



Investigation of Analytical Solutions of the Nonlinear Mathematical Model Representing Gas Overflowing

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Received: 26.01.2021

Accepted: 28.05.2021

Published: 30.06.2021

Abstract

In this study, traveling wave soliton solutions of hyperbolic and trigonometric functions are successfully obtained by using the modified exponential function method of the Buckley-Leverett equation. In addition to these, there are also rational function solutions. Two and three-dimensional graphs of real and imaginary parts are included with contour simulations to physically analysis of the solution functions of the equation analyzed as a mathematical model using Mathematica software.

Keywords: Buckley-Leverett Equation; Traveling wave solution; Modified exponential function method.

Gaz Taşmasını Temsil Eden Doğrusal Olmayan Matematiksel Modelin Analitik Çözümlerinin Araştırılması

Öz



Bu çalışmada, modifiye edilmiş üstel fonksiyon metodu kullanılarak Buckley-Leverett denkleminin hiperbolik ve trigonometrik fonksiyon gezici dalga soliton çözümleri başarıyla elde edilmiştir. Bunların yanı sıra rasyonel fonksiyon çözümleri de elde edilmiştir. Matematiksel model olarak incelenen denklemin çözüm fonksiyonlarını fiziksel olarak analiz etmek için Mathematica yazılımı kullanılarak kontur simülasyonları ile birlikte gerçek ve sanal parçaların iki ve üç boyutlu grafiklerine yer verilmiştir.

Anahtar Kelimeler: Buckley-Leverett denklemi; Yürüyen dalga çözümleri; Geliştirilmiş Üstel Fonksiyon Metodu.

1. Introduction

Nonlinear partial differential equations have an important place in all areas of life. This equation is based on the motion, density, fluidity of behavior or object in fields such as physics, chemistry, engineering, health, biology, etc. It provides the opportunity to evaluate. It is essential to obtain analytical solutions to such mathematical models. Because, with the solutions of these types of mathematical models, it will be easier to evaluate the physical conditions of the object being investigated in the desired time interval. For example, various mathematical models have been developed for the Covid-19 outbreak, which has affected worldwide recently. With the help of these models, obtaining information about the course of the epidemic in advance has emerged. For this reason, there are various methods or techniques in the literature about establishing such mathematical models and analyzing their solutions [1-14].

This article aims to obtain the analytical solution or solutions of the Buckley-Leverett equation with the help of the modified exponential function method (MEFM) by determining the appropriate parameters for gas flooding under varying constant pressure boundary conditions [15]. The Buckley-Leverett equation analyzed in the article is as follows [16-18],

$$U_t + (U - U^3)_x = \beta U_{xx} + \alpha U_{xxt}. \quad (1)$$

2. Analysis of Modified Exponential Function Method (MEFM)

In this section, general information about MEFM is given as follows. According to the method, the general form of the analyzed nonlinear partial differential equation is as follows;

$$P(U, U^3, U_x, U_t, U_{xx}, U_{xxt}, \dots) = 0, \quad (2)$$

where $U = U(x, t)$, is unknown function.

Step 1. If the following nonlinear wave transformation is applied to partial differential equations representing the mathematical model;

$$U(x, t) = u(\xi), \quad \xi = k \cdot (x - ct), \quad (3)$$

c and k are real values that are not zero. The k term in wave transformation represents the height of the wave, and c represents the frequency of the wave. If the required derivative terms in Eqn. (1) are obtained from the wave transform Eqn. (3) and replaced by the following, the following nonlinear ordinary differential equation is obtained,

$$N(u, u', u'', \dots) = 0. \quad (4)$$

Step 2: The solution function U analyzed in the Eqn. (4) obtained above is as follows;

$$U(\xi) = \frac{\sum_{i=0}^n A_i [\exp(-\Omega(\xi))]^{-i}}{\sum_{j=0}^m B_j [\exp(-\Omega(\xi))]^j} = \frac{A_0 + A_1 \exp(-\Omega) + \dots + A_n \exp(n(-\Omega))}{B_0 + B_1 \exp(-\Omega) + \dots + B_m \exp(m(-\Omega))}, \quad (5)$$

where $A_i \neq 0, (0 \leq i \leq n)$ and $B_j \neq 0, (0 \leq j \leq m)$ are real and complex constants. The term $\Omega = \Omega(\xi)$ given above provides the following differential equation,

$$\Omega'(\xi) = \exp(-\Omega(\xi)) + \mu \exp(\Omega(\xi)) + \lambda. \quad (6)$$

If Eqn. (6) is solved under the following conditions [19],

Family 1: If $\mu \neq 0, \lambda^2 - 4\mu > 0$,

$$\Omega(\xi) = \ln \left(\frac{-\sqrt{\lambda^2 - 4\mu}}{2\mu} \tanh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + E) \right) - \frac{\lambda}{2\mu} \right). \quad (7)$$

Family 2: If $\mu \neq 0, \lambda^2 - 4\mu < 0$,

$$\Omega(\xi) = \ln \left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2\mu} \tan \left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2} (\xi + E) \right) - \frac{\lambda}{2\mu} \right). \quad (8)$$

Family 3: If $\mu = 0, \lambda \neq 0$ and $\lambda^2 - 4\mu > 0$,

$$\Omega(\xi) = -\ln \left(\frac{\lambda}{\exp(\lambda(\xi + E)) - 1} \right). \quad (9)$$

Family 4: If $\mu \neq 0$, $\lambda \neq 0$ and $\lambda^2 - 4\mu = 0$,

$$\Omega(\xi) = \ln\left(-\frac{2\lambda(\xi + E) + 4}{\lambda^2(\xi + E)}\right). \quad (10)$$

Family 5: If $\mu = 0$, $\lambda = 0$ and $\lambda^2 - 4\mu = 0$,

$$\Omega(\xi) = \ln(\xi + E). \quad (11)$$

Step 3: Using the wave transformation, the equation obtained by substituting the required terms in Eqn. (2) is found in the algebraic equation system according to the powers of $\Omega(\xi)$. With the solving of this algebraic equation system, the values of the constants used in the solution function are obtained.

3. Application of Modified Exponential Function Method to Buckley-Leverett Equation

In this section, wave solutions of the Buckley-Leverett equation are obtained through using MEFM. For this, first of all, for Eqn. (1), the wave transformation Eqn. (3) is used to find the nonlinear ordinary differential equation given below,

$$(1-c)u - u^3 - \beta k u' + \alpha c k^2 u'' = 0. \quad (12)$$

Considering the balancing procedure for Eqn. (12), the following equation is obtained regarding the upper limits of the symbols of the sum in Eqn. (5):

$$n = m + 1.$$

According to the equation obtained above, $n = 2$ for $m = 1$. In this case, Eqn. (5) and its derivatives terms,

$$\begin{aligned} u(\xi) &= \frac{\psi}{\varphi} = \frac{A_0 + A_1 e^{-\Omega(\xi)} + A_2 e^{-2\Omega(\xi)}}{B_0 + B_1 e^{-\Omega(\xi)}}, \\ u'(\xi) &= \frac{\psi' \varphi - \psi \varphi'}{\varphi^2}, \\ u''(\xi) &= \frac{\psi'' \varphi^3 - \varphi^2 \psi' \varphi' - (\psi \varphi'' + \psi' \varphi') \varphi^2 + 2(\psi')^2 \psi \varphi}{\varphi^4}, \end{aligned} \quad (13)$$

are obtained. By replacing Eqn. (13) in Eqn. (12), the U solution function that provides the nonlinear ordinary differential equation is obtained. It is checked that the solution function found

with the help of Mathematica software provides Eqn. (1). By determining the appropriate parameters, two and three-dimensional graphs of the solution function are obtained with the help of the same program.

CASE 1:

$$\begin{aligned}
 A_0 &= \frac{1}{2}i\sqrt{-1+c-8ck^2\alpha\mu}B_0 - \frac{1}{2}\sqrt{-(-1+c)B_0^2}, \\
 A_1 &= \frac{-4ck^2\alpha B_0^2 + A_2\left(-i\sqrt{-1+c-8ck^2\alpha\mu}B_0 + \sqrt{-(-1+c)B_0^2}\right)}{2\sqrt{2}\sqrt{ck}\sqrt{\alpha}B_0}, \\
 B_1 &= -\frac{A_2}{\sqrt{2}\sqrt{ck}\sqrt{\alpha}}, \beta = \frac{3\sqrt{c}\sqrt{\alpha}\sqrt{-(-1+c)B_0^2}}{\sqrt{2}B_0}, \lambda = -\frac{i\sqrt{-1+c-8ck^2\alpha\mu}}{\sqrt{2}\sqrt{ck}\sqrt{\alpha}}.
 \end{aligned} \tag{14}$$

If the coefficients given above are used, the following solution functions are obtained.

Family 1:

$$U_{1,1}(x,t) = \frac{1}{2} \frac{\left(-\frac{\sqrt{-(-1+c)B_0^2}}{B_0} + \left(\begin{array}{l} \zeta\lambda\mu + i\sqrt{-1+c-8ck^2\alpha\mu} \\ 4\mu + (\lambda^2 - 4\mu) \operatorname{Sech}\omega^2 \\ -\zeta\sqrt{\lambda^2 - 4\mu}\mu \operatorname{Tanh}\omega \end{array} \right) \right)}{(4\mu + (\lambda^2 - 4\mu) \operatorname{Sech}\omega^2)}, \tag{15}$$

where $\omega(x,t) = \left[\frac{1}{2}(EE + \xi)\sqrt{\lambda^2 - 4\mu} \right]$, $\zeta(x,t) = 4\sqrt{2}\sqrt{ck}\sqrt{\alpha}$.

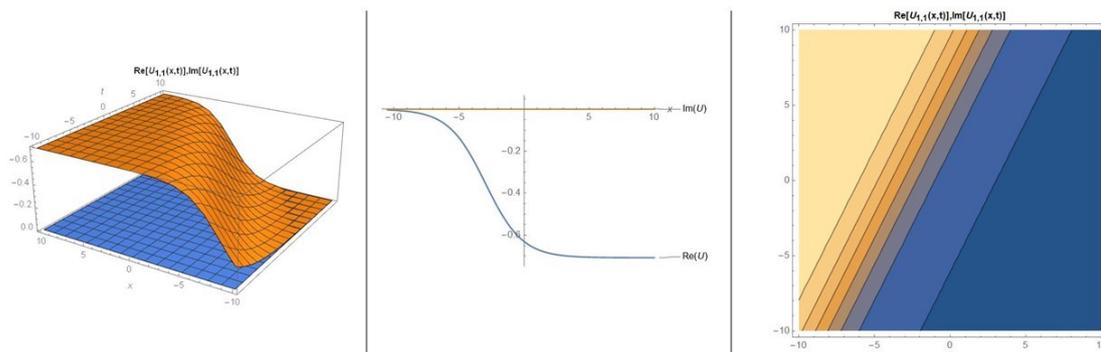


Figure 1: Three-dimensional, contour graphs of real and imaginary parts of Eqn. (15) for values $c = 0$, $\alpha = 0.1$, $B_0 = 1$, $k = 1$, $\mu = 0.1$, $\beta = \frac{3}{2\sqrt{2}}$, $\lambda = 0.948683$, $EE = 0.75$ and two-dimensional graph $t = 1$

Family 2:

$$U_{1,2}(x,t) = \left(\frac{1}{2} i \sqrt{-1+c-8ck^2\alpha\mu} + \frac{\xi}{4} \mu \left(\lambda + \lambda \cos[\kappa] + \sqrt{-\lambda^2 + 4\mu \sin[\kappa]} \right) \right) \frac{1}{\lambda^2 - 2\mu + 2\mu \cos[\kappa]}, \quad (16)$$

where $\kappa(x,t) = (EE + \xi) \sqrt{-\lambda^2 + 4\mu}$.

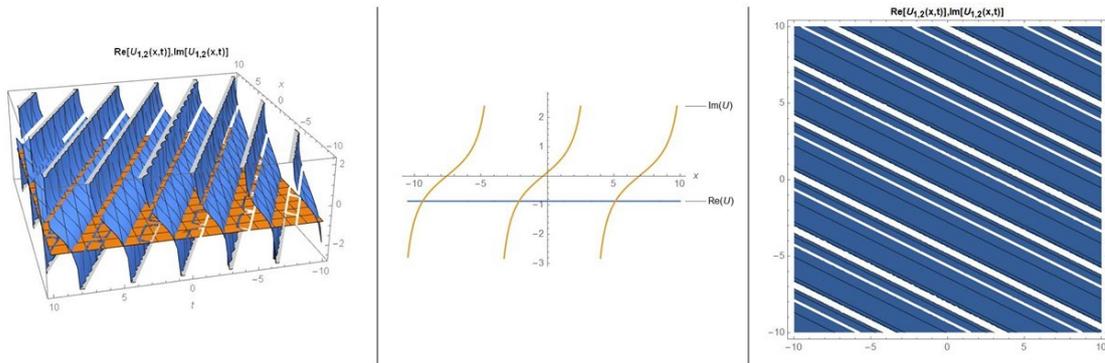


Figure 2: Three-dimensional, contour graphs of real and imaginary parts of Eqn. (16) for values $c = -2, \alpha = 1, B_0 = 3, k = -1, \mu = 1, \beta = 3i\sqrt{3}, \lambda = \frac{\sqrt{13}}{2}, EE = 0.65$ and two-dimensional graph $t = 1$

Family 3:

$$U_{1,3}(x,t) = \left(\frac{1}{2} i \sqrt{-1+c} - \frac{\xi \lambda}{4(-1+e^{(EE+\xi)\lambda})} - \frac{\sqrt{-(-1+c)B_0^2}}{2B_0} \right). \quad (17)$$

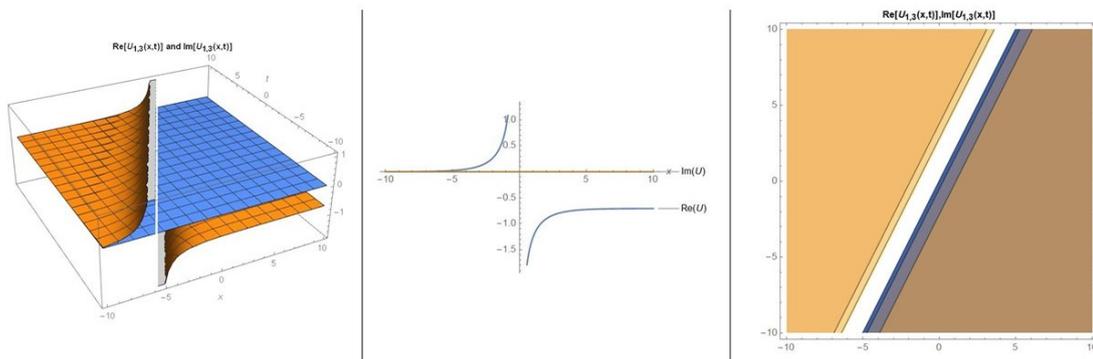


Figure 3: Three-dimensional, contour graphs of real and imaginary parts of Eqn. (17) for values $c = \frac{1}{2}, \alpha = 1, B_0 = 3, k = 1, \mu = 0, \beta = 3, \lambda = \frac{1}{\sqrt{2}}, EE = 0.65$ and two-dimensional graph $t = 1$

Family 4:

$$U_{1,4}(x,t) = \frac{1}{2} \left(\zeta \lambda \left(1 - \frac{2}{2 + EE \lambda + \xi \lambda} \right) + i \sqrt{-1 + c - 8ck^2 \alpha \mu} - \frac{\sqrt{-(-1+c)B_0^2}}{B_0} \right). \quad (18)$$

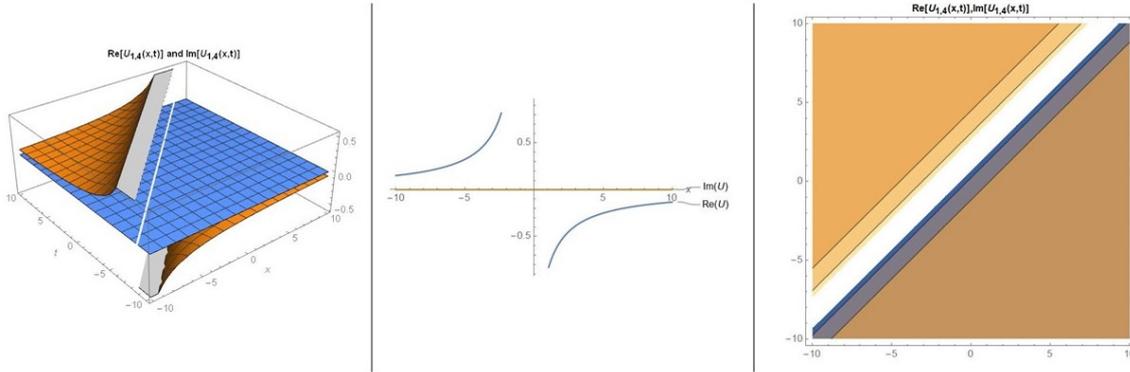


Figure 4: Three-dimensional, contour graphs of real and imaginary parts of Eqn. (18) for values $c = 1$, $\alpha = 1$, $B_0 = 3$, $k = 1$, $\mu = 1$, $\beta = \frac{3}{2\sqrt{2}}$, $\lambda = 2$, $EE = 0.65$ and two-dimensional graph $t = 1$

Family 5:

$$U_{1,5}(x,t) = \frac{1}{2} \left(i \sqrt{-1 + c} - \frac{\zeta}{2(EE + \xi)} - \frac{\sqrt{-(-1+c)B_0^2}}{B_0} \right). \quad (19)$$

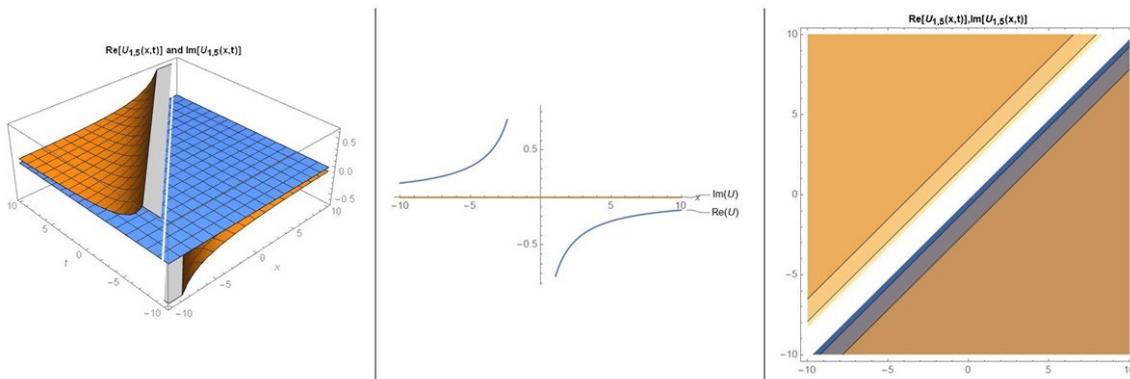


Figure 5: Three-dimensional, contour graphs of real and imaginary parts of Eqn. (19) for values $c = 1$, $\alpha = 1$, $B_0 = 3$, $k = 1$, $\mu = 0$, $\beta = \frac{3}{2\sqrt{2}}$, $\lambda = 0$, $EE = 0.65$ and two-dimensional graph $t = 1$

4. Conclusions

By Bruzón, M. S., et al., The solution function of the Buckley-Leverett equation has been studied by using Lie theory. They converted the nonlinear partial differential equation into ordinary differential equation format with the help of transformation. Then they obtained the Jacobi Elliptic function as a solution function. When the solution functions obtained in our study are analyzed, it is seen that hyperbolic, trigonometric, and rational solution functions are obtained. In order to understand the physical meanings of the solutions in this study, two and three-dimensional graphics were drawn with contour simulations. Mathematica software program was used to obtain all these numerical operations and to draw graphs. When the algebraic equation system is solved according to MEFM, the analyzed family cases are determined for the coefficients obtained, and the real and imaginary solution functions of the Buckley-Leverett equation are obtained. It has been seen that the method used in this study is a highly effective technique in obtaining analytical solutions of nonlinear partial differential equations.

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