathbf{k}=(k_0,k_1,\ldots,k_m)\in\Lambda,\ f\in C(\Delta_m).$$

Let supp $\mathbf{k}=\{i_0,\ldots,i_p\}\subset\{0,1,\ldots,m\}$, $p\geq 0$. If $p=0$, then relation (3.23) is immediate. Now consider that $p\geq 1$. We introduce simplified notations as follows. Denote $\mu_j=k_{i_j}$, $(0\leq j\leq p)$. Recall that $D_{\mathbf{k}}=\left\{\sum_{j=0}^{p}y_j\mathbf{e}_{i_j}|y_j\geq 0,\ (0\leq j\leq p),\ \sum_{j=0}^{p}y_j=1\right\}$. Define $\varphi:\Delta_p\to\mathbb{R}$ by

$$\varphi(\mathbf{y})=\prod_{j=1}^{p}y_j^{\mu_j}(1-|\mathbf{y}|)^{\mu_0},\ \mathbf{y}=(y_1,\ldots,y_p)\in\Delta_p.$$

We have $\varphi\geq 0$ on Δ_p. Since $\mu_j\geq 1$, $0\leq j\leq p$, it follows that $\varphi=0$ on the frontier of Δ_p. Consequently, the maximum of φ is reached in the interior of domain Δ_p. It is simple to show that the unique interior critical point of φ is $\mathbf{y}^*=\left(\frac{\mu_1}{n},\ldots,\frac{\mu_p}{n}\right)\in\Delta_p$. Then, $\mathbf{y}^*$ is the unique maximum point of φ.
Define $g\in C(\Delta_p)$, $g=f\circ\theta_{\mathbf{k}}$, where $\theta_{\mathbf{k}}$ was defined in (2.1).
Let $\varepsilon>0$ arbitrarily chose. We can choose a number $r>0$, such that $B_r(\mathbf{y}^*)=\{\mathbf{y}\in\mathbb{R}^p|\ \|\mathbf{y}-\mathbf{y}^*\|<r\}\subset\mathrm{Int}\Delta_p$ and $|g(\mathbf{y})-g(\mathbf{y}^*)|<\frac{\varepsilon}{2}$, for all $\mathbf{y}\in B_r(\mathbf{y}^*)$. Define $M=\max\{\varphi(\mathbf{y})|\ \mathbf{y}\in\overline{\Delta_p\setminus B_r(\mathbf{y}^*)}\}$. Then $M<\varphi(\mathbf{y}^*)$. Choose $M<M_1<\varphi(\mathbf{y}^*)$. There is $\delta>0$, such that $0<\delta<r$ and $\varphi(\mathbf{y})\geq M_1$, for all $\mathbf{y}\in B_\delta(\mathbf{y}^*)$.
For $\rho>1$, define $\Psi=\Psi_{\rho,\mathbf{k}}\in C(\Delta_p)$, $\Psi=Q_{\mathbf{k}}^{\rho}\circ\theta_{\mathbf{k}}$, where $Q_{\mathbf{k}}^{\rho}$ and $\theta_{\mathbf{k}}$ are defined in (2.5) and (2.1), respectively. Then we can write $\Psi=\varphi^{\rho-1}\cdot\eta$, where

$$\eta(\mathbf{y})=\prod_{j=1}^{p}y_j^{\mu_j-1}(1-|\mathbf{y}|)^{\mu_0-1},\ \mathbf{y}=(y_1,\ldots,y)\in\Delta_p.$$

We have

$$\int_{\Delta_p \setminus B_r(y^*)} \Psi = \int_{\Delta_p \setminus B_r(y^*)} (\varphi)^{\rho-1}\eta \leq \|\eta\| M^{\rho-1} \mathrm{vol}(\Delta_p)$$

and

$$\int_{B_r(y^*)} \Psi \geq \int_{B_\delta(y^*)} (\varphi)^{\rho-1}\eta \geq h \cdot M_1^{\rho-1} \mathrm{vol}(B_\delta(\mathbf{y}^*)),$$

where $h = \min\{\eta(\mathbf{y}) |\ \mathbf{y} \in \overline{B_r(\mathbf{y}^*)}\} > 0$. Then

$$\frac{\int_{\Delta_p \setminus B_r(\mathbf{y}^*)} \Psi}{\int_{B_r(y^*)} \Psi} \leq \frac{\|\eta\| \cdot \mathrm{vol}(\Delta_p)}{h \cdot \mathrm{vol}(B_\delta(\mathbf{y}^*))} \left(\frac{M}{M_1}\right)^{\rho-1}.$$

It is possible to choose $\rho_0 > 1$, such that

$$2\|f\| \cdot \frac{\int_{\Delta_p \setminus B_r(\mathbf{y}^*)} \Psi}{\int_{B_r(y^*)} \Psi} < \frac{\varepsilon}{2},\ \forall \rho > \rho_0.$$

Using formula (2.2) or formula (2.3) depending on the condition $0 \in \mathrm{supp}\ \mathbf{k}$ or $0 \notin \mathrm{supp}\ \mathbf{k}$, in both cases it results

$$F_{n,\mathbf{k}}^\rho(f) = \frac{\int_{D_\mathbf{k}} f \cdot Q_\mathbf{k}^\rho\, d\sigma}{\int_{D_\mathbf{k}} Q_\mathbf{k}^\rho\, d\sigma} = \frac{\int_{\Delta_p} (f \cdot Q_\mathbf{k}^\rho) \circ \theta_\mathbf{k}}{\int_{\Delta_p} Q_\mathbf{k}^\rho \circ \theta_\mathbf{k}} = \frac{\int_{\Delta_p} g \cdot \Psi}{\int_{\Delta_p} \Psi}.$$

Then, for $\rho > \rho_0$:

$$\begin{aligned}
|F_{n,\mathbf{k}}^\rho(f) - g(\mathbf{y}^*)| &= \left| \frac{\int_{\Delta_p} g \cdot \Psi}{\int_{\Delta_p} \Psi} - g(\mathbf{y}^*) \right| \\
&\leq \frac{\int_{\Delta_p} |(g - g(y^*)| \cdot \Psi}{\int_{\Delta_p} \Psi} \\
&\leq \frac{\int_{\Delta_p \setminus B_r(\mathbf{y}^*)} |g - g(y^*)| \cdot \Psi}{\int_{\Delta_p} \Psi} + \frac{\int_{B_r(\mathbf{y}^*)} |(g - g(y^*)| \cdot \Psi}{\int_{\Delta_p} \Psi} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Finally note that $g(\mathbf{y}^*) = f\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)$. □

4. Convergence properties

The moments of operators play a crucial role in the study of the convergence properties of a sequence of linear positive operators. The computation of moments of operators $\mathbb{M}_n^{\rho,\mathbf{a}}$ and $\mathbb{U}_n^\rho$ can be reduced to the moments of the Bernstein operators $\mathbb{B}_n$.

Define the functions $\mathbf{1}_{\Delta_m} \in C(\Delta_m)$, $\mathbf{1}_{\Delta_m}(\mathbf{x}) = 1$ and $\mathrm{pr}_i \in C(\Delta_m)$, $(1 \leq i \leq m)$, $\mathrm{pr}_i(\mathbf{x}) = x_i$, $\mathbf{x} = (x_1, \ldots, x_m) \in \Delta_m$.

Define

$$\| \bullet - \overline{x}\|(t_1, \ldots, t_m) = \sqrt{\sum_{i=1}^m (t_i - x_i)^2}.$$

Lemma 4.2. *For $m \in \mathbb{N}$, $\mathbf{a} = (a_0, \ldots, a_m) \in \mathbb{R}^{m+1}$, $\mathbf{a} > -\mathbf{1}$, $\rho \geq 1$, $n \in \mathbb{N}$ and $\mathbf{x} = (x_1, \ldots, x_m) \in \Delta_m$:*

i) $\mathbb{M}_n^{\rho,\mathbf{a}}(\mathbf{1}_{\Delta_m})(\mathbf{x}) = 1$,

ii) $\mathbb{M}_n^{\rho,\mathbf{a}}(\mathrm{pr}_i)(\mathbf{x}) = \frac{n\rho x_i + a_i + 1}{\rho n + |\mathbf{a}| + m + 1}$, $(1 \leq i \leq m)$, *iii)* $\mathbb{M}_n^{\rho,\mathbf{a}}(\|\bullet - \mathbf{x}\|^2)(\mathbf{x}) = \sum_{i=1}^m \frac{n\rho(\rho+1)x_i(1-x_i)+\lambda_i(\mathbf{a},m,\mathbf{x})}{(\rho n+|\mathbf{a}|+m+1)(\rho n+|\mathbf{a}|+m+2)}$, *where*

$$\lambda_i(\mathbf{a},m,\mathbf{x}) := (|\mathbf{a}|+m+1)(|\mathbf{a}|+m+2)x_i^2 - 2(|\mathbf{a}|+m+2)(a_i+1)x_i + (a_i+1)(a_i+2).$$

Proof. For any $\mathbf{k} \in \Lambda$, $\mathbf{k} = (k_0, \dots, k_m)$, we have

a) $F_{n,\mathbf{k}}^{\rho}(\mathbf{1}_{\Delta_m}) = 1$;

b) $F_{n,\mathbf{k}}^{\rho}(\mathrm{pr}_i) = \frac{\rho k_i + a_i + 1}{\rho n + |\mathbf{a}| + m + 1}$, $(1 \le i \le m)$;

c) $F_{n,\mathbf{k}}^{\rho}(\mathrm{pr}_i^2) = \frac{(\rho k_i + a_i + 1)(\rho k_i + a_i + 2)}{(\rho n + |\mathbf{a}| + m + 1)(\rho n + |\mathbf{a}| + m + 2)}$, $(1 \le i \le m)$.

Then, we can apply the known results for Bernstein operator on a simplex. □

By passing to limit $\mathbf{a} \to -\mathbf{1}$ and using Lemma 4.2 and Theorem 3.1, we obtain:

Corollary 4.1. *For $m \in \mathbb{N}$, $\rho \ge 1$, $n \in \mathbb{N}$ and $\mathbf{x} = (x_1, \dots, x_m) \in \Delta_m$:*

i) $\mathbb{U}_n^{\rho}(\ell) = \ell$, *for any affine function,*

ii) $\mathbb{U}_n^{\rho}(\|\bullet - \mathbf{x}\|^2)(\mathbf{x}) = \frac{\rho+1}{n\rho+1} \sum_{i=1}^m x_i(1-x_i)$.

Lemma 4.3. *For $m \ge 2$, we have*

$$\max\left\{ \sum_{i=1}^m x_i(1-x_i) \Big| (x_1, \dots, x_m) \in \Delta_m \right\} = \frac{m-1}{m}.$$

Proof. We can apply for instance the Kuhn-Tucker conditions for this maximization problem and the optimum is obtained for $x_i = \frac{1}{m}$, $(1 \le i \le m)$. □

For $f \in C(\Delta_m)$, $h > 0$, define

$$\omega_1(f,h) = \sup\{|f(\mathbf{x}) - f(\mathbf{y})|,\ \mathbf{x}, \mathbf{y} \in \Delta_m,\ \|\overline{x} - \overline{y}\| \le h\}.$$

Theorem 4.3. *For $m \in \mathbb{N}$, $\mathbf{a} = (a_0, \dots, a_m) \in \mathbb{R}^{m+1}$, $\mathbf{a} > -\mathbf{1}$, and $\rho \ge 1$, we have*

$$\|\mathbb{M}_n^{\rho,\mathbf{a}}(f) - f\| \le 2\omega_1(f, \sqrt{\mu_n}),\ f \in C(\Delta_m),\ n \in \mathbb{N}, \tag{4.25}$$

where

$$\mu_n = \sup_{\mathbf{x} \in \Delta_m} \mathbb{M}_n^{\rho,\mathbf{a}}(\|\bullet - \mathbf{x}\|^2)(\mathbf{x})$$

and

$$\mu_n = \mathrm{O}\left(\frac{1}{n}\right), \text{ uniformly with regard to } \rho \in [1, \infty).$$

Proof. For $m \ge 2$, from Lemma 4.2, Lemma 4.3, since $|\mathbf{a}| + m + 1 > 0$, it follows:

$$\mu_n \le \sum_{i=1}^m \frac{n\rho(\rho+1)x_i(1-x_i) + \|\lambda_i(\mathbf{a}, m, \bullet)\|}{(n\rho)^2}$$
$$\le \frac{1}{n}\left[\frac{\rho+1}{\rho} \cdot \frac{m-1}{m} + \frac{1}{n\rho^2} \sum_{i=1}^m \|\lambda_i(\mathbf{a}, m, \bullet)\|\right] \le \frac{1}{n}\left(2 + \sum_{i=1}^m \|\lambda_i(\mathbf{a}, m, \bullet)\|\right).$$

This final estimate exists also in the case $m = 1$. Then, we apply the generalized theorem of Shisha and Mond, given in Altomare and Campiti [2]- Proposition 5.1.5. in the following form:

$$|L(f)(\mathbf{x}) - f(\mathbf{x})| \le \left(1 + \frac{1}{\delta^2}(L(e)(\mathbf{x}) - e(\mathbf{x}))\right)\omega_1(f, \delta),$$

where $L : C(K) \to B(K)$ is a positive linear operator which preserves affine functions, K is a compact set in an inner product space, $e(\mathbf{x}) = \|\mathbf{x}\|^2$, $\mathbf{x} \in K$, $f \in C(K)$ and $\delta > 0$. Here, we take $K = \Delta_m$, $L = \mathbb{M}_n^{\rho,\mathbf{a}}$ and $\delta = \sqrt{\mu_n}$. □

Theorem 4.4. *For $m \in \mathbb{N}$ and $\rho \geq 1$, we have*

$$|\mathbb{U}_n^\rho(f)(\mathbf{x}) - f(\mathbf{x})| \leq 2\omega_1\left(f, \sqrt{\frac{\rho+1}{n\rho+1}\sum_{i=1}^m x_i(1-x_i)}\right), \ f \in C(\Delta_m), \ n \in \mathbb{N}, \ \mathbf{x} \in \Delta_m,$$

$$\|\mathbb{U}_n^\rho(f) - f\| \leq 2\omega_1\left(f, \sqrt{\frac{\rho+1}{n\rho+1} \cdot \max\left\{\frac{1}{4}, \frac{m-1}{m}\right\}}\right), \ f \in C(\Delta_m), \ n \in \mathbb{N}.$$

Proof. We apply Corollary 4.1, Lemma 4.3 and the generalized theorem of Shisha and Mond as in the proof of Theorem 4.3. □

Corollary 4.2. *For any $f \in C(\Delta_m)$, we have*

i) $\lim_{n\to\infty} \|\mathbb{M}_n^{\rho,\mathbf{a}}(f) - f\| = 0$, *where* $\mathbf{a} > -\mathbf{1}$, $\rho \geq 0$,
ii) $\lim_{n\to\infty} \|\mathbb{U}_n^\rho(f) - f\| = 0$, *where* $\rho \geq 0$, $m \geq 2$.

Corollary 4.3. *For any $m \in \mathbb{N}$, $\rho \geq 1$ and $n \in \mathbb{N}$, operator $\mathbb{U}_n^\rho$ interpolates each function $f \in C(\Delta_m)$ in the vertices of the simplex Δ_m, i.e.,*

$$\mathbb{U}_n^\rho(f)(\mathbf{e}_i) = f(\mathbf{e}_i), \ (0 \leq i \leq m).$$

More refined estimates with second order moduli can be given for operators $\mathbb{U}_n^\rho$ because they reproduce the affine functions.

For $f \in C(\Delta_m, Y)$, $h > 0$, define

$$\omega_2(f,h) = \sup\left\{\left|f(\mathbf{x}) - 2f\left(\frac{1}{2}(\mathbf{x}+\mathbf{y})\right) + f(\mathbf{y})\right|, \ \mathbf{x}, \mathbf{y} \in \Delta_m, \ \|\mathbf{x}-\mathbf{y}\| < h\right\}.$$

We apply the following scalar version of a theorem given in [23, Th. 7.2.4].

Theorem A. *Let $D \subset \mathbb{R}^m$ be a compact convex set. Let $F : C(D) \to \mathbb{R}$ be a functional given by a positive Borel measure μ. Suppose $\mu(D) = 1$. Let $\mathbf{x} \in D$ be the barycenter of μ. Then*

$$|F(f) - f(\mathbf{x})| \leq \left[m + \frac{1}{2}h^{-2}F(\|\bullet - \mathbf{x}\|^2)\right]\omega_2(f,h)$$

for $f \in C(D)$, $h > 0$.

Theorem 4.5. *For $n \in \mathbb{N}$, $\rho > 0$, $f \in C(\Delta_m)$, $m \geq 2$ and $h > 0$*

$$\|\mathbb{U}_n^\rho(f) - f\| \leq \left(m + \frac{1}{2h^2}\frac{\rho+1}{\rho n+1} \cdot \frac{m-1}{m}\right)\omega_2(f,h).$$

Proof. For any fixed $\mathbf{x} \in \Delta_m$, define the functional on $C(\Delta_m)$, $F(f) = \mathbb{U}_n^\rho(f, \mathbf{x})$. This is a functional defined by a positive Borel measure, say μ. From Corollary 4.1 - i), it follows that $\mathbf{x}$ is the barycenter of μ. Then, we can apply Theorem A. □

An other second modulus can be defined as follows. For $f \in C(\Delta_m)$ and $h > 0$, define

$$\tilde{\omega}_2(f,h) = \sup\left\{\left|\sum_{i=1}^p \lambda_i f(\mathbf{y}_i) - f(\mathbf{x})\right|, \ p \in \mathbb{N}, \ \mathbf{x}, \mathbf{y}_i \in \Delta_m,\right.$$

$$\left.\mathbf{x} = \sum_{i=1}^p \lambda_i \mathbf{y}_i, \ \lambda_i \in (0,1), \ \sum_{i=1}^p \lambda_i = 1, \ \|\mathbf{x} - \mathbf{y}_i\| \leq h\right\}.$$

The theorem below is a scalar version of a result given in [23, Th. 6.2.9].

Theorem B. *Let $D \subset \mathbb{R}^m$ be a compact convex set. Let $F : C(D) \to \mathbb{R}$ be a functional given by a positive Borel measure μ. Suppose $\mu(D) = 1$. Let $\mathbf{x} \in D$ be the barycenter of μ. Then*

$$|F(f) - f(\mathbf{x})| \leq \left[1 + h^{-2}F(\| \bullet - \overline{x}\|^2)\right]\tilde{\omega}_2(f, h)$$

for $f \in C(D)$ and $h > 0$.

In a similar mode as in the proof of Theorem 4.5, we obtain

Theorem 4.6. *For $n \in \mathbb{N}$, $\rho > 0$, $f \in C(\Delta_m)$, $m \in \mathbb{N}$ and $h > 0$,*

$$\|\mathbb{U}_n^\rho(f) - f\| \leq \left(1 + h^{-2}\frac{\rho + 1}{\rho n + 1} \cdot \max\left\{\frac{1}{4}, \frac{m-1}{m}\right\}\right)\tilde{\omega}_2\left(f, h\right).$$

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