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Quaternionic $(1,3)-$Bertrand Curves According to Type 2-Quaternionic Frame in $\mathbb{R}^4$

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Abstract

If there exists a quaternionic Bertrand curve in $\mathbb{E}^4$, then its torsion or bitorsion vanishes. So we can say that there is no quaternionic Bertrand curves whose torsion and bitorsion are non-zero. Hence by using the method which is given by Matsuda and Yorozu [13], we give the definition of quaternionic $(1,3)-$Bertrand curve according to Type 2-Quaternionic Frame and obtain some results about these curves.

Keywords: *Bertrand curve, Quaternions, Quaternionic Frame, Euclidean Space.*
2010 Mathematics Subject Classification: *53A04, 11R52.*

1. Introduction

Bertrand curve was introduced by *Bertrand* in 1850 (see [1]). When a curve is given, if there exists a second curve whose the principal normal is the principal normal of that curve, then the first curve is called Bertrand curve and the second curve is called the Bertrand mate of the first curve. The most important properties of Bertrand curves in Euclidean 3-space are that the distance between corresponding points is constant and there is a linear relation between the curvature functions of the first curve, that is, for $\lambda,\mu\in\mathbb{R}$, $\lambda\kappa+\mu\tau=1$, where κ is curvature and τ is the torsion of the first curve. Also the absolute value of the real number λ in this linear relation is equal to the distance between corresponding points of Bertrand curves. The Bertrand curves in Euclidean 3-space was extended by L. R. Pears to Riemannian $n-$space and gave general results for Bertrand curves [16]. If these general results were applied to Euclidean $n-$space, then either torsion or bitorsion of the curve vanishes. Otherwise, for $n\geq 4$, then no special Frenet curve in $\mathbb{E}^n$ is a Bertrand curve [13]. Hence, Matsuda and Yorozu gave a new definition of Bertrand curve which is called $(1,3)-$Bertrand curve and obtain a characterization of $(1,3)-$Bertrand curve. [13]. After then many researchers have made a lot of papers about $(1,3)-$Bertrand curves [4], [6], [9], [18], [19], [20].

In 1987, Bharathi and Nagaraj introduced the Serret-Frenet formulas for spatial quaternionic curves in $\mathbb{R}^3$ and quaternionic curves in $\mathbb{R}^4$ [2]. Since the quaternionic multiplication of two orthogonal vectors in $\mathbb{R}^3$ becomes vector product of these vectors, they reconsider the Serret-Frenet formulae of any curve in $\mathbb{R}^3$ which is well known in differential geometry by using the quaternionic multiplication and then they compose the Serret- Frenet formulae of quaternionic curves in $\mathbb{R}^4$ by means of the the Serret-Frenet formulas of spatial quaternionic curves in $\mathbb{R}^3$ [2]. After then various studies have been carried out on the adaptation of some special curves to quaternionic curves [3], [5], [7], [8], [11], [14], [15], [17], [21], [22], [23]. Keçilioğlu and İlarslan defined $(1,3)-$Bertrand curves for quaternionic curves in Euclidean 4-space and obtained a characterization for such curves [12].

Also, Kahraman Aksoyak defined a new type of quaternionic frame for quaternionic curves in Euclidean 4- space which is called Type 2-Quaternionic Frame [10].

In this paper, by using the method which is given by Matsuda and Yorozu [13], we give the definition of quaternionic $(1,3)-$Bertrand curve according to Type 2-Quaternionic Frame and obtain some results about these curves.

2. Preliminaries

A real quaternion is defined as:

$$q=q_0+q_1e_1+q_2e_2+q_3e_3$$

where $q_t\in\mathbb{R}$ for $0\leq t\leq 3$ and e_1, e_2, e_3 are unit vectors in usual three dimensional real vector space. Any quaternion q can be divided into two parts such that the scalar part denoted by S_q and the vectorial part denoted by V_q, where $S_q=q_0$ and $V_q=q_1e_1+q_2e_2+q_3e_3$. So, we

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can rewrite any real quaternion as $q = S_q + V_q$. If $q = S_q + V_q$ and $q' = S_{q'} + V_{q'}$ are any two quaternions, addition, the multiplication by a real scalar c and the conjugate of q denoted by γq are defined as, respectively:

$$\begin{aligned} q+q' &= \left(S_q + S_{q'}\right) + \left(V_q + V_{q'}\right) \\ cq &= cS_q + cV_q \\ \gamma q &= S_q - V_q \end{aligned}$$

Let denote the set of quaternions by Q. Q is a real vector space according to this addition and scalar multiplication. A basis of this vector space is $\{1,\ e_1,\ e_2,\ e_3\}$ and it is a four dimensional vector space. Hence we can think of any quaternion q as an element (q_0, q_1, q_2, q_3) of $\mathbb{R}^4$. Even a quaternion whose the scalar part is zero (it is called spatial quaternion) can be considered as a ordered triple (q_1, q_2, q_3) of $\mathbb{R}^3$. The product of two quaternions is defined by means of the multiplication rule between the units $e_1,\ e_2,\ e_3$ are given by:

$$e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1 \tag{2.1}$$

So, by using (2.1), quaternionic multiplication is obtained as:

$$q \times q' = S_q S_{q'} - \langle V_q, V_{q'} \rangle + S_q V_{q'} + S_{q'} V_q + V_q \wedge V_{q'} \text{ for every } q,\ q' \in Q,$$

where $\langle , \rangle$ and $\wedge$ denote the inner product and cross products in $\mathbb{R}^3$, respectively. The quaternion multiplication is associative and distributed but non-commutative. So Q is a real algebra and it is called quaternion algebra.
Now, the symetric, non-degenerate, bilinear form h on Q is defined as :

$$h : Q \times Q \to \mathbb{R},$$

$$h(q,q') = \frac{1}{2}(q \times \gamma q' + q' \times \gamma q) \text{ for } q,\ q' \in Q$$

and the norm of any real quaternion q is determined as:

$$\|q\|^2 = h(q,q) = q \times \gamma q = S_q^2 + \left\langle V_q, V_q \right\rangle.$$

So the mapping h is called the quaternion (or Euclidean) inner product [2].
In this paper, a quaternionic curve in $\mathbb{R}^4$ is denoted by $\alpha^{(4)}$ and the spatial quaternionic curve in $\mathbb{R}^3$ associated with $\alpha^{(4)}$ in $\mathbb{R}^4$ is denoted by α. Bharathi and Nagaraj introduced the Serret-Frenet formulas for spatial quaternionic curves in $\mathbb{R}^3$ and quaternionic curves in $\mathbb{R}^4$ follow as:

Theorem 2.1. *(see [2])Let $I = [0,1]$ denote the unit interval in the real line $\mathbb{R}$ and S be the set of spatial quaternionic curve*

$$\alpha : I \subset \mathbb{R} \longrightarrow S,$$

$$s \longrightarrow \alpha(s) = \alpha_1(s)e_1 + \alpha_2(s)e_2 + \alpha_3(s)e_3$$

be an arc-lenghted curve. Then the Frenet equations of α are as:

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

where $t = \alpha'$ is unit tangent, n is unit principal normal, $b = t \times n$ is binormal, where $\times$ denotes the quaternion product. $k = \|t'\|$ is the principal curvature and r is the torsion of the curve γ. Morever these Frenet vectors hold the following equations:

$$\begin{aligned} h(t,t) &= h(n,n) = h(b,b) = 1, \\ h(t,n) &= h(t,b) = h(n,b) = 0. \end{aligned}$$

Theorem 2.2. *(see [2]) Let $I = [0,1]$ denote the unit interval in the real line $\mathbb{R}$ and*

$$\alpha^{(4)} : I \subset \mathbb{R} \longrightarrow Q,$$

$$s \longrightarrow \alpha^{(4)}(s) = \alpha_0^{(4)}(s) + \alpha_1^{(4)}(s)e_1 + \alpha_2^{(4)}(s)e_2 + \alpha_3^{(4)}(s)e_3$$

be an arc-length curve in $\mathbb{R}^4$. Then Frenet equations of $\alpha^{(4)}$ are given by

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (K-r) \\ 0 & 0 & -(K-r) & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

where $T = \frac{d\alpha^{(4)}}{ds}$, N_1, N_2, N_3 are the Frenet vectors of the curve $\alpha^{(4)}$ and $K = \left\| T' \right\|$ is the principal curvature, k is the torsion and $(K-r)$ is the bitorsion of the curve $\alpha^{(4)}$. There exists following relation between the Frenet vectors of $\alpha^{(4)}$ and the Frenet vectors of α

$$N_1(s) = t(s) \times T(s),\ N_2(s) = n(s) \times T(s),\ N_3(s) = b(s) \times T(s)$$

and these Frenet vectors satisfy the following equations:

$$\begin{aligned} h(T,T) &= h(N_1,N_1) = h(N_2,N_2) = h(N_3,N_3) = 1, \\ h(T,N_1) &= h(T,N_2) = h(T,N_3) = h(N_1,N_2) = h(N_1,N_3) = h(N_2,N_3) = 0. \end{aligned}$$

Type 2-Quaternionic Frame for a quaternionic curve in $\mathbb{R}^4$ is defined as:

Theorem 2.3. *(see [10])Let $I=[0,1]$ denote the unit interval in the real line $\mathbb{R}$ and*

$$\alpha^{(4)}: I\subset\mathbb{R}\longrightarrow Q,$$

$$s\longrightarrow\alpha^{(4)}(s)=\alpha_0^{(4)}(s)+\alpha_1^{(4)}(s)e_1+\alpha_2^{(4)}(s)e_2+\alpha_3^{(4)}(s)e_3$$

be an arc-length curve in $\mathbb{R}^4$. Then Frenet equations of $\alpha^{(4)}$ are given by

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix}=\begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & -r & 0 \\ 0 & r & 0 & (K-k) \\ 0 & 0 & -(K-k) & 0 \end{bmatrix}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} \tag{2.2}$$

where $T=\frac{d\alpha^{(4)}}{ds}$, N_1, N_2, N_3 are the Frenet vectors of the curve $\alpha^{(4)}$ and $K=\left\|T'\right\|$ is the principal curvature, $-r$ is the torsion and $(K-k)$ is the bitorsion of the curve $\alpha^{(4)}$. There exists following relation between the Frenet vectors of $\alpha^{(4)}$ and the Frenet vectors of α

$$N_1(s)=b(s)\times T(s),\ N_2(s)=n(s)\times T(s),\ N_3(s)=t(s)\times T(s)$$

and these Frenet vectors satisfy the following equations:

$$\begin{aligned} h(T,T) &= h(N_1,N_1)=h(N_2,N_2)=h(N_3,N_3)=1, \\ h(T,N_1) &= h(T,N_2)=h(T,N_3)=h(N_1,N_2)=h(N_1,N_3)=h(N_2,N_3)=0. \end{aligned}$$

3. Characterizations of the Quaternionic $(1,3)$-Bertrand Curve in Euclidean Space $\mathbb{R}^4$

If there exists a quaternionic Bertrand curve in $\mathbb{R}^4$, then the torsion $-r$ or bitorsion $K-k$ vanishes. So we can say that there is no quaternionic Bertrand curves whose torsion and bitorsion are non-zero. Hence by using the method which is given by Matsuda and Yorozu [13], we give the definition of quaternionic $(1,3)-$Bertrand curve according to Type 2-Quaternionic Frame and then obtain a characterization for such curves.

Definition 3.1. *Let $\alpha^{(4)}: I\subset\mathbb{R}\to\mathbb{R}^4$ and $\beta^{(4)}:\bar{I}\subset\mathbb{R}\to\mathbb{R}^4$ be a quaternionic curves. There exists a regular $C^\infty-$function $\varphi: I\to\bar{I}, s\to\varphi(s)=\bar{s}$ such that it corresponds each point $\alpha^{(4)}(s)$ of $\alpha^{(4)}$ to the point $\beta^{(4)}(s)$ of $\beta^{(4)}$, for all $s\in I$. If $(1,3)-$normal plane spanned by the normal vectors $N_1(s)$ and $N_3(s)$ at the each point $\alpha^{(4)}(s)$ of $\alpha^{(4)}$ coincides with $(1,3)-$normal plane spanned by the normal vectors $\bar{N}_1(\bar{s})$ and $\bar{N}_3(\bar{s})$ at the corresponding point $\beta^{(4)}(\bar{s})=\beta^{(4)}(\varphi(s))$ of $\beta^{(4)}$then we called $\alpha^{(4)}$ is a quaternionic $(1,3)-$Bertrand curve in $\mathbb{E}^4$ and $\beta^{(4)}$ is called a quaternionic $(1,3)-$Bertrand mate of $\alpha^{(4)}$.*

Theorem 3.2. *Let $\alpha^{(4)}: I\subset\mathbb{R}\to\mathbb{R}^4$ be a quaternionic curve whose the curvatures functions K, $-r$, $K-k$ and $\alpha: I\subset\mathbb{R}\to\mathbb{R}^3$ be a spatial quaternionic curve associated with quaternionic curve $\alpha^{(4)}$ in $\mathbb{R}^4$ with the curvatures k and r. Then $\alpha^{(4)}$ is a quaternionic $(1,3)-$Bertrand curve if and only if there exists constant real numbers $a\neq 0$, $b\neq 0$, c, d satifying*

$$ar(s)+b(K-k)(s)\neq 0, \tag{3.1}$$

$$aK(s)-c\left[ar(s)+b(K-k)(s)\right]=1, \tag{3.2}$$

$$cK(s)+r(s)=d(K-k)(s) \tag{3.3}$$

$$\left(1-c^2\right)K(s)r(s)+c\left(K^2(s)-r^2(s)-(K-k)^2(s)\right)\neq 0, \tag{3.4}$$

for all $s\in I$.

Proof. We suppose that $\alpha^{(4)}$ is a quaternionic $(1,3)$ Bertrand curve given by arc-lenght parameter s and $\beta^{(4)}$ is a quaternionic $(1,3)$-Bertrand mate of $\alpha^{(4)}$ with arc-lenght parameter $\bar{s}$. Then we have

$$\beta^{(4)}(\bar{s})=\beta^{(4)}(\varphi(s))=\alpha^{(4)}(s)+a(s)N_1(s)+b(s)N_3(s) \tag{3.5}$$

for all $s\in I$, where $a,b: I\to\mathbb{R}$ are differentiable functions. Taking the derivative of (3.5) with respect to s and using (2.2), we have

$$\begin{aligned} \bar{T}(\bar{s})\varphi'(s)= &\ [1-a(s)K(s)]T(s)+a'(s)N_1(s) \\ &-[a(s)r(s)+b(s)(K-k)(s)]N_2(s)+b'(s)N_3(s) \end{aligned} \tag{3.6}$$

for all $s\in I$.
Since $Sp\{N_1(s),N_3(s)\}=Sp\{\bar{N}_1(\bar{s}),\bar{N}_3(\bar{s})\}$, we can write

$$\bar{N}_1(\bar{s})=\cos\theta(s)N_1(s)+\sin\theta(s)N_3(s), \tag{3.7}$$

$$\bar{N}_3(\bar{s})=-\sin\theta(s)N_1(s)+\cos\theta(s)N_3(s). \tag{3.8}$$

We notice that $\sin\theta(s)\neq 0$. Otherwise, $\bar{N}_1(\bar{s})=\pm N_1(s)$. By using (3.6) and (3.7), we get

$$h(\bar{T}(\bar{s})\varphi'(s),\bar{N}_1(\bar{s}))=\cos\theta(s)a'(s)+\sin\theta(s)b'(s)=0. \tag{3.9}$$

By using (3.6) and (3.8), we get

$$h(\bar{T}(\bar{s})\varphi'(s),\bar{N}_3(\bar{s}))=-\sin\theta(s)a'(s)+\cos\theta(s)b'(s)=0. \tag{3.10}$$

From (3.9) and (3.10), since $\begin{vmatrix}\cos\theta(s) & \sin\theta(s)\\ -\sin\theta(s) & \cos\theta(s)\end{vmatrix}=1$, we find

$$a'(s)=0,\ b'(s)=0.$$

From above equalites, we obtain that a and b are real constants.
So, we can rewrite $\beta^{(4)}$ given by (3.5) as:

$$\beta^{(4)}(\bar{s})=\alpha^{(4)}(s)+aN_1(s)+bN_3(s) \tag{3.11}$$

and the unit tangent vector of $\beta^{(4)}$ is following:

$$\bar{T}(\bar{s})\varphi'(s)=(1-aK(s))T(s)-(ar(s)+b(K-k)(s))N_2(s), \tag{3.12}$$

where

$$(\varphi'(s))^2=(1-aK(s))^2+(ar(s)+b(K-k)(s))^2\neq 0 \tag{3.13}$$

for all $s\in I$, if we denote by

$$\cos\tau(s)=\left(\frac{1-aK(s)}{\varphi'(s)}\right),\ \sin\tau(s)=-\left(\frac{ar(s)+b(K-k)(s)}{\varphi'(s)}\right), \tag{3.14}$$

where τ is differentiable function on I, so we can rewrite (3.12) as:

$$\bar{T}(\bar{s})=\cos\tau(s)T(s)+\sin\tau(s)N_2(s) \tag{3.15}$$

If we calculate the derivative of (3.15) with respect to s and use (2.2), we obtain

$$\begin{aligned}\bar{K}(\bar{s})\bar{N}_1(\bar{s})\varphi'(s)&=(\cos\tau(s))'T(s)+[\cos\tau(s)K(s)+\sin\tau(s)r(s)]N_1(s)\\&\quad+(\sin\tau(s))'N_2(s)+\sin\tau(s)(K-k)(s)N_3(s)\end{aligned}$$

From (3.7), we know that $\bar{N}_1(\bar{s})\in Sp\{N_1(s),N_3(s)\}$. So, from the above equation

$$(\cos\tau(s))'=0,\ (\sin\tau(s))'=0,$$

and it means that $\tau=\tau_0$ is a real constant. Then we can rewrite (3.15) as:

$$\bar{T}(\bar{s})=\cos\tau_0(s)T(s)+\sin\tau_0(s)N_2(s) \tag{3.16}$$

and from (3.14), we get

$$\cos\tau_0\varphi'(s)=1-aK(s) \tag{3.17}$$

and

$$\sin\tau_0\varphi'(s)=-(ar(s)+b(K-k)(s)) \tag{3.18}$$

From (3.17) and (3.18)

$$(1-aK(s))\sin\tau_0=-(ar(s)+b(K-k)(s))\cos\tau_0 \tag{3.19}$$

If $\sin\tau_0$ vanishes, then $\cos\tau_0=\pm1$. And from (3.16), we get $\bar{T}(\bar{s})=\pm T(s)$. If we differentiate this equality and use (2.2), we have $\bar{N}_1(\bar{s})=\pm 1N_1(s)$. It is a contradiction. So $\sin\tau_0\neq 0$, that is, from (3.18) implies that

$$ar(s)+b(K-k)(s)\neq 0.$$

Hence we obtain the relation (3.1).
If we denote the constant c by $c=\frac{\cos\tau_0}{\sin\tau_0}$, from (3.19),

$$aK(s)-c(ar(s)+b(K-k)(s))=1$$

for all $s\in I$. Thus we find the relation (3.2). Differentiating (3.16) with respect to s and using the equations of Type 2- Quaternionic Frame given by (2.2), we have

$$\bar{K}(\bar{s})\bar{N}_1(\bar{s})\varphi'(s)=(\cos\tau_0K(s)+\sin\tau_0r(s))N_1(s)+\sin\tau_0(K-k)(s)N_3(s). \tag{3.20}$$

By using (3.20) we have

$$\left(\bar{K}(\bar{s})\varphi^{\prime}(s)\right)^2=(\sin\tau_0)^2\left[\left(\frac{\cos\tau_0}{\sin\tau_0}K(s)+r(s)\right)^2+((K-k)(s))^2\right].$$

By using (3.17) and (3.18) in above equality,

$$\left(\bar{K}(\bar{s})\varphi^{\prime}(s)\right)^2=(ar(s)+b(K-k)(s))^2\left[(cK(s)+r(s))^2+((K-k)(s))^2\right]\left(\varphi^{\prime}(s)\right)^{-2}. \tag{3.21}$$

On the other hand, from (3.2) and (3.13), we obtain

$$\left(\varphi^{\prime}(s)\right)^2=\left(1+c^2\right)(ar(s)+b(K-k)(s))^2 \tag{3.22}$$

Then if we consider with (3.21) and (3.22), we get

$$\left(\bar{K}(\bar{s})\varphi^{\prime}(s)\right)^2=\frac{1}{1+c^2}\left[(cK(s)+r(s))^2+((K-k)(s))^2\right] \tag{3.23}$$

By using (3.17), (3.18) and the ralation (3.2), we rewrite (3.20) as:

$$\bar{N}_1(\bar{s})=\cos\eta(s)N_1(s)+\sin\eta(s)N_3(s), \tag{3.24}$$

where

$$\cos\eta(s)=\frac{-(ar(s)+b(K-k)(s))(cK(s)+r(s))}{\bar{K}(\bar{s})\left(\varphi^{\prime}(s)\right)^2}, \tag{3.25}$$

and

$$\sin\eta(s)=\frac{-(ar(s)+b(K-k)(s))(K-k)(s)}{\bar{K}(\bar{s})\left(\varphi^{\prime}(s)\right)^2} \tag{3.26}$$

for $s\in I$. Here, η is differentiable function on I.
Taking the derivative of (3.24) and using the equations of Type 2- Quaternionic Frame given by (2.2), we have

$$\begin{aligned}\left(-\bar{K}(\bar{s})\bar{T}(\bar{s})-\bar{r}(\bar{s})\bar{N}_2(\bar{s})\right)\varphi^{\prime}(s) &= -\cos\eta(s)K(s)T(s)+(\cos\eta(s))^{\prime}N_1(s)\\ &\quad+(-\cos\eta(s)r(s)-\sin\eta(s)(K-k)(s))N_2(s)\\ &\quad+(\sin\eta(s))^{\prime}N_3(s)\end{aligned} \tag{3.27}$$

From (3.27), it satisfies

$$(\cos\eta(s))^{\prime}=0 \text{ and } (\sin\eta(s))^{\prime}=0,$$

that is, $\eta=\eta_0$ is a constant function on I. Let $d=\frac{\cos\eta_0}{\sin\eta_0}$ be a constant then from (3.25) and (3.26), we find following relation:

$$cK(s)+r(s)=d(K-k)(s).$$

Thus we obtain the relation (3.3).
Since $\eta=\eta_0$ is a constant function, we rewrite (3.27)

$$\begin{aligned}\left(-\bar{K}(\bar{s})\bar{T}(\bar{s})-\bar{r}(\bar{s})\bar{N}_2(\bar{s})\right)\varphi^{\prime}(s) &= -\cos\eta_0K(s)T(s)+\\ &\quad+(-\cos\eta_0r(s)-\sin\eta_0(K-k)(s))N_2(s)\end{aligned}$$

By considering above equation with (3.12), we get

$$\begin{aligned}-\bar{r}(\bar{s})\bar{N}_2(\bar{s})\varphi^{\prime}(s) &= \left(\bar{K}(\bar{s})\varphi^{\prime}(s)\frac{(1-aK(s))}{\varphi^{\prime}(s)}-\cos\eta_0K(s)\right)T(s)\\ &\quad+\left(\begin{array}{c}-\bar{K}(\bar{s})\varphi^{\prime}(s)\frac{(ar(s)+b(K-k)(s))}{\varphi^{\prime}(s)}\\ -\cos\eta_0r(s)-\sin\eta_0(K-k)(s)\end{array}\right)N_2(s)\\ &= \frac{1}{\bar{K}(\bar{s})\left(\varphi^{\prime}(s)\right)^2}\left\{A(s)T(s)+B(s)N_2(s)\right\},\end{aligned}$$

where

$$\begin{aligned}A(s) &= \left(\bar{K}(\bar{s})\varphi^{\prime}(s)\right)^2(1-aK(s))+(ar(s)+b(K-k)(s))(cK(s)+r(s))K(s),\\ B(s) &= -\left(\bar{K}(\bar{s})\varphi^{\prime}(s)\right)^2(ar(s)+b(K-k)(s))+(ar(s)+b(K-k)(s))(cK(s)+r(s))r(s)\\ &\quad+(ar(s)+b(K-k)(s))((K-k)(s))^2\end{aligned}$$

By using (3.23) and the ralation (3.2), we can rewrite $A(s)$ and $B(s)$ as:

$$A(s)=\left(1+c^2\right)^{-1}(ar(s)+b(K-k)(s))\left\{\left(1-c^2\right)K(s)r(s)+c\left(K^2(s)-r^2(s)-(K-k)^2(s)\right)\right\}$$

and

$$B(s)=-c\left(1+c^{2}\right)^{-1}(ar(s)+b(K-k)(s))\left\{\left(1-c^{2}\right)K(s)\,r(s)+c\left(K^{2}(s)-r^{2}(s)-(K-k)^{2}(s)\right)\right\}$$

Since $\bar{r}(\bar{s})\bar{N}_2(\bar{s})\varphi^{\shortmid}(s)\neq 0$ for $\forall s\in I$, we have

$$\left(1-c^{2}\right)K(s)\,r(s)+c\left(K^{2}(s)-r^{2}(s)-(K-k)^{2}(s)\right)\neq 0$$

for all $s\in I$. Thus we obtain the relation (3.4).
Conversely, let $\alpha^{(4)}:I\subset\mathbb{R}\to\mathbb{E}^4$ be a quaternionic curve with curvatures K, $-r$, $(K-k)\neq 0$ satisfying the equations (3.1), (3.2), (3.3), (3.4) for constant numbers a, b,c, d and $\beta^{(4)}$ be a quaternionic curve such that

$$\beta^{(4)}(s)=\alpha^{(4)}(s)+aN_1(s)+bN_3(s)$$

for all $s\in I$. Differentiating above equality with respect to s and using the equations of Type 2- Quaternionic Frame given by (2.2), we have

$$\frac{d\beta^{(4)}(s)}{ds}=(1-aK(s))\,T(s)-(ar(s)+b(K-k)(s))\,N_2(s),$$

thus, by using the relation (3.2), we obtain

$$\frac{d\beta^{(4)}(s)}{ds}=-\,(ar(s)+b(K-k)(s))\,(cT(s)+N_2(s))$$

for all $s\in I$. From the relation (3.1), since $ar(s)+b(K-k)(s)\neq 0$, the curve $\beta^{(4)}$ is a regular curve. Then there exists a regular $C^{\infty}-$function $\varphi:I\to\bar{I}$ defined by

$$\bar{s}=\varphi(s)=\int\left\|\frac{d\beta^{(4)}(s)}{ds}\right\|ds$$

where $\bar{s}$ denotes the arc-length parameter of $\beta^{(4)}$. Then

$$\varphi^{\shortmid}(s)=\varepsilon\sqrt{1+c^{2}}\,(ar(s)+b(K-k)(s)) \tag{3.28}$$

where if $ar(s)+b(K-k)(s)>0$ then $\varepsilon=1$, if $ar(s)+b(K-k)(s)<0$ then $\varepsilon=-1$ for all $s\in I$. Hence we can express $\beta^{(4)}$ again as:

$$\beta^{(4)}(\bar{s})=\beta^{(4)}(\varphi(s))=\alpha^{(4)}(s)+aN_1(s)+bN_3(s)$$

Differentiating the above equality with respect to s, we have

$$\varphi^{\shortmid}(s)\,\frac{d\beta^{(4)}(\bar{s})}{d\bar{s}}=-\,(ar(s)+b(K-k)(s))\,(cT(s)+N_2(s)) \tag{3.29}$$

Considering (3.28) and (3.29) with together, we can write

$$\bar{T}(\bar{s})=\frac{1}{\bar{\varepsilon}\sqrt{1+c^{2}}}\,(cT(s)+N_2(s)), \tag{3.30}$$

where $\bar{\varepsilon}=-\varepsilon$. Differentiating (3.30) with respect to s and using the equations of Type 2-Quaternionic Frame, we get

$$\varphi^{\shortmid}(s)\,\frac{d\bar{T}(\bar{s})}{d\bar{s}}=\frac{1}{\bar{\varepsilon}\sqrt{1+c^{2}}}\,((cK(s)+r(s))\,N_1(s)+(K-k)(s)N_3(s))$$

Then we can calculate curvature of $\beta^{(4)}$ as:

$$\bar{K}(\bar{s})=\left\|\frac{d\bar{T}(\bar{s})}{d\bar{s}}\right\|=\frac{\sqrt{(cK(s)+r(s))^{2}+((K-k)(s))^{2}}}{\varphi^{\shortmid}(s)\sqrt{1+c^{2}}}. \tag{3.31}$$

for all $s\in I$. From using the equations of Type 2-Quaternionic Frame given by (2.2), we can determine the unit normal vector $\bar{N}_1$ along $\beta^{(4)}$

$$\begin{aligned}\bar{N}_1(\bar{s}) &= \frac{1}{\bar{K}(\bar{s})}\frac{d\bar{T}(\bar{s})}{d\bar{s}}\\ &= \frac{((cK(s)+r(s))\,N_1(s)+(K-k)(s)N_3(s))}{\bar{\varepsilon}\sqrt{(cK(s)+r(s))^{2}+((K-k)(s))^{2}}}\end{aligned}$$

for all $s\in I$. Thus we can put

$$\bar{N}_1(\bar{s})=\cos\gamma(s)N_1(s)+\sin\gamma(s)N_3(s), \tag{3.32}$$

where

$$\cos\gamma(s)=\frac{cK(s)+r(s)}{\bar{\varepsilon}\sqrt{(cK(s)+r(s))^{2}+((K-k)(s))^{2}}} \tag{3.33}$$

and

$$\sin\gamma(s)=\frac{(K-k)(s)}{\bar{\varepsilon}\sqrt{(cK(s)+r(s))^2+((K-k)(s))^2}}, \tag{3.34}$$

So differentiating (3.32) with respect to s and using (2.2), we have

$$\begin{aligned}\frac{\bar{N}_1(\bar{s})}{d\bar{s}}\varphi^{'}(s) &= -K(s)\cos\gamma(s)T(s)+(\cos\gamma(s))^{'}N_1(s)\\ &+(-r(s)\cos\gamma(s)-(K-k)(s)\sin\gamma(s))N_2(s)+(\sin\gamma(s))^{'}N_3(s)\end{aligned}$$

On the other hand, from the relation (3.3), we get

$$\frac{cK(s)+r(s)}{(K-k)(s)}=d$$

Calculating the derivative of the last equation with respect to s, we find the following equality:

$$\left(cK^{'}(s)+r^{'}(s)\right)(K-k)(s)-(cK(s)+r(s))(K-k)^{'}(s)=0 \tag{3.35}$$

Taking the derivatives of (3.33) and (3.34) and using (3.35), we obtain

$$(\cos\gamma(s))^{'}=0 \text{ and } (\sin\gamma(s))^{'}=0,$$

that is, γ is a real constant with value γ_0. Thus we have

$$\cos\gamma_0=\frac{cK(s)+r(s)}{\bar{\varepsilon}\sqrt{(cK(s)+r(s))^2+(K-k)^2(s)}} \tag{3.36}$$

and

$$\sin\gamma_0=\frac{(K-k)(s)}{\bar{\varepsilon}\sqrt{(cK(s)+r(s))^2+(K-k)^2(s)}} \tag{3.37}$$

Hence we can rewrite (3.32) as:

$$\bar{N}_1(\bar{s})=\cos\gamma_0 N_1(s)+\sin\gamma_0 N_3(s) \tag{3.38}$$

Differentiating (3.38) with respect to s and using the equations of Type 2- Quaternionic Frame given by (2.2), (3.36), (3.37), we have

$$\begin{aligned}\frac{d\bar{N}_1(\bar{s})}{d\bar{s}} &= -\frac{(cK(s)+r(s))K(s)}{\bar{\varepsilon}\varphi^{'}(s)\sqrt{(cK(s)+r(s))^2+((K-k)(s))^2}}T(s)\\ &-\frac{(cK(s)+r(s))r(s)+((K-k)(s))^2}{\bar{\varepsilon}\varphi^{'}(s)\sqrt{(cK(s)+r(s))^2+((K-k)(s))^2}}N_2(s)\end{aligned}$$

By using (3.30) and (3.31), we have

$$\bar{K}(\bar{s})\bar{T}(\bar{s})=\frac{(cK(s)+r(s))^2+((K-k)(s))^2}{\bar{\varepsilon}\varphi^{'}(s)\left(1+c^2\right)\sqrt{(cK(s)+r(s))^2+((K-k)(s))^2}}(cT(s)+N_2(s))$$

By using the above equalities, we have

$$\frac{d\bar{N}_1(\bar{s})}{d\bar{s}}+\bar{K}(\bar{s})\bar{T}(\bar{s})=\frac{P(s)}{R(s)}T(s)+\frac{Q(s)}{R(s)}N_2(s),$$

where we can easily show

$$\begin{aligned}P(s) &= -\left[\left(1-c^2\right)K(s)r(s)+c\left\{K^2(s)-r^2(s)-(K-k)^2(s)\right\}\right]\\ Q(s) &= c\left[\left(1-c^2\right)K(s)r(s)+c\left\{K^2(s)-r^2(s)-(K-k)^2(s)\right\}\right]\\ R(s) &= \bar{\varepsilon}\varphi^{'}(s)\left(1+c^2\right)\sqrt{(cK(s)+r(s))^2+((K-k)(s))^2}\neq 0.\end{aligned}$$

Since $\frac{d\bar{N}_1(\bar{s})}{d\bar{s}}+\bar{K}(\bar{s})\bar{T}(\bar{s})=-\bar{r}(\bar{s})\bar{N}_2(\bar{s})$, we obtain the torsion of $\beta^{(4)}$

$$\begin{aligned}-\bar{r}(\bar{s}) &= \left\|\frac{d\bar{N}_1(\bar{s})}{d\bar{s}}+\bar{K}(\bar{s})\bar{T}(\bar{s})\right\|\\ &= \frac{1}{R(s)}\sqrt{P^2(s)+Q^2(s)}\\ &= \frac{\left|\left(1-c^2\right)K(s)r(s)+c\left\{K^2(s)-r^2(s)-(K-k)^2(s)\right\}\right|}{\varphi^{'}(s)\bar{\varepsilon}\sqrt{1+c^2}\sqrt{(cK(s)+r(s))^2+((K-k)(s))^2}}.\end{aligned} \tag{3.39}$$

Now we can define unit vector field $\bar{N}_2(\bar{s})$ along $\beta^{(4)}$,

$$\bar{N}_2(\bar{s}) = -\frac{1}{\bar{r}(\bar{s})}\left(\frac{d\bar{N}_1(\bar{s})}{d\bar{s}} + \bar{K}(\bar{s})\bar{T}(\bar{s})\right),$$

that is,

$$\bar{N}_2(\bar{s}) = \frac{1}{\bar{\varepsilon}\sqrt{1+c^2}}\left(-T(s) + cN_2(s)\right) \tag{3.40}$$

Also, we can define the unit vector field $\bar{N}_3(\bar{s})$ along $\beta^{(4)}$ as:

$$\begin{aligned} \bar{N}_3(\bar{s}) &= -\sin\gamma_0 N_1(s) + \cos\gamma_0 N_3(s) \\ &= \frac{1}{\bar{\varepsilon}\sqrt{(cK(s)+r(s))^2 + ((K-k)(s))^2}}\begin{pmatrix} -(K-k)(s)N_1(s) \\ +(cK(s)+r(s))N_3(s) \end{pmatrix} \end{aligned} \tag{3.41}$$

Finally we define the bitorsion of $\beta^{(4)}$

$$\begin{aligned} \left(\bar{K}-\bar{k}\right)(\bar{s}) &= \left\langle \frac{d\bar{N}_2(\bar{s})}{d\bar{s}}, \bar{N}_3(\bar{s}) \right\rangle \\ &= \frac{(K-k)(s)K(s)\sqrt{1+c^2}}{\varphi'(s)\sqrt{(cK(s)+r(s))^2 + ((K-k)(s))^2}} \end{aligned} \tag{3.42}$$

for all $s \in I$. Using the Frenet vectors $\bar{T}, \bar{N}_1, \bar{N}_2, \bar{N}_3$ we can easily see that

$$h(\bar{T},\bar{T}) = h(\bar{N}_1,\bar{N}_1) = h(\bar{N}_2,\bar{N}_2) = h(\bar{N}_3,\bar{N}_3) = 1,$$

and

$$h(\bar{T},\bar{N}_1) = h(\bar{T},\bar{N}_2) = h(\bar{T},\bar{N}_3) = h(\bar{N}_1,\bar{N}_2) = h(\bar{N}_1,\bar{N}_3) = h(\bar{N}_2,\bar{N}_3) = 0,$$

for all $s \in I$ where $\{\bar{T}(\bar{s}),\bar{N}_1(\bar{s}),\bar{N}_2(\bar{s}),\bar{N}_3(\bar{s})\}$ is Frenet frame along quaternionic curve β^4 in $\mathbb{E}^4$. And it is fact that $(1,3)$ normal plane $Sp\{N_1,N_3\}$ of $\alpha^{(4)}$ coincides $(1,3)$ normal plane $Sp\{\bar{N}_1,\bar{N}_3\}$ of $\beta^{(4)}$. Consequently, $\alpha^{(4)}$ is a quaternionic $(1,3)$ Bertrand curve in $\mathbb{E}^4$ and $\beta^{(4)}$ is quaternionic $(1,3)$ Bertrand mate of it. This completes the proof. □

Theorem 3.3. *Let $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ be a quaternionic $(1,3)$ Bertrand curve and $\beta^{(4)}$ be a quaternionic $(1,3)$ Bertrand mate of $\alpha^{(4)}$ and $\varphi : I \to \bar{I}$, $\bar{s} = \varphi(s)$ is a regular C^∞−function such that s and $\bar{s}$ are arc-length parameter of $\alpha^{(4)}$ and $\beta^{(4)}$, respectively. Then the distance between the points $\alpha^{(4)}(s)$ and $\beta^{(4)}(\bar{s})$ is constant for all $s \in I$.*

Proof. Let $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ be quaternionic $(1,3)$-Bertrand curve in $\mathbb{E}^4$ and $\beta^{(4)} : \bar{I} \subset \mathbb{R} \to \mathbb{E}^4$ be a quaternionic $(1,3)$-Bertrand mate of $\alpha^{(4)}$. Then we can write,

$$\beta^{(4)}(\bar{s}) = \alpha^{(4)}(s) + aN_1(s) + bN_3(s)$$

where a and b are non-zero constants. Thus, we can write

$$\beta^{(4)}(\bar{s}) - \alpha^{(4)}(s) = aN_1(s) + bN_3(s)$$

and

$$\left\|\beta^{(4)}(\bar{s}) - \alpha^{(4)}(s)\right\| = \sqrt{a^2+b^2}.$$

□

Theorem 3.4. *Let $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ be a quaternionic $(1,3)$-Bertrand curve such that $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ is a spatial quaternionic curve associated with $\alpha^{(4)}$. If $\beta^{(4)}$ is a quaternionic $(1,3)$-Bertrand mate of $\alpha^{(4)}$ then the curvature functions of $\beta^{(4)}$ are determined in terms of the principal curvature K of the curve $\alpha^{(4)}$ and the principal curvature k of the curve α as follows:*

$$\begin{aligned} \bar{K}(\bar{s}) &= \frac{c\sqrt{1+d^2}\,(K-k)(s)}{\bar{\varepsilon}\delta\left(1+c^2\right)(1-aK(s))}, \\ -\bar{r}(\bar{s}) &= \frac{c\left|\left(c\left(1+d^2\right)(K-k)(s) - \left(1+c^2\right)dK(s)\right)\right|}{\bar{\varepsilon}\left(1+c^2\right)\sqrt{1+d^2}\,(1-aK(s))}, \\ \bar{K}(\bar{s}) - \bar{k}(\bar{s}) &= \frac{cK(s)}{\bar{\varepsilon}\delta\sqrt{1+d^2}\,(1-aK(s))}, \end{aligned}$$

where δ is the signature of the curvature $K-k$, that is, $\delta(K-k) > 0$.

Proof. We suppose that $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ is a quaternionic curve whose the curvatures functions K, $-r$, $K-k$ and $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a spatial quaternionic curve associated with quaternionic curve $\alpha^{(4)}$ in $\mathbb{E}^4$ with the curvatures k and r. In that case for constant real numbers $a \neq 0$, $b \neq 0$, c, d hold the relations (3.1), (3.2), (3.3), (3.4). If $\beta^{(4)}$ is a quaternionic $(1,3)$-Bertrand mate of $\alpha^{(4)}$ then the curvature functions of $\beta^{(4)}$ are defined by the equations (3.31), (3.39) and (3.42) in Theorem 3.2. If we consider (3.31), (3.39) and (3.42) with the relations (3.1), (3.2), (3.3), (3.4), we obtain these curvature functions in terms of the principal curvature K of the curve $\alpha^{(4)}$ and the principal curvature k of the curve α. □

Remark 3.5. *We note that if $\alpha^{(4)}$ is a quaternionic $(1,3)$-Bertrand curve and $\beta^{(4)}$ is a quaternionic $(1,3)$-Bertrand mate of $\alpha^{(4)}$ then the curvature functions of $\beta^{(4)}$ is independent of the torsion $-r$ of $\alpha^{(4)}$.*

Theorem 3.6. *Let $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ be a quaternionic $(1,3)$-Bertrand curve and $\beta^{(4)}$ be a quaternionic $(1,3)$-Bertrand mate of $\alpha^{(4)}$. Then the curvature functions of the curve β which is a spatial quaternionic curve associated with $\beta^{(4)}$ are defined by*

$$\begin{aligned} \bar{k}(\bar{s}) &= \frac{c\left[\left(1+d^2\right)(K-k)(s)-\left(1+c^2\right)K(s)\right]}{\bar{\varepsilon}\delta\left(1+c^2\right)\sqrt{1+d^2}\,(1-aK(s))}, \\ \bar{r}(\bar{s}) &= -\frac{c\left|\left(c\left(1+d^2\right)(K-k)(s)-\left(1+c^2\right)dK(s)\right)\right|}{\bar{\varepsilon}\left(1+c^2\right)\sqrt{1+d^2}\,(1-aK(s))}. \end{aligned}$$

Proof. It is obvious from Theorem (3.4). □

Example 3.7. *We will examine a special case of the example which is given by Matsuda and Yorozu in [13] for quaternionic $(1,3)$-Bertrand curve according to Type 2-Quaternionic Frame and we will see that the Theorem (3.2) is provided with this example.*
Let consider a quaternionic curve $\alpha^{(4)}(s)$ in $\mathbb{R}^4$ defined by

$$\alpha^{(4)}(s) = \begin{pmatrix} \cos\left(\frac{3}{\sqrt{10}}s\right), \\ \sin\left(\frac{3}{\sqrt{10}}s\right), \\ \cos\left(\frac{1}{\sqrt{10}}s\right), \\ \sin\left(\frac{1}{\sqrt{10}}s\right) \end{pmatrix}.$$

for $s \in I$. The curve $\alpha^{(4)}$ is a unit speed regular curve. With the help of Type-2 Quaternionic Frame, we get α spatial quaternionic curves in $\mathbb{R}^3$ associated with quaternionic curve $\alpha^{(4)}$ in $\mathbb{R}^4$ as:

$$\alpha(s) = \frac{1}{\sqrt{82}}\left(\frac{6}{\sqrt{10}}s, -7\cos\left(\frac{4}{\sqrt{10}}s\right), -7\sin\left(\frac{4}{\sqrt{10}}s\right)\right).$$

The principal curvature and torsion of α are given

$$k = \frac{112}{10\sqrt{82}} \text{ and } r = \frac{24}{10\sqrt{82}}$$

Then we have the curvatures of $\alpha^{(4)}$ as :

$$K = \frac{82}{10\sqrt{82}},\ -r = -\frac{24}{10\sqrt{82}},\ K-k = -\frac{30}{10\sqrt{82}}.$$

For $a = 10\sqrt{82}$, $b = -10\sqrt{82}$, $c = \frac{3}{2}$, $d = -\frac{49}{10}$, the curvatures of quaternionic curve $\alpha^{(4)}$ satisfy the relations (3.1), (3.2), (3.3), (3.4). Hence $\alpha^{(4)}$ is a quaternionic $(1,3)$-Bertrand curve and its quaternionic $(1,3)$-Bertrand mate $\beta^{(4)}$ is obtained as:

$$\beta^{(4)}(\bar{s}) = 9\begin{pmatrix} -11\cos\left(\frac{3\bar{s}}{27\sqrt{130}}\right), \\ -11\sin\left(\frac{3\bar{s}}{27\sqrt{130}}\right), \\ 9\cos\left(\frac{\bar{s}}{27\sqrt{130}}\right), \\ 9\sin\left(\frac{\bar{s}}{27\sqrt{130}}\right) \end{pmatrix}.$$

where $\bar{s} = \varphi(s) = 27\sqrt{13}s$. By using Type-2 Quaternionic Frame given by (2.2), β spatial quaternionic curves in $\mathbb{R}^3$ associated with quaternionic $\beta^{(4)}$ in $\mathbb{R}^4$ is obtained as:

$$\beta(\bar{s}) = \frac{1}{\sqrt{122}}\left(-\frac{22}{\sqrt{130}}\bar{s}, -837\cos\left(\frac{4\bar{s}}{27\sqrt{130}}\right), -837\sin\left(\frac{4\bar{s}}{27\sqrt{130}}\right)\right)$$

The principal curvature and torsion of β are given

$$\bar{k} = \frac{496}{3510\sqrt{122}} \text{ and } \bar{r} = -\frac{88}{3510\sqrt{122}}$$

So, the curvatures of $\beta^{(4)}$ are computed as :

$$\bar{K} = \frac{366}{3510\sqrt{122}},\ -\bar{r} = \frac{88}{3510\sqrt{122}},\ \bar{K}-\bar{k} = -\frac{130}{3510\sqrt{122}}$$

From (3.31), (3.39) and (3.42), we can compute the curvatures of $\beta^{(4)}$ by using the curvatures of $\alpha^{(4)}$ and the real numbers a, b, c, d, too. Hence we see that Theorem (3.2) is provided.

4. Conclusion

Since there is no Bertrand curves whose torsion and bitorsion are non-zero in $\mathbb{R}^4$, Matsuda and Yorozu defined a new type of Bertrand curves which is called (1,3)-Bertrand curve. Keçilioğlu and İlarslan introduced quaternionic (1,3)-Bertrand curves in Euclidean 4-space by using the quaternionic frame given by Bharathi and Nagaraj. Kahraman Aksoyak defined a new type of quaternionic frame in $\mathbb{R}^4$ which is called Type 2-Quaternionic Frame. In this paper, we investigate quaternionic (1,3)-Bertrand curve according to Type 2-Quaternionic Frame. The most important point of working on quaternionic curves is this: if a quaternionic curve is given in $\mathbb{R}^4$, a spatial quaternionic curve in $\mathbb{R}^3$ is determined individually. So, when we study about curves in $\mathbb{R}^4$, we get an idea of curves in $\mathbb{R}^3$. It allows us to work with curves in both $\mathbb{R}^3$ and $\mathbb{R}^4$

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