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# A New Faster Four step Iterative Algorithm for Suzuki Generalized Nonexpansive Mappings with an Application

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**Abstract**

The focus of this paper is to introduce a four step iterative algorithm, called $A^*$ iterative method, for approximating the fixed points of Suzuki generalized nonexpansive mappings. We prove analytically and numerically that our new iterative algorithm converges faster than some leading iterative algorithms in the literature for almost contraction mappings and Suzuki generalized nonexapansive mapping. Furthermore, we prove weak and strong convergence theorems of our new iterative method for Suzuki generalized nonexpansive mappings in uniformly convex Banach spaces. Again, we show analytically and numerically that our new iterative algorithm is $G$-stable and data dependent. Finally, to illustrate the applicability of our iterative method, we will find the solution of a functional Volterra–Fredholm integral equation with a deviating argument via our new iterative method. Hence, our results generalize and improve several well known results in the existing literature.

*Keywords:* Banach space, fixed point, stability, almost contraction map, Suzuki generalized nonexpansive mapping, data dependence, convergence, iterative scheme, Volterra Fredholm integral equation.
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## 1. Introduction

Let $\Omega$ be a real Banach space and $\Lambda$ be a nonempty closed convex subset of $\Omega$. Let $\mathbb{N}$ denote the set of natural numbers and $\Re$ be the set of real numbers. By a fixed point of a mapping $G : \Lambda \to \Lambda$, we mean an

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element $\psi \in \Lambda$ satisfying $G\psi = \psi$. We denote the set of all fixed point of $G$ by $F(G)$. A mapping $G$ is said to be a contraction if there exists a constant $\gamma \in (0,1)$ such that $\|G\psi - G\eta\| \leq \gamma\|\psi - \eta\|$. The mapping $G$ is said to be nonexpansive if $\|G\psi - G\eta\| \leq \|\psi - \eta\|$ (i.e. every contraction mapping is a nonexpansive mapping with $\gamma = 1$).

Fixed point theory has received massive attention for some decades now. This is as a result of its application to certain areas in applied science and engineering such as: Optimization theory, Game theory, Approximation theory, Dynamic theory, Fractals and many other subjects.

One of the first fixed point theorems is the Banach fixed point theorem. This theorem is also known as the Banach contraction principle. Banach contraction principle is important as a source of existence and uniqueness theorem in diverse branches of sciences. This theorem gives a demonstration of the unifying power of functional analytic methods and usefulness of fixed point theory.

The Banach contraction principle uses the Picard iterative method which is defined as follows:

$$\psi_{s+1} = G\psi_s, \ \forall \ s \in \mathbb{N}, \tag{1}$$

for contraction mappings in a complete metric space. It is well known that this principle does not hold for nonexpanive mappings since Picard iteration method fails to converge to the fixed point of nonexpansive mappings even when the existence of fixed point is guaranteed in a complete metric space.

So many authors have constructed several iterative methods for approximating the fixed points of nonexpansive mappings and other wider classes of mappings. An efficient iterative method is one which; converges to the fixed point of an operator, has a better rate of convergence, gives data dependent result and guarantees stability with respect to $G$.

Some notable iterative schemes in the existing literature includes: Mann iteration [17], Ishikawa iteration [14], Noor iteration [20], Argawal et al. iteration [2], Abbas and Nazir iteration [1], SP iteration [23], S* iteration [13], CR iteration [8], Normal-S iteration [24], Picard-S iteration [11], Thakur iteration [30], M iteration [32], M* iteration [31], Garodia and Uddin iteration [9], Two-Step Mann iteration [29] and many others.

Let $\{r_s\}$ and $\{p_s\}$ be two nonnegative real sequences in [0,1]. The following iteration processes are known as S iteration process [2], Picard-S iteration process [11], Thakur iteration process [30] and K* iteration process [33], respectively:

$$\begin{cases} w_0 \in \Lambda, \\ \mu_s = (1-p_s)w_s + p_s G w_s, \\ w_{s+1} = (1-r_s)Gw_s + r_s G\mu_s, \end{cases} \quad \forall s \geq 1. \tag{2}$$

$$\begin{cases} u_0 \in \Lambda, \\ \varphi_s = (1-p_s)u_s + p_s G u_s, \\ \varrho_s = (1-r_s)Gu_s + r_s G\varphi_s, \\ u_{s+1} = G\varrho_s, \end{cases} \quad \forall s \geq 1. \tag{3}$$

$$\begin{cases} \omega_0 \in \Lambda, \\ \rho_s = (1-p_s)w_s + p_s G w_s, \\ v_s = G((1-r_s)\omega_s + r_s\rho_s), \\ \omega_{s+1} = Gv_s, \end{cases} \quad \forall s \geq 1. \tag{4}$$

$$\begin{cases} \ell_0 \in \Lambda, \\ m_s = (1-p_s)\ell_s + p_s G\ell_s, \\ \eta_s = G((1-r_s)m_s + r_s G m_s), \\ \ell_{s+1} = G\eta_s, \end{cases} \quad \forall s \geq 1. \tag{5}$$

In 2014, Gursoy and Karakaya [11] introduced the Picard-S iteration process (3), the authors showed analytically and with the aid of a numerical example that Picard-S iteration process (3) converges at a rate faster than all of Picard, Mann, Ishikawa, Noor, SP, CR, S, S*, Abbas and Nazir, Normal-S and Two-Step Mann iteration processes for contraction mappings.

In 2016, Thakur et al. [30] introduced the iteration process (4). The authors used a numerical example to show that (4) converges faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration process for Suzuki generalized nonexpansive mappings.

Very recently, Ullah and Arshad [33] introduced the $K^*$ iteration process (5). The authors proved both analytically and numerically that K* iteration process (5) converges faster than S iteration process (2), Thakur iteration process (4) and Picard-S iteration process (3) for Suzuki generalized nonexpansive mapping. Also, they noted that the speed of convergence of Picard-S iteration process (3) and Thakur iteration (4) are almost same.

On the other hand, several problems which arise in mathematical physics, engineering, biology, economics and etc., lead to mathematical models described by nonlinear integral equations (see [18] and the references therein). In particular, Volterra-Fredholm integral equations arise from parabolic boundary value problems, from the mathematical modeling of the spatio-temporal development of an epidemic, and from various physical and biological models (see [19, 34]). Recently, some iterative approaches for solution of nonlinear integral equations have been studied by several authors (see for example [10, 3, 16, 21, 22] and the references therein).

Motivated and inspired by the ongoing research in this direction, we introduce the following four steps iteration process, called $A^*$ iteration process, to obtain better rate of convergence for almost contraction mappings and Suzuki generalized nonexpansive mappings:

$$\begin{cases} \psi_0 \in \Lambda, \\ g_s = G((1-p_s)\psi_s + p_s G\psi_s), \\ k_s = G((1-r_s)g_s + r_s G g_s), \qquad \forall s \geq 1. \\ \eta_s = G k_s, \\ \psi_{s+1} = G\eta_s, \end{cases} \tag{6}$$

where $\{r_s\}$ and $\{p_s\}$ are sequences in [0,1].

The aim of this paper is to prove analytically that $A^*$ iteration process (6) converges at rate faster than $K^*$ iteration process (5) for almost contraction mappings. Also, we provide numerical examples to show that (6) converges faster than the iteration processes (2)–(5) for almost contraction mappings and Suzuki generalized nonexpansive mappings. Furthermore, we prove weak and strong convergence theorems for $A^*$ iteration process (6) in uniformly convex Banach spaces. Again, we show analytically and numerically that our new iterative algorithm is $G$-stable. Furthermore, we prove that our new iterative method (6) is data dependent. Finally, to illustrate the applicability of our iterative method, we will find the solution of a functional Volterra–Fredholm integral equation with a deviating argument by using our new iterative method (6).

## 2. Preliminaries

The following definitions, propositions and lemmas will be useful in proving our main results.

**Definition 2.1.** *A mapping $G : \Lambda \to \Lambda$ is said to be a Suzuki generalized nonexpansive mapping if for all $\psi, \eta \in \Lambda$, we have*

$$\frac{1}{2}\|\psi - G\psi\| \leq \|\psi - \eta\| \Longrightarrow \|G\psi - G\eta\| \leq \|\psi - \eta\|. \tag{7}$$

Suzuki generalized nonexpansive mapping is also known as mapping satisfying condition (C). In [28], Suzuki showed that the class of mapping satisfying condition (C) is more general than the class of nonexpansive mapping and obtained some fixed points and convergence theorems.

In 2003, Berinde [5] introduced the concept of weak contraction mapping which is also known as almost contraction mapping. He showed that the class of almost contraction mapping is more general than the class of Zamfirescu mapping [36] which includes contraction mapping, Kannan mapping [15] and Chatterjea mapping [7].

**Definition 2.2.** *A mapping $G : \Lambda \to \Lambda$ is called almost contraction mapping if there exists a constant $\gamma \in (0,1)$ and some constant $L \geq 0$, such that*

$$\|G\psi - G\eta\| \leq \gamma\|\psi - \eta\| + L\|\psi - G\psi\|, \ \forall\, \psi, \eta \in \Lambda. \tag{8}$$

**Definition 2.3.** *A Banach space $\Omega$ is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that for $\psi, \eta \in \Omega$ satisfying $\|\psi\| \leq 1$, $\|\eta\| \leq 1$ and $\|\psi - \eta\| > \epsilon$, we have $\left\|\frac{\psi+\eta}{2}\right\| < 1 - \delta$.*

**Definition 2.4.** *A Banach space $\Omega$ is said to satisfy Opial's condition if for any sequence $\{\psi_s\}$ in $\Omega$ which converges weakly to $\psi \in \Omega$ implies*

$$\limsup_{s\to\infty} \|\psi_s - \psi\| < \limsup_{s\to\infty} \|\psi_s - \eta\|, \, \forall\, \eta \in \Omega \text{ with } \eta \neq \psi.$$

**Definition 2.5.** *Let $\{\psi_s\}$ be a bounded sequence in $\Omega$. For $\psi \in \Lambda \subset \Omega$, we put*

$$r(\psi, \{\psi_s\}) = \limsup_{s\to\infty} \|\psi_s - \psi\|.$$

*The asymptotic radius of $\{\psi_s\}$ relative to $\Lambda$ is defined by*

$$r(\Lambda, \{\psi_s\}) = \inf\{r(\psi, \{\psi_s\}) : \psi \in \Lambda\}.$$

*The asymptotic center of $\{\psi_s\}$ relative to $\Lambda$ is given as:*

$$A(\Lambda, \{\psi_s\}) = \{\psi \in \Lambda : r(\psi, \{\psi_s\}) = r(\Lambda, \{\psi_s\})\}.$$

In a uniformly convex Banach space, it is well known that $A(\Lambda, \{\psi_s\})$ consist of exactly one point.

**Definition 2.6.** *[4] Let $\{a_s\}$ and $\{b_s\}$ be two sequences of real numbers that converge to $a$ and $b$ respectively, and assume that there exists*

$$\ell = \lim_{s\to\infty} \frac{\|a_s - a\|}{\|b_s - b\|}.$$

*Then,*

*($R_1$) if $\ell = 0$, we say that $\{a_s\}$ converges faster to $a$ than $\{b_s\}$ does to $b$.*

*($R_2$) If $0 < \ell < \infty$, we say that $\{a_s\}$ and $\{b_s\}$ have the same rate of convergence.*

**Definition 2.7.** *[4] Let $\{\Theta_s\}$ and $\{\Xi_s\}$ be two fixed point iteration processes that converge to the same point $z$, the error estimates*

$$\begin{aligned} \|\Theta_s - z\| &\leq a_s, \ \forall\, s \geq 1 \\ \|\Xi_s - z\| &\leq b_s, \ \forall\, s \geq 1 \end{aligned}$$

*are available where $\{a_s\}$ and $\{b_s\}$ are two sequences of positive numbers converging to zero. Then we say that $\{\Theta_s\}$ converges faster to $z$ than $\{\Xi_s\}$ does if $\{a_s\}$ converges faster than $\{b_s\}$.*

**Definition 2.8.** *[4] Let $G$, $\tilde{G} : \Lambda \to \Lambda$ be two operators. We say that $\tilde{G}$ is an approximate operator for $G$ if for some $\epsilon > 0$, we have*

$$\|G\psi - \tilde{G}\psi\| \leq \epsilon, \ \forall \, \psi \in \Lambda.$$

**Definition 2.9.** *[12] Let $\{\zeta_s\}$ be any sequence in $\Lambda$. Then, an iteration process $\psi_{s+1} = f(G, \psi_s)$, which converges to fixed point $z$, is said to be $G$-stable, if for $\varepsilon_s = \|\zeta_{s+1} - f(G, \zeta_s)\|$, $\forall \, s \in \mathbb{N}$, we have*

$$\lim_{s\to\infty} \varepsilon_s = 0 \Leftrightarrow \lim_{s\to\infty} \zeta_s = z.$$

**Definition 2.10.** *[26] A mapping $G : \Lambda \to \Lambda$ is said to satisfy condition $(I)$ if a nondecreasing function $f : [0, \infty) \to [0, \infty)$ exists with $f(0) = 0$ and for all $r > 0$ then $f(r) > 0$ such that $\|\psi - G\psi\| \geq f(d(\psi, F(G))))$ for all $\psi \in \Lambda$, where $d(\psi, F(G)) = \inf_{z\in F(G)} \|\psi - z\|$.*

**Proposition 2.11.** *[28] Suppose $G : \Lambda \to \Lambda$ is any mapping. Then*

*(i) If $G$ is nonexpansive, it follows that $G$ is a Suzuki generalized nonexpansive mapping.*

*(ii) Every Suzuki generalized nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive.*

*(iii) If $G$ is a Suzuki generalized nonexpansive mapping, then the following inequality holds:*

$$\|\psi - G\eta\| \leq 3\|G\psi - \psi\| + \|\psi - \eta\|, \ \forall \ \psi, \eta \in \Lambda.$$

**Lemma 2.12.** *[28] Let $G$ be a self mapping on a subset $\Lambda$ of a Banach space $\Omega$ which satisfies Opial's condition. Suppose $G$ is a Suzuki generalized nonexpansive mapping. If $\{\psi_s\}$ converges weakly to $z$ and $\lim_{s\to\infty} \|G\psi_s - \psi_s\| = 0$, then $Gz = z$. That is, $I - G$ is demiclosed at zero.*

**Lemma 2.13.** *[28] Let $G$ be a self mapping on a weakly compact convex subset $\Lambda$ of a Banach space $\Omega$ with the Opial's property. If $G$ is a Suzuki generalized nonexpansive mapping, then $G$ has a fixed point.*

**Lemma 2.14.** *[35] Let $\{\theta_s\}$ and $\{\lambda_s\}$ be nonnegative real sequences satisfying the following inequalities:*

$$\theta_{s+1} \leq (1 - \sigma_s)\theta_s + \lambda_s,$$

*where $\sigma_s \in (0, 1)$ for all $s \in \mathbb{N}$, $\sum_{s=0}^{\infty} \sigma_s = \infty$ and $\lim_{s\to\infty} \frac{\lambda_s}{\sigma_s} = 0$, then $\lim_{s\to\infty} \theta_s = 0$.*

**Lemma 2.15.** *[27] Let $\{\theta_s\}$ and $\{\lambda_s\}$ be nonnegative real sequences satisfying the following inequalities:*

$$\theta_{s+1} \leq (1 - \sigma_s)\theta_s + \sigma_s\lambda_s,$$

*where $\sigma_s \in (0, 1)$ for all $s \in \mathbb{N}$, $\sum_{s=0}^{\infty} \sigma_s = \infty$ and $\lambda_s \geq 0$ for all $s \in \mathbb{N}$, then*

$$0 \leq \limsup_{s\to\infty} \theta_s \leq \limsup_{s\to\infty} \lambda_s.$$

**Lemma 2.16.** *[25] Suppose $\Omega$ is a uniformly convex Banach space and $\{\iota_s\}$ is any sequence satisfying $0 < p \leq \iota_s \leq q < 1$ for all $s \geq 1$. Suppose $\{\psi_s\}$ and $\{\eta_s\}$ are any sequences of $\Omega$ such that $\limsup_{s\to\infty} \|\psi_s\| \leq \alpha$, $\limsup_{s\to\infty} \|\eta_s\| \leq \alpha$ and $\limsup_{s\to\infty} \|\iota_s\psi_s + (1 - \iota_s)\eta_s\| = \alpha$ hold for some $\alpha \geq 0$. Then $\lim_{s\to\infty} \|\psi_s - \eta_s\| = 0$.*

## 3. Rate of Convergence

In this section, we will prove that $A^*$ iteration process (6) converges faster than the iteration process (5) for almost contraction mappings.

**Theorem 3.1.** *Let $\Omega$ be a Banach space and let $\Lambda$ be closed convex subset of $\Omega$. Let $G : \Lambda \to \Lambda$ be a mapping satisfying (8) with $F(G) \neq \emptyset$. Let $\{\psi_s\}$ be the iterative algorithm defined by (6) with sequences $\{r_s\}$, $\{p_s\} \in [0,1]$ such that $\sum_{s=0}^{\infty} r_s = \infty$, then $\{\psi_s\}$ converges strongly to a unique fixed point of $G$.*

*Proof.* Let $z \in F(G)$. Then from (6), we have get

$$\begin{aligned} \|g_s - z\| &= \|G((1-p_s)\psi_s + p_s G\psi_s) - z\| \\ &= \|Gz - G((1-p_s)\psi_s + p_s G\psi_s)\| \\ &\leq \gamma\|z - ((1-p_s)\psi_s + p_s G\psi_s)\| + L\|z - Tz\| \\ &= \gamma\|(1-p_s)\psi_s + p_s G\psi_s - z\| \\ &\leq \gamma((1-p_s)\|\psi_s - z\| + p_s\|G\psi_s - z\|) \\ &\leq \gamma((1-p_s)\|\psi_s - z\| + p_s\gamma\|\psi_s - z\|) \\ &= \gamma(1-(1-\gamma)p_s)\|\psi_s - z\|. \end{aligned} \tag{9}$$

Using (6) and (9), we have

$$\begin{aligned} \|k_s - z\| &= \|G((1-r_s)g_s + r_s Gg_s) - z\| \\ &\leq \gamma\|(1-r_s)g_s + r_s Gg_s) - z\| \\ &\leq \gamma((1-r_s)\|g_s - z\| + r_s\|Gg_s - z\|) \\ &\leq \gamma((1-r_s)\|g_s - z\| + r_s\gamma\|g_s - z\|) \\ &= \gamma(1-(1-\gamma)r_s)\|g_s - z\| \\ &\leq \gamma^2(1-(1-\gamma)r_s)(1-(1-\gamma)p_s)\|\psi_s - z\|. \end{aligned} \tag{10}$$

From (6) and (10), we obtain

$$\begin{aligned} \|\eta_s - z\| &= \|Gk_s - z\| \\ &\leq \gamma\|k_s - z\| \\ &\leq \gamma^3(1-(1-\gamma)r_s)(1-(1-\gamma)p_s)\|\psi_s - z\|. \end{aligned} \tag{11}$$

Using (6) and (11), we have

$$\begin{aligned} \|\psi_{s+1} - z\| &= \|G\eta_s - z\| \\ &\leq \gamma\|\eta_s - z\| \\ &\leq \gamma^4(1-(1-\gamma)r_s)(1-(1-\gamma)p_s)\|\psi_s - z\|. \end{aligned} \tag{12}$$

Since $\gamma \in (0,1)$ and $p_s \in [0,1]$, for all $s \in \mathbb{N}$, it follows that $(1-(1-\gamma)p_s) < 1$. Then from (12), we obtain

$$\|\psi_{s+1} - z\| \leq \gamma^4(1-(1-\gamma)r_s)\|\psi_s - z\|. \tag{13}$$

From (13), we have the following inequalities:

$$\begin{aligned} \|\psi_{s+1} - z\| &\leq \gamma^4(1-(1-\gamma)r_s)\|\psi_s - z\| \\ &\leq \gamma^4(1-(1-\gamma)r_{s-1})\|\psi_{s-1} - z\| \\ &\vdots \\ \|\psi_1 - z\| &\leq \gamma^4(1-(1-\gamma)r_0)\|\psi_0 - z\|. \end{aligned} \tag{14}$$

From (14), we get

$$\|\psi_{s+1} - z\| \leq \|\psi_0 - z\|\gamma^{4(s+1)} \prod_{t=0}^{s}(1-(1-\gamma)r_t). \tag{15}$$

Since $\gamma \in (0,1)$, $r_t \in [0,1]$ for all $t \in \mathbb{N}$, it follows that $(1-(1-\gamma)r_t) \in (0,1)$. Since from classical analysis we know that $1-\psi \leq e^{-\psi}$ for all $\psi \in [0,1]$. Thus from (15), we have

$$\|\psi_{s+1} - z\| \leq \frac{\gamma^{4(s+1)}\|\psi_0 - z\|}{e^{(1-\gamma)\sum_{t=0}^{s} r_t}}. \tag{16}$$

If we take the limits of both sides of (16), we get $\lim_{s\to\infty}\|\psi_s - z\| = 0$.

Next, we show that $z$ is unique. Let $z, z_1 \in F(G)$, such that $z \neq z_1$, using the definition of $G$, we get

$$\begin{aligned} \|z - z_1\| &= \|Gz - Gz_1\| \\ &\leq \gamma\|z - z_1\| + L\|z - Tz\| \\ &= \gamma\|z - z_1\|. \end{aligned} \tag{17}$$

Obviously, from (17) we have that $\|z - z_1\| = \|z - z_1\|$, if not we have a contradiction $\|z - z_1\| < \|z - z_1\|$. Hence, we have that $z = z_1$.

□

**Theorem 3.2.** *Let $\Omega$ be a Banach space and let $\Lambda$ be closed convex subset of $\Omega$. Let $G : \Lambda \to \Lambda$ be a mapping satisfying* (8) *with $F(G) \neq \emptyset$. Let $\{\psi_s\}$ be iterative algorithm defined by* (6) *with sequences $\{r_s\}$, $\{p_s\} \in [0,1]$ such that $r \leq r_s \leq 1$, for some $r > 0$ and for all $s \in \mathbb{N}$. Then $\{\psi_s\}$ converges faster to $z$ than the iteration process* (5).

*Proof.* From (15) in Theorem 3.1 and the assumption $r \leq r_s \leq 1$, for some $r > 0$ and for all $s \in \mathbb{N}$, we have

$$\begin{aligned} \|\psi_{s+1} - z\| &\leq \|\psi_0 - z\|\gamma^{4(s+1)} \prod_{t=0}^{s}(1-(1-\gamma)r_t) \\ &= \|\psi_0 - z\|\gamma^{4(s+1)}(1-(1-\gamma)r)^{s+1}. \end{aligned} \tag{18}$$

Similarly, in (Ullah and Arshad [33], Theorem 3.2), the authors showed that the iteration process (5) takes the form

$$\|\ell_{s+1} - z\| \leq \|\ell_0 - z\|\gamma^{2(s+1)} \prod_{t=0}^{s}(1-(1-\gamma)r_t). \tag{19}$$

Since $r \leq r_s \leq 1$, for some $r > 0$ and for all $s \in \mathbb{N}$, then from (19), we have

$$\begin{aligned} \|\ell_{s+1} - z\| &\leq \|\ell_0 - z\|\gamma^{2(s+1)} \prod_{t=0}^{s}(1-(1-\gamma)r_t) \\ &= \|\ell_0 - z\|\gamma^{2(s+1)}(1-(1-\gamma)r)^{s+1}. \end{aligned} \tag{20}$$

Set

$$a_s = \|\psi_0 - z\|\gamma^{4(s+1)}(1-(1-\gamma)r)^{s+1}, \tag{21}$$

and

$$b_s = \|\ell_0 - z\|\gamma^{2(s+1)}(1-(1-\gamma)r)^{s+1}. \tag{22}$$

Define

$$\begin{aligned} \vartheta_s &= \frac{a_s}{b_s} \\ &= \frac{\|\psi_0 - z\|\gamma^{4(s+1)}(1-(1-\gamma)r)^{s+1}}{\|\ell_0 - z\|\gamma^{2(s+1)}(1-(1-\gamma)r)^{s+1}} \\ &= \gamma^{2(s+1)}. \end{aligned} \tag{23}$$

Since $\gamma \in (0,1)$, we have $\lim_{s\to\infty} \vartheta_s = 0$, which implies that $\{\psi_s\}$ converges faster than $\{\ell_s\}$ to $z$. □

To show the validity of the analytical prove in Theorem 3.2, we give the following numerical example.

**Example 3.3.** *Let $\Omega = \Re$ and $\Lambda = [0, 50]$. Let $G : \Lambda \to \Lambda$ be a mapping defined by $G(\psi) = \sqrt{\psi^2 - 9\psi + 54}$. Obviously, 6 is the fixed point of $G$. Take $r_s = p_s = \frac{3}{4}$, with an initial value of $\psi_0 = 11$. Then we obtain the following table and graph for comparison of various iterative method.*

By writing all the codes in MATLAB (R2015a) and running them on PC with Intel(R) Core(TM)2 Duo CPU @ 2.26GHz 2.27 GHz, we obtain the comparison Table 1 and Figure 1 below.

We observe here that Thakur and Picard-S iterative schemes converge at almost the rate.

Table 1: Comparison of speed of convergence of $A^*$ iterative scheme with S, Thakur and $K^*$ iterative schemes.

| Step | S | Thakur | $K^*$ | $A^*$ |
|---|---|---|---|---|
| 1 | 11.0000000000 | 11.0000000000 | 11.0000000000 | 11.0000000000 |
| 2 | 7.8258228926 | 6.6850984699 | 6.23580353950 | 6.0169328397 |
| 3 | 6.4101626968 | 6.0303937423 | 6.00300497860 | 6.0000127259 |
| 4 | 6.0664027976 | 6.0011083301 | 6.00003597710 | 6.0000000095 |
| 5 | 6.0097817373 | 6.0000400605 | 6.00000043040 | 6.0000000000 |
| 6 | 6.0014177612 | 6.0000014475 | 6.00000000510 | 6.0000000000 |
| 7 | 6.0002049947 | 6.0000000523 | 6.00000000010 | 6.0000000000 |
| 8 | 6.0000296299 | 6.0000000019 | 6.0000000000 | 6.0000000000 |
| 9 | 6.0000042825 | 6.0000000001 | 6.00000000000 | 6.0000000000 |
| 10 | 6.0000006190 | 6.0000000000 | 6.00000000000 | 6.0000000000 |
| 11 | 6.0000000895 | 6.0000000000 | 6.00000000000 | 6.0000000000 |
| 12 | 6.0000000129 | 6.0000000000 | 6.00000000000 | 6.0000000000 |
| 13 | 6.0000000019 | 6.0000000000 | 6.00000000000 | 6.0000000000 |
| 14 | 6.0000000003 | 6.0000000000 | 6.00000000000 | 6.0000000000 |
| 15 | 6.0000000000 | 6.0000000000 | 6.00000000000 | 6.0000000000 |

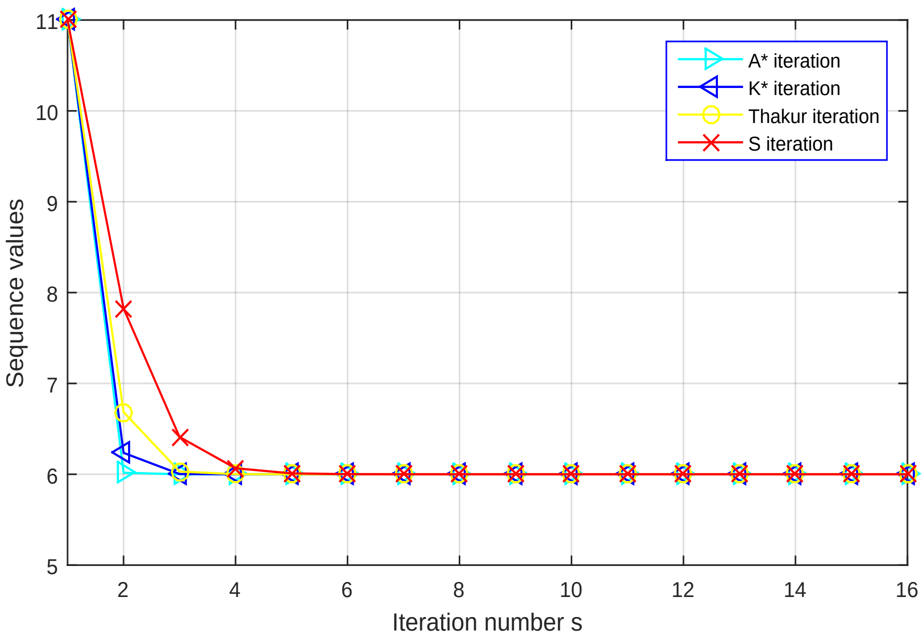


Figure 1: Graph corresponding to Table 1.

## 4. Convergence Results

In this section, we will prove the weak and strong convergence of $A^*$ iteration algorithm (6) for Suzuki generalized nonexpansive mappings in the framework of uniformly convex Banach spaces.

Firstly, we will state and prove the following lemmas which will be useful in obtaining our main results.

**Lemma 4.1.** *Let $\Omega$ be a Banach space and $\Lambda$ be a closed convex subset of $\Omega$. Let $G : \Lambda \to \Lambda$ be a Suzuki generalized nonexpansive mapping with $F(G) \neq \emptyset$. If $\{\psi_s\}$ is the iterative sequence defined by (6), then $\lim_{s\to\infty} \|\psi_s - z\|$ exists for all $z \in F(G)$.*

*Proof.* Let $z \in F(G)$ and $\varsigma \in \Lambda$. By Proposition 2.11(ii), we know that every Suzuki generalized nonexpansive mapping with $F(G) \neq \emptyset$ is quasi-nonexpansive mapping, so

$$\frac{1}{2}\|z - Gz\| = 0 \leq \|z - \varsigma\| \text{ implies that } \|Gz - G\varsigma\| \leq \|z - \varsigma\|. \tag{24}$$

Now, from (6), we have

$$\begin{aligned} \|g_s - z\| &= \|G((1-p_s)\psi_s + p_s G\psi_s) - z\| \\ &\leq \|(1-p_s)\psi_s + p_s G\psi_s - z\| \\ &\leq (1-p_s)\|\psi_s - z\| + p_s\|G\psi_s - z\| \\ &\leq (1-p_s)\|\psi_s - z\| + p_s\|\psi_s - z\| \\ &= \|\psi_s - z\|. \end{aligned} \tag{25}$$

Using (6) and (25), we obtain

$$\begin{aligned} \|k_s - z\| &= \|G((1-r_s)g_s + r_s G_s) - z\| \\ &\leq \|(1-r_s)g_s + r_s Gg_s - z\| \\ &\leq (1-r_s)\|g_s - z\| + r_s\|Gg_s - z\| \\ &\leq (1-r_s)\|g_s - z\| + r_s\|g_s - z\| \\ &= \|g_s - z\| \leq \|\psi_s - z\|. \end{aligned} \tag{26}$$

Again, using (6) and (26), we get

$$\begin{aligned}\|\eta_s - z\| &= \|Gg_s - z\| \\ &\le \|g_s - z\| \\ &\le \|\psi_s - z\|. \end{aligned} \tag{27}$$

Lastly, from (6) and (27), we have

$$\begin{aligned}\|\psi_{s+1} - z\| &= \|G\eta_s - z\| \\ &\le \|\eta_s - z\| \\ &\le \|\psi_s - z\|. \end{aligned} \tag{28}$$

This implies that $\{\|\psi_s - z\|\}$ is bounded and nondecreasing for all $z \in F(G)$. Hence, $\lim_{s\to\infty} \|\psi_s - z\|$ exists. □

**Lemma 4.2.** *Let $\Omega$ be a uniformly convex Banach space and $\Lambda$ be a nonempty closed convex subset of $\Omega$. Let $G : \Lambda \to \Lambda$ be a Suzuki generalized nonexpansive mapping. Suppose $\{\psi_s\}$ is the iterative sequence defined by (6). Then, $F(G) \neq \emptyset$ if and only if $\{\psi_s\}$ is bounded and $\lim_{s\to\infty} \|G\psi_s - \psi_s\| = 0$.*

*Proof.* Suppose $F(G) \neq \emptyset$ and let $z \in F(G)$. Then, by Lemma 4.1, $\lim_{s\to\infty} \|\psi_s - z\|$ exists and $\{\psi_s\}$ is bounded. Put

$$\lim_{s\to\infty} \|\psi_s - z\| = \alpha. \tag{29}$$

From (28) and (25), we obtain

$$\limsup_{s\to\infty} \|g_s - z\| \le \limsup_{s\to\infty} \|\psi_s - z\| = \alpha. \tag{30}$$

From Proposition 2.11(ii), we know that every Suzuki generalized nonexpansive mapping with $F(G) \neq \emptyset$ is quasi-nonexpansive mapping. So that we have

$$\limsup_{s\to\infty} \|G\psi_s - z\| \le \limsup_{s\to\infty} \|\psi_s - z\| = \alpha. \tag{31}$$

Again, using (6) and (25), we get

$$\begin{aligned}\|\psi_{s+1} - z\| &= \|G\eta_s - z\| \\ &\le \|\eta_s - z\| \\ &= \|Gk_s - z\| \\ &\le \|k_s - z\| \\ &= \|G((1-r_s)g_s + r_sGg_s) - z\| \\ &\le \|(1-r_s)g_s + r_sGg_s - z\| \\ &\le (1-r_s)\|g_s - z\| + r_s\|Gg_s - z\| \\ &\le (1-r_s)\|\psi_s - z\| + r_s\|Gg_s - z\| \\ &\le \|\psi_s - z\| - r_s\|\psi_s - z\| + r_s\|g_s - z\|. \end{aligned} \tag{32}$$

From (32), we have

$$\frac{\|\psi_{s+1} - z\| - \|\psi_s - z\|}{r_s} \le \|g_s - z\| - \|\psi_s - z\|. \tag{33}$$

Since $r_s \in [0, 1]$, then from (33), we have

$$\|\psi_{s+1} - z\| - \|\psi_s - z\| \le \frac{\|\psi_{s+1} - z\| - \|\psi_s - z\|}{r_s} \le \|g_s - z\| - \|\psi_s - z\|,$$

which implies that

$$\|\psi_{s+1} - z\| \leq \|g_s - z\|.$$

Therefore, from (29), we obtain

$$\alpha \leq \liminf_{s\to\infty} \|g_s - z\|. \tag{34}$$

From (30) and (34) we obtain

$$\begin{aligned}
\alpha &= \lim_{s\to\infty} \|g_s - z\| \\
&= \lim_{s\to\infty} \|G((1-p_s)\psi_s + p_s G\psi_s) - z\| \\
&\leq \lim_{s\to\infty} \|(1-p_s)\psi_s + p_s G\psi_s - z\| \\
&= \lim_{s\to\infty} \|(1-p_s)(\psi_s - z) + p_s(Gg_s - z)\| \\
&= \lim_{s\to\infty} \|p_s(Gg_s - z) + (1-p_s)(\psi_s - z)\|. 
\end{aligned} \tag{35}$$

From (29), (31), (35) and Lemma 2.16, we obtain

$$\lim_{s\to\infty} \|G\psi_s - \psi_s\| = 0. \tag{36}$$

Conversely, assume that $\{\psi_s\}$ is bounded and $\lim_{s\to\infty} \|G\psi_s - \psi_s\| = 0$. Let $z \in A(\Lambda, \{\psi_s\})$, by definition 2.5 and Proposition 2.11(iii), we have

$$\begin{aligned}
(Gz, \{\psi_s\}) &= \limsup_{s\to\infty} \|\psi_s - Gz\| \\
&\leq \limsup_{s\to\infty} (3\|G\psi_s - \psi_s\| + \|\psi_s - z\|) \\
&= \limsup_{s\to\infty} \|\psi_s - z\| \\
&= r(z, \{\psi_s\}). 
\end{aligned} \tag{37}$$

This implies that $z \in A(\Lambda, \{\psi_s\})$. Since $\Omega$ is uniformly convex, $A(\Lambda, \{\psi_s\})$ is singleton, thus we have $Gz = z$. □

**Theorem 4.3.** *Let $\Omega$, $\Lambda$, $G$ be as in Lemma 4.2. Suppose that $\Omega$ satisfies Opial's condition and $F(G) \neq \emptyset$. Then, the sequence $\{\psi_s\}$ defined by (6) converges weakly to a fixed point of $G$.*

*Proof.* Let $z \in F(G)$, then by Lemma 4.1, we have $\lim_{s\to\infty} \|\psi_s - z\|$ exists. Now we show that $\{\psi_s\}$ has weak sequential limit in $F(G)$. Let $\psi$ and $\eta$ be weak limits of the subsequences $\{\psi_{s_j}\}$ and $\{\psi_{s_k}\}$ of $\{\psi_s\}$ respectively. By Lemma 4.2, we have $\lim_{s\to\infty} \|G\psi_s - \psi_s\| = 0$ and from Lemma 2.12, $I - G$ is demiclosed at zero. It follows that $(I - G)\psi = 0$ implies $\psi = G\psi$, similarly $G\eta = \eta$.

Next we show uniqueness. Suppose $\psi \neq \eta$, then by Opial's property, we obtain

$$\begin{aligned}
\lim_{s\to\infty} \|\psi_s - \psi\| &= \lim_{s_j\to\infty} \|\psi_{s_j} - \psi\| \\
&< \lim_{s_j\to\infty} \|\psi_{s_j} - \eta\| \\
&= \lim_{s\to\infty} \|\psi_s - \eta\| \\
&= \lim_{s_k\to\infty} \|\psi_{s_k} - \eta\| \\
&< \lim_{s_k\to\infty} \|\psi_{s_k} - \psi\| \\
&= \lim_{s\to\infty} \|\psi_s - \psi\|, 
\end{aligned} \tag{38}$$

which is a contradiction, so $\psi = \eta$. Hence, $\{\psi_s\}$ converges weakly to a fixed point of $G$. □

**Theorem 4.4.** *Let $\Omega$ be a uniformly convex Banach space and $\Lambda$ be a nonempty compact convex subset of $\Omega$. Let $G : \Lambda \to \Lambda$ be a Suzuki generalized nonexpansive mapping. Suppose $\{\psi_s\}$ is the iterative sequence defined by* (6)*. Then $\{\psi_s\}$ converges strongly to a fixed point of $G$.*

*Proof.* From Lemma 2.13, we have $F(G) \neq \emptyset$ and from Lemma 4.2, we have $\lim_{s\to\infty} \|G\psi_s - \psi_s\| = 0$. Since $\Lambda$ is compact, so a subsequence $\{\psi_{s_k}\}$ of $\{\psi_s\}$ exists such that $\psi_{s_k} \to z$ for some $z \in \Lambda$. From Proposition 2.11(iii), we obtain

$$\|\psi_{s_k} - Gz\| \leq 3\|G\psi_{s_k} - \psi_{s_k}\| + \|\psi_{s_k} - z\|, \text{ for all } s \geq 1. \tag{39}$$

Letting $k \to \infty$, we have $Gz = z$, i.e., $z \in F(G)$. Again, from Lemma 4.1, $\lim_{s\to\infty} \|\psi_s - z\|$ exists for all $z \in F(G)$, thus $\psi_s \to z$ strongly. □

**Theorem 4.5.** *Let $\Omega$, $\Lambda$, $G$ be as in Lemma 4.2. Then, the $\{\psi_s\}$ defined by* (6) *converges strongly to a point of $F(G)$ if and only if $\liminf_{s\to\infty} d(\psi_s, F(G)) = 0$, where $d(\psi, F(G)) = \inf\{\|\psi - z\| : z \in F(G)\}$.*

*Proof.* Necessity is obvious. Assume that $\liminf_{s\to\infty} d(\psi_s, F(G)) = 0$. From Lemma 4.1, we have $\lim_{s\to\infty} \|\psi_s - z\|$ exists for all $z \in F(G)$, it follows that $\liminf_{s\to\infty} d(\psi_s, F(G))$ exists. But by hypothesis, $\liminf_{s\to\infty} d(\psi_s, F(G)) = 0$, thus $\lim_{s\to\infty} d(\psi_s, F(G)) = 0$. Next we prove that $\{\psi_s\}$ is a Cauchy sequence in $\Lambda$. Since $\liminf_{s\to\infty} d(\psi_s, F(G)) = 0$, then given $\varepsilon > 0$, there exists $s_0 \in \mathbb{N}$ such that, for all $s, n \geq s_0$, we have

$$\begin{aligned} d(\psi_s, F(G)) &\leq \frac{\epsilon}{2}, \\ d(\psi_n, F(G)) &\leq \frac{\epsilon}{2}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|\psi_s - \psi_n\| &\leq \|\psi_s - z\| + \|\psi_n - z\| \\ &\leq d(\psi_s, F(G)) + d(\psi_n, F(G)) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $\{\psi_s\}$ is a Cauchy sequence in $\Lambda$. Since $\Lambda$ is closed, therefore there exists a point $\psi_1 \in \Lambda$ such that $\lim_{s\to\infty} \psi_s = \psi_1$. Since $\lim_{s\to\infty} d(\psi_s, F(G)) = 0$, it implies that $\lim_{s\to\infty} d(\psi_1, F(G)) = 0$. Hence, $\psi_1 \in F(G)$ since $F(G)$ closed. □

**Theorem 4.6.** *Let $\Omega$, $\Lambda$, $G$ be as in Lemma 4.2. If $G$ satisfies condition $(I)$, then the sequence $\{\psi_s\}$ defined by* (6) *converges strongly to a fixed point of $G$.*

*Proof.* We have shown in Lemma 4.2 that

$$\lim_{s\to\infty} \|G\psi_s - \psi_s\| = 0. \tag{40}$$

Using condition (I) in Definition 2.10 and (40), we get

$$\lim_{s\to\infty} f(d(\psi_s, F(G))) \leq \lim_{s\to\infty} \|G\psi_s - \psi_s\| = 0, \tag{41}$$

i.e., $\lim_{s\to\infty} f(d(\psi_s, F(G))) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$, we have

$$\lim_{s\to\infty} d(\psi_s, F(G)) = 0. \tag{42}$$

From Theorem 4.5, then sequence $\{\psi_s\}$ converges strongly to a point of $F(G)$. □

## 5. Numerical Illustration

In this section, we provide an example of a mapping which satisfies condition $(C)$, but not nonexpansive. With the aid of the numerical example, we will prove that $A^*$ iterative algorithm (6) outperform some leading iterative algorithms in the existing literature in terms of speed of convergence.

**Example 5.1.** *Let the mapping $G : [0, 1] \to [0, 1]$ be defined by*

$$\begin{cases} 1-\psi & \text{if } \psi \in [0, \frac{1}{11}), \\ \frac{\psi+10}{11} & \text{if } \psi \in [\frac{1}{11}, 1]. \end{cases} \tag{43}$$

*We now show that $G$ is a Suzuki generalized nonexpansive mapping, but not nonexpansive. If we take $\psi = \frac{9}{100}$ and $\eta = \frac{1}{11}$, then*

$$\begin{aligned} \|G\psi - G\eta\| &= |G\psi - G\eta| = \left|1 - \psi - \left(\frac{\eta + 10}{11}\right)\right| \\ &= \left|\frac{91}{100} - \frac{111}{121}\right| = \frac{89}{12100}. \end{aligned}$$

*And*

$$\|\psi - \eta\| = |\psi - \eta| = \left|\frac{9}{100} - \frac{1}{11}\right| = \frac{1}{1100}.$$

*This implies that $\|G\psi - G\eta\| > \|\psi - \eta\|$. Hence, $G$ is not a nonexpansive mapping.*

*Next we show that $G$ is a Suzuki generalized nonexpansive mapping by considering the following cases:*

***Case I:*** *Let $\psi \in [0, \frac{1}{11})$, then $\frac{1}{2}\|\psi - G\psi\| = \frac{1}{2}|2\psi - 1| = \frac{1-2\psi}{2} \in \left(\frac{9}{22}, \frac{1}{2}\right]$. For $\frac{1}{2}\|\psi - G\psi\| \leq \|\psi - \eta\|$, we must have $\frac{1-2\psi}{2} \leq \|\psi - \eta\|$, i.e., $\frac{1-2\psi}{2} \leq |\psi - \eta|$. The case $\eta < \psi$ is not possible. Thus, we are left with $\eta > \psi$, which gives $\frac{1-2\psi}{2} \leq \eta - \psi$ , which implies $\eta \geq \frac{1}{2}$ and hence $\eta \in [\frac{1}{2}, 1]$. Now,*

$$\|G\psi - G\eta\| = \left|\frac{\eta + 10}{11} - (1 - \psi)\right| = \left|\frac{\eta + 10\psi - 1}{11}\right| < \frac{1}{11}.$$

*And*

$$\|\psi - \eta\| = |\psi - \eta| = \left|\frac{1}{11} - \frac{1}{2}\right| = \frac{9}{22} > \frac{1}{11}.$$

*Hence, $\frac{1}{2}\|\psi - G\psi\| \leq \|\psi - \eta\| \Longrightarrow \|G\psi - G\eta\| \leq \|\psi - \eta\|$.*

***Case II:*** *Let $\psi \in \left[\frac{1}{11}, 1\right]$, then $\frac{1}{2}\|\psi - G\psi\| = \frac{1}{2}\left|\frac{\psi+10}{11} - \psi\right| = \frac{10-10\psi}{22} \in \left[0, \frac{50}{121}\right]$. For $\frac{1}{2}\|\psi - G\psi\| \leq \|\psi - \eta\|$, we have $\frac{10-10\psi}{22} \leq |\psi - \eta|$, which gives two possibilities:*

***(a)*** *For $\psi < \eta$, we have $\frac{10-10\psi}{22} \leq \eta - \psi \Longrightarrow \eta \geq \frac{10+12\psi}{22} \Longrightarrow \eta \in \left[\frac{122}{242}, 1\right] \subset \left[\frac{1}{11}, 1\right]$. So*

$$\|G\psi - G\eta\| = \left|\frac{\psi + 10}{11} - \frac{\eta + 10}{11}\right| = \frac{1}{11}|\psi - \eta| \leq |\psi - \eta|.$$

*Hence, $\frac{1}{2}\|\psi - G\psi\| \leq \|\psi - \eta\| \Longrightarrow \|G\psi - G\eta\| \leq \|\psi - \eta\|$.*

***(b)*** *For $\psi > \eta$, we have $\frac{10-10\psi}{22} \leq \psi - \eta \Longrightarrow \eta \leq \frac{32\psi-10}{22} \Longrightarrow \eta \in \left[\frac{-78}{242}, 1\right]$. Since $\eta \in [0, 1]$ and $\eta \leq \frac{32\psi-10}{22}$, we get $\frac{22\eta+10}{32} \leq \psi \Longrightarrow \psi \in \left[\frac{10}{32}, 1\right]$.*

*Notice that for $\psi \in \left[\frac{10}{32}, 1\right]$ and $\eta \in \left[\frac{1}{11}, 1\right]$ have been considered in case (a). So, we now consider when $\psi \in \left[\frac{10}{32}, 1\right]$ and $\eta \in \left[0, \frac{1}{11}\right)$. Then*

$$\|G\psi - G\eta\| = \left|\frac{\psi + 10}{11} - (1 - \eta)\right| = \left|\frac{\psi + 11\eta - 1}{11}\right| < \frac{1}{11},$$

*and*

$$\|\psi - \eta\| = |\psi - \eta| > \left|\frac{10}{32} - \frac{1}{11}\right| = \frac{78}{352} > \frac{1}{11}.$$

*Thus, $\frac{1}{2}\|\psi - G\psi\| \leq \|\psi - \eta\| \Longrightarrow \|G\psi - G\eta\| \leq \|\psi - \eta\|$. Hence, $G$ is a generalized nonexpansive mapping.*

By using the above example, we will show that $A^*$ iteration process (6) converges faster than $S$, Tharkur and $K^*$ iteration processes. With the aid of MATLAB (R2015a), we observe that Picard-S and Thakur iteration have almost the same speed of convergence and we obtain the comparison Table 2 and Figure 2 for various iterative schemes with control sequences $r_s = p_s = \frac{s}{s+1}$ and initial guess $\psi_0 = 0.9$.

Table 2: Comparison of speed of convergence of $A^*$ iterative scheme with S, Thakur and $K^*$ iterative schemes.

| Step | $S$ | Thakur | $K^*$ | $A^*$ |
|---|---|---|---|---|
| 1 | 0.0200000000 | 0.0200000000 | 0.0200000000 | 0.0200000000 |
| 2 | 0.9115784441 | 0.9919616767 | 0.9931842144 | 0.9999436712 |
| 3 | 0.9920220698 | 0.9999340667 | 0.9999525970 | 0.9999999968 |
| 4 | 0.9992801827 | 0.9999994592 | 0.9999996703 | 1.0000000000 |
| 5 | 0.9999350537 | 0.9999999956 | 0.9999999977 | 1.0000000000 |
| 6 | 0.9999941402 | 1.0000000000 | 1.0000000000 | 1.0000000000 |
| 7 | 0.9999994713 | 1.0000000000 | 1.0000000000 | 1.0000000000 |
| 8 | 0.9999999523 | 1.0000000000 | 1.0000000000 | 1.0000000000 |
| 9 | 0.9999999957 | 1.0000000000 | 1.0000000000 | 1.0000000000 |
| 10 | 0.9999999996 | 1.0000000000 | 1.0000000000 | 1.0000000000 |
| 11 | 1.0000000000 | 1.0000000000 | 1.0000000000 | 1.0000000000 |

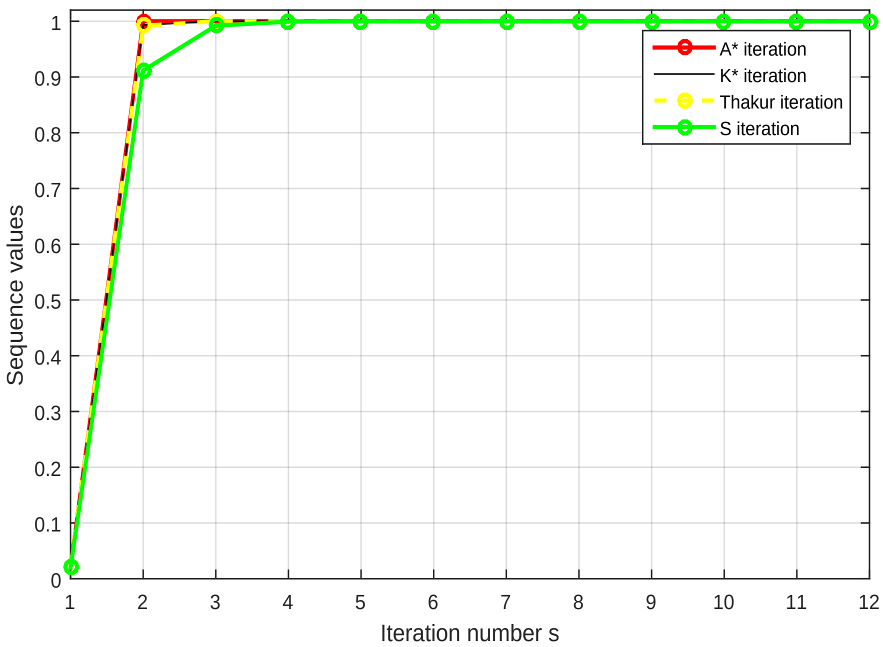


Figure 2: Graph corresponding to Table 2.

## 6. Stability result

Our aim in this section is to show that our new iterative method (6) is G–Stable.

**Theorem 6.1.** *Let $\Omega$ be a Banach space and $\Lambda$ be a closed convex subset of $\Omega$. Let $G$ be a mapping satisfy* (8)*. Let $\{\psi_s\}$ be the iterative method defined by* (6) *with sequences $r_s$ and $p_s \in [0,1]$ such that $\sum_{s=0}^{\infty} r_s = \infty$. Then the iterative method* (6) *is G–stable.*

*Proof.* Let $\{\zeta_s\} \subset \Omega$ be an arbitrary sequence in $\Lambda$ and suppose that the sequence iteratively generated by (6) is $\psi_{s+1} = f(G, \psi_s)$ converging to a unique point $z$ and that $\varepsilon_s = \|\zeta_{s+1} - f(G, \zeta_s)\|$. To prove that $G$ is stable, we have to show that $\lim_{s\to\infty} \varepsilon_s = 0 \Leftrightarrow \lim_{s\to\infty} \zeta_s = z$.

Let $\lim_{s\to\infty} \varepsilon_s = 0$. Then from (6) and (13), we obtain

$$\begin{aligned}
\|\zeta_{s+1} - z\| &= \|\zeta_{s+1} - f(G,\zeta_s) + f(G,\zeta_s) - z\| \\
&\le \|\zeta_{s+1} - f(G,\zeta_s)\| + \|f(G,\zeta_s) - z\| \\
&= \varepsilon_s + \|f(G,\zeta_s) - z\| \\
&= \varepsilon_s + \|G(G(G((1-r_n)G((1-p_s)\zeta_s + p_sG\zeta_s) \\
&\quad + r_sG(G((1-p_s)\zeta_s + p_sG\zeta_s)))) - z\| \\
&= \gamma^4(1-(1-\gamma)r_s)\|\zeta_s - z\| + \varepsilon_s. \qquad (44)
\end{aligned}$$

For all $s \ge 1$, put

$$\begin{aligned}
\theta_s &= \|\zeta_s - z\|, \\
\sigma_s &= (1-\gamma)r_s \in (0,1), \\
\lambda_s &= \varepsilon_s.
\end{aligned}$$

Since $\lim_{s\to\infty} \varepsilon_s = 0$, this implies that $\frac{\lambda_s}{\sigma_s} = \frac{\varepsilon_s}{(1-\gamma)r_s} \to 0$ as $s \to \infty$. Apparently, all the conditions of Lemma 2.14 are fulfilled. Hence, from Lemma 2.14 we have $\lim_{s\to\infty} \zeta_s = z$.

Conversely, let $\lim_{s\to\infty} \zeta_s = z$. The we have

$$\begin{aligned}
\varepsilon_s &= \|\zeta_{s+1} - f(G,\zeta_s)\| \\
&\leq \|\zeta_{s+1} - z + z - f(G,\zeta_s)\| \\
&\leq \|\zeta_{s+1} - z\| + \|f(G,\zeta_s) - z\| \\
&\leq \|\zeta_{s+1} - z\| + \gamma^4(1-(1-\gamma)r_s)\|\zeta_s - z\|. \qquad (45)
\end{aligned}$$

From (45), it follows that $\lim_{s\to\infty} \varepsilon_s = 0$. Hence, our new iterative scheme (6) is stable with respect to $G$. □

We now provide the following numerical example to support of analytic prove in Theorem 6.1.

**Example 6.2.** *Let $\Lambda = [0,1]$ and $G\psi = \frac{\psi}{4}$. Obviously, the fixed point of $G$ is 0. Firstly, we have to show that $G$ satisfies (8). To do this, with $\gamma = \frac{1}{4}$ and for $L \geq 0$, we have*

$$\begin{aligned}
\|G\psi - G\eta\| - \gamma\|\psi-\eta\| - L\|\psi - \eta\| &= \frac{1}{4}|\psi-\eta| - \frac{1}{4}|\psi-\eta| - L|\psi - \frac{\psi}{4}| \\
&= -L\left(\frac{3\psi}{4}\right) \leq 0.
\end{aligned}$$

*Now, we show that $A^*$ iterative method (6) is $G$–stable with respect with $G$.*

*Let $r_s = p_s = \frac{1}{s+2}$ and $\psi_0 \in [0,1]$, then we have*

$$\begin{aligned}
g_s &= \frac{1}{4}\left(1 - \frac{1}{s+2} + \frac{1}{4(s+2)}\right)\psi_s = \left(1 - \frac{3}{4(s+2)}\right)\psi_s \\
k_s &= \frac{1}{16}\left(1 - \frac{6}{4(s+2)} + \frac{1}{4^2(s+2)^2}\right)\psi_s \\
\eta_s &= \frac{1}{64}\left(1 - \frac{6}{4(s+2)} + \frac{9}{4^2(s+2)^2}\right)\psi_s \\
\psi_{s+1} &= \frac{1}{156}\left(1 - \frac{6}{4(s+2)} + \frac{9}{4^2(s+2)^2}\right)\psi_s \\
&= \left(1 - \left(\frac{254}{256} + \frac{6}{4^3(s+2)} - \frac{9}{4^2(s+2)^2}\right)\right)\psi_s.
\end{aligned}$$

*Let $\zeta_s = \frac{254}{256} + \frac{6}{4^3(s+2)} - \frac{9}{4^2(s+2)^2}$. Obviously, $\zeta_s \in (0,1)$ for all $s \in \mathbb{N}$ and $\sum_{s=0}^{\infty} \zeta_s = \infty$. By Lemma 2.14, we obtain $\lim_{s\to\infty} \psi_s = 0$. Let $y_s = \frac{1}{s+3}$, we have*

$$\begin{aligned}
\varepsilon_s &= |y_{s+1} - f(G, y_s)| \\
&= \left| y_{s+1} - \left(\frac{1}{256} - \frac{6}{4^5(s+2)} + \frac{9}{4^6(s+2)^2}\right) y_s \right| \\
&= \left| \frac{1}{s+4} - \left(\frac{1}{4^4(s+3)} - \frac{6}{4^5(s+2)(s+3)} + \frac{9}{4^6(s+2)^2(s+3)}\right)\right|.
\end{aligned}$$

*Obviously, $\lim_{s\to\infty} \varepsilon_s = 0$.*

Hence, our iterative algorithm (6) is stable with respect to $G$.

## 7. Data Dependence result

In this section, we obtain data dependence result for the mapping $G$ satisfying (8) using our new iterative algorithm (6).

**Theorem 7.1.** *Let $\tilde{G}$ be an approximate operator of a mapping $G$ satisfying* (8)*. Let $\{\psi_s\}$ be an iterative sequence generated by* (6) *for $G$ and define an iterative sequence as follows:*

$$\begin{cases} \tilde{\psi}_0 \in \Lambda, \\ \tilde{g}_s = \tilde{G}((1-p_s)\tilde{\psi}_s + p_s\tilde{G}\tilde{\psi}_s), \\ \tilde{k}_s = \tilde{G}((1-r_s)\tilde{g}_s + r_s\tilde{G}\tilde{g}_s), \qquad \forall s \geq 1. \\ \tilde{\eta}_s = \tilde{G}\tilde{k}_s, \\ \tilde{\psi}_{s+1} = \tilde{G}\tilde{\eta}_s, \end{cases} \tag{46}$$

*where $\{r_s\}$ and $\{p_s\}$ are sequences in $[0,1]$ satisfying the following conditions:*

*(i)* $\frac{1}{2} \leq r_s, \forall s \in \mathbb{N}$,

*(ii)* $\sum\limits_{s=0}^{\infty} r_s = \infty$.

*If $Tz = z$ and $\tilde{T}\tilde{z} = \tilde{z}$ such that $\lim\limits_{s\to\infty} \tilde{\psi}_s = \tilde{z}$, we have*

$$\|z - \tilde{z}\| \leq \frac{11\epsilon}{1-\gamma}, \tag{47}$$

*where $\epsilon > 0$ is a fixed number.*

*Proof.* Using (6), (8) and (46), we have

$$\begin{aligned} \|g_s - \tilde{g}_s\| &= \|G((1-p_s)\psi_s + p_sG\psi_s) - \tilde{G}((1-p_s)\tilde{\psi}_s + p_s\tilde{G}\tilde{\psi}_s)\| \\ &\leq \|G((1-p_s)\psi_s + p_sG\psi_s) - G((1-p_s)\tilde{\psi}_s + p_s\tilde{G}\tilde{\psi}_s)\| \\ &\quad + \|G((1-p_s)\tilde{\psi}_s + p_s\tilde{G}\tilde{\psi}_s) - \tilde{G}((1-p_s)\tilde{\psi}_s + p_s\tilde{G}\tilde{\psi}_s)\| \\ &\leq \gamma((1-p_s)\|\psi_s - \tilde{\psi}_s\| + p_s\|G\psi_s - \tilde{G}\tilde{\psi}_s\|) \\ &\quad + L\|(1-p_s)\psi_s + p_sG\psi_s - G((1-p_s)\psi_s + p_sG\psi_s)\| + \epsilon \\ &\leq \gamma((1-p_s)\|\psi_s - \tilde{\psi}_s\| + p_s(\|G\psi_s - G\tilde{\psi}_s\| + \|G\tilde{\psi}_s - \tilde{G}\tilde{\psi}_s\|)) \\ &\quad + L\|(1-p_s)\psi_s + p_sG\psi_s - G((1-p_s)\psi_s + p_sG\psi_s)\| + \epsilon \\ &\leq \gamma((1-p_s)\|\psi_s - \tilde{\psi}_s\| + \gamma p_s\|\psi_s - \tilde{\psi}_s\| + p_sL\|\psi_s - G\psi_s\| + p_s\epsilon) \\ &\quad + L\|(1-p_s)\psi_s + p_sG\psi_s - G((1-p_s)\psi_s + p_sG\psi_s)\| + \epsilon \\ &\leq \gamma(1-(1-\gamma)p_s)\|\psi_s - \tilde{\psi}_s\| + \gamma p_sL\|\psi_s - G\psi_s\| + \gamma p_s\epsilon \\ &\quad + L\|(1-p_s)\psi_s + p_sG\psi_s) - G((1-p_s)\psi_s + p_sG\psi_s)\| + \epsilon. \end{aligned} \tag{48}$$

Similarly, using (6), (8) and (46), we have

$$\begin{aligned}
\|k_s - \tilde{k}_s\| &= \|G((1-r_s)g_s + r_sGg_s) - \tilde{G}((1-r_s)\tilde{g}_s + r_s\tilde{G}\tilde{g}_s)\| \\
&\leq \|G((1-r_s)g_s + r_sGg_s) - G((1-r_s)\tilde{g}_s + r_s\tilde{G}\tilde{g}_s)\| \\
&\quad + \|G((1-r_s)\tilde{g}_s + r_s\tilde{G}\tilde{g}_s) - \tilde{G}((1-r_s)\tilde{g}_s + r_s\tilde{G}\tilde{g}_s)\| \\
&\leq \gamma((1-r_s)\|g_s - \tilde{g}_s\| + r_s\|Gg_s - \tilde{G}\tilde{g}_s\|) \\
&\quad + L\|(1-r_s)g_s + p_sGg_s - G((1-r_s)\psi_s + r_sGg_s)\| + \epsilon \\
&\leq \gamma((1-r_s)\|g_s - \tilde{g}_s\| + r_s(\|Gg_s - G\tilde{g}_s\| + \|G\tilde{g}_s - \tilde{G}\tilde{g}_s\|)) \\
&\quad + L\|(1-r_s)g_s + r_sGg_s - G((1-r_s)g_s + r_sGg_s)\| + \epsilon \\
&\leq \gamma((1-r_s)\|g_s - \tilde{g}_s\| + \gamma r_s\|g_s - \tilde{g}_s\| + r_sL\|g_s - Gg_s\| + r_s\epsilon) \\
&\quad + L\|(1-r_s)g_s + r_sGg_s) - G((1-r_s)\psi_s + r_sGg_s)\| + \epsilon \\
&\leq \gamma(1-(1-\gamma)r_s)\|g_s - \tilde{g}_s\| + \gamma r_sL\|g_s - Gg_s\| + \gamma r_s\epsilon \\
&\quad + L\|(1-r_s)\psi_s + r_sGg_s - G((1-r_s)g_s + r_sGg_s)\| + \epsilon. \qquad (49)
\end{aligned}$$

Putting (48) in (49), we have

$$\begin{aligned}
\|k_s - \tilde{k}_s\| &\leq \gamma(1-(1-\gamma)r_s)\{\gamma(1-(1-\gamma)p_s)\|\psi_s - \tilde{\psi}_s\| + \gamma p_sL\|\psi_s - G\psi_s\| \\
&\quad + \gamma p_s\epsilon + L\|(1-p_s)\psi_s + p_sG\psi_s) - G((1-p_s)\psi_s + p_sG\psi_s)\| + \epsilon\} \\
&\quad + \gamma r_sL\|g_s - Gg_s\| + \gamma r_s\epsilon \\
&\quad + L\|(1-r_s)\psi_s + r_sGg_s) - G((1-r_s)g_s + r_sGg_s)\| + \epsilon
\end{aligned}$$

$$\begin{aligned}
&= \gamma^2(1-(1-\gamma)r_s)(1-(1-\gamma)p_s)\|\psi_s - \tilde{\psi}_s\| \\
&\quad + \gamma^2(1-(1-\gamma)r_s)p_sL\|\psi_s - G\psi_s\| + \gamma^2p_s\epsilon - \gamma r_sp_s\epsilon + \gamma^3r_sp_s\epsilon \\
&\quad + \gamma(1-(1-\gamma)r_s)L\|(1-p_s)\psi_s + p_sG\psi_s) - G((1-p_s)\psi_s + p_sG\psi_s)\| \\
&\quad + \gamma\epsilon - \gamma r_s\epsilon + \gamma^2r_s\epsilon + \gamma r_sL\|g_s - Gg_s\| + \gamma r_s\epsilon \\
&\quad + L\|(1-r_s)\psi_s + r_sGg_s) - G((1-r_s)g_s + r_sGg_s)\| + \epsilon \\
&= \gamma^2(1-(1-\gamma)r_s)(1-(1-\gamma)p_s)\|\psi_s - \tilde{\psi}_s\| \\
&\quad + \gamma^2(1-(1-\gamma)r_s)p_sL\|\psi_s - G\psi_s\| + \gamma r_sL\|g_s - Gg_s\| \\
&\quad + \gamma(1-(1-\gamma)r_s)L\|(1-p_s)\psi_s + p_sG\psi_s) - G((1-p_s)\psi_s + p_sG\psi_s)\| \\
&\quad + L\|(1-r_s)\psi_s + r_sGg_s) - G((1-r_s)g_s + r_sGg_s)\| \\
&\quad + \gamma^2p_s\epsilon + \gamma^2r_sp_s(\gamma-1) + \gamma\epsilon + \gamma^2r_s\epsilon + \epsilon. \qquad (50)
\end{aligned}$$

From (6), (46), (8) and (50) we obtain

$$\begin{aligned}
\|\eta_s - \tilde{\eta}_s\| &= \|Gk_s - \tilde{G}\tilde{k}_s\| \\
&= \|Gk_s - G\tilde{k}_s + G\tilde{k}_s - \tilde{G}\tilde{k}_s\| \\
&\leq \|Gk_s - G\tilde{k}_s\| + \|G\tilde{k}_s - \tilde{G}\tilde{k}_s\| \\
&\leq \gamma\|k_s - \tilde{k}_s\| + L\|k_s - Gk_s\| + \epsilon \\
&\leq \gamma^3(1-(1-\gamma)r_s)(1-(1-\gamma)p_s)\|\psi_s - \tilde{\psi}_s\| \\
&\quad + \gamma^3(1-(1-\gamma)r_s)p_sL\|\psi_s - G\psi_s\| + \gamma^2r_sL\|g_s - Gg_s\| \\
&\quad + \gamma^2(1-(1-\gamma)r_s)L\|(1-p_s)\psi_s + p_sG\psi_s) - G((1-p_s)\psi_s + p_sG\psi_s)\| \\
&\quad + \gamma L\|(1-r_s)\psi_s + r_sGg_s) - G((1-r_s)g_s + r_sGg_s)\| \\
&\quad + \gamma^3p_s\epsilon + \gamma^3r_sp_s(\gamma-1) + \gamma^2\epsilon + \gamma^3r_s\epsilon + \gamma\epsilon + L\|k_s - Gk_s\| + \epsilon. \qquad (51)
\end{aligned}$$

From (6), (46), (8) and (51), we have

$$\begin{aligned}
\|\psi_{s+1}-\tilde{\psi}_{s+1}\| &= \|G\eta_s-\tilde{G}\tilde{\eta}_s\| \\
&= \|G\eta_s-G\tilde{\eta}_s+G\tilde{\eta}_s-\tilde{G}\tilde{\eta}_s\| \\
&\leq \|G\eta_s-G\tilde{\eta}_s\|+\|G\tilde{\eta}_s-\tilde{G}\tilde{\eta}_s\| \\
&\leq \gamma\|\eta_s-\tilde{\eta}_s\|+L\|\eta_s-G\eta_s\|+\epsilon \\
&\leq \gamma^4(1-(1-\gamma)r_s)(1-(1-\gamma)p_s)\|\psi_s-\tilde{\psi}_s\| \\
&\quad +\gamma^4(1-(1-\gamma)r_s)p_sL\|\psi_s-G\psi_s\|+\gamma^3r_sL\|g_s-Gg_s\| \\
&\quad +\gamma^3(1-(1-\gamma)r_s)L\|(1-p_s)\psi_s+p_sG\psi_s)-G((1-p_s)\psi_s+p_sG\psi_s)\| \\
&\quad +\gamma^2L\|(1-r_s)\psi_s+r_sGg_s)-G((1-r_s)g_s+r_sGg_s)\| \\
&\quad +\gamma^4p_s\epsilon+\gamma^4r_sp_s(\gamma-1)+\gamma^3\epsilon+\gamma^4r_s\epsilon+\gamma^2\epsilon+\gamma L\|k_s-Gk_s\| \\
&\quad +\gamma\epsilon+L\|\eta_s-G\eta_s\|+\epsilon. \qquad (52)
\end{aligned}$$

Since $r_n$, $p_n \in [0,1]$ and $\gamma \in (0,1)$, it implies that

$$\begin{cases}
(1-(1-\gamma)r_s)<1, \\
(1-(1-\gamma)p_s)<1, \\
\gamma-1<0, \\
\gamma^4,\gamma^3,\gamma^2<1, \\
\gamma^4p_s<1.
\end{cases} \qquad (53)$$

From (52) and (53), we obtain

$$\begin{aligned}
\|\psi_{s+1}-\tilde{\psi}_{s+1}\| &\leq (1-(1-\gamma)r_s)\|\psi_s-\tilde{\psi}_s\| \\
&\quad +L\|\psi_s-G\psi_s\|+r_sL\|g_s-Gg_s\| \\
&\quad +L\|(1-p_s)\psi_s+p_sG\psi_s-G((1-p_s)\psi_s+p_sG\psi_s)\| \\
&\quad +L\|(1-r_s)\psi_s+r_sGg_s-G((1-r_s)g_s+r_sGg_s)\| \\
&\quad +L\|k_s-Gk_s\|+L\|\eta_s-G\eta_s\|+r_s\epsilon+5\epsilon. \qquad (54)
\end{aligned}$$

By our assumption (i) that $\frac{1}{2} \leq r_s$, we have

$$1-r_s \leq r_s \Rightarrow 1 = 1-r_s+r_s \leq r_s+r_s = 2r_s.$$

$$\begin{aligned}
\|\psi_{s+1}-\tilde{\psi}_{s+1}\| &\leq (1-(1-\gamma)r_s)\|\psi_s-\tilde{\psi}_s\| \\
&\quad +2r_sL\|\psi_s-G\psi_s\|+r_sL\|g_s-Gg_s\| \\
&\quad +2r_sL\|(1-p_s)\psi_s+p_sG\psi_s-G((1-p_s)\psi_s+p_sG\psi_s)\| \\
&\quad +2r_sL\|(1-r_s)\psi_s+r_sGg_s-G((1-r_s)g_s+r_sGg_s)\| \\
&\quad +2r_sL\|k_s-Gk_s\|+2r_sL\|\eta_s-G\eta_s\|+r_s\epsilon+10r_s\epsilon \\
&= (1-(1-\gamma)r_s)\|\psi_s-\tilde{\psi}_s\| \\
&\quad +r_s(1-\gamma)\times\Bigg\{\frac{2L\|\psi_s-G\psi_s\|+L\|g_s-Gg_s\|}{(1-\gamma)} \\
&\quad +\frac{2L\|(1-p_s)\psi_s+p_sG\psi_s-G((1-p_s)\psi_s+p_sG\psi_s)\|}{(1-\gamma)} \\
&\quad +\frac{2L\|(1-r_s)\psi_s+r_sGg_s-G((1-r_s)g_s+r_sGg_s)\|}{(1-\gamma)} \\
&\quad +\frac{2L\|k_s-Gk_s\|+2L\|\eta_s-G\eta_s\|+11\epsilon}{(1-\gamma)}\Bigg\}. \qquad (55)
\end{aligned}$$

Set

$$\begin{aligned} \theta_s &= \|\psi_s - \tilde{\psi}_s\| \\ \sigma_s &= (1-\gamma)r_s \in (0,1) \end{aligned}$$

$$\begin{aligned} \lambda_s &= \Bigg\{\frac{2L\|\psi_s - G\psi_s\| + L\|g_s - Gg_s\|}{(1-\gamma)} \\ &+\frac{2L\|(1-p_s)\psi_s + p_sG\psi_s - G((1-p_s)\psi_s + p_sG\psi_s)\|}{(1-\gamma)} \\ &+\frac{2L\|(1-r_s)\psi_s + r_sGg_s - G((1-r_s)g_s + r_sGg_s)\|}{(1-\gamma)} \\ &+\frac{2L\|k_s - Gk_s\| + 2L\|\eta_s - G\eta_s\| + 11\epsilon}{(1-\gamma)}\Bigg\}. \end{aligned}$$

From Theorem 3.1, we know that $\lim_{s\to\infty} \psi_s = z$ and since $Gz = z$, it follows that

$$\begin{aligned} \lim_{s\to\infty} \|\psi_s - G\psi_s\| &= \lim_{s\to\infty} \|g_s - Gg_s\| = \lim_{s\to\infty} \|k_s - Gk_s\| = \lim_{s\to\infty} \|\eta_s - G\eta_s\| \\ &= \lim_{s\to\infty} \|(1-p_s)\psi_s + p_sG\psi_s - G((1-p_s)\psi_s + p_sG\psi_s)\| \end{aligned}$$

$$\begin{aligned} &= \lim_{s\to\infty} \|(1-r_s)\psi_s + r_sGg_s - G((1-r_s)g_s + r_sGg_s)\| \\ &= 0. \end{aligned}$$

Using Lemma 2.15, we get

$$0 \le \limsup_{s\to\infty} \|\psi_s - \tilde{\psi}_s\| \le \limsup_{s\to\infty} \frac{11\epsilon}{(1-\gamma)}. \tag{56}$$

Since by Theorem 3.1, we have that $\lim_{s\to\infty} \psi_s = z$ and using our that hypothesis $\lim_{s\to\infty} \tilde{\psi}_s = \tilde{z}$, then from (56) we have

$$\|z - \tilde{z}\| \le \frac{11\epsilon}{(1-\gamma)}.$$

This completes the proof. □

## 8. Application

In this section, we will use our new iterative method (6) to solve the following Volterra-Fredholm integral equation which have been considered by Lungu and Rus [16]:

$$u(\psi,\eta) = g(\psi,\eta,h(u(\psi,\eta))) + \int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,u(m,n))dmdn, \tag{57}$$

for all $\psi,\eta \in \Re_+$. Let Let $(\Omega, |\cdot|)$ be a Banach space. Let $\tau > 0$ and

$$X_\tau = \{u \in C(\Re_+^2, \Omega) | \exists\, M(u) > 0 : |u(\psi,\eta)|e^{-\tau(\psi+\eta)} \le M(u)\}.$$

We now consider BieleckiâĂŹs norm on $X_\tau$ as follows:

$$\|u\|_\tau = \sup_{\psi,\eta\in\Re_+} \left(|u(\psi,\eta)|e^{-\tau(\psi+\eta)}\right).$$

Obviously, $(X_\tau, ||\cdot||_\tau)$ is a Banach space (see [6]).

The following result which was given by Lungu and Rus [16] will be useful in proving our main result.

**Theorem 8.1.** *[16] Suppose the following conditions are fulfilled:*

*(V₁)* $g \in C(\Re_+^2 \times \Omega, \Omega)$, $K \in C(\Re_+^4 \times \Omega, \Omega)$;

*(V₂)* $h : X_\tau \to X_\tau$ *is such that*

$$\exists l_h > 0 : |h(u(\psi,\eta)) - h(v(\psi,\eta))| \le l_h \|u - v\| \cdot e^{\tau(\psi+\eta)},$$

*for all* $\psi, \eta \in \Re_+$ *and* $u, v \in X_\tau$;

*(V₃)*

$$\exists l_g > 0 : |g(\psi,\eta,e_1) - g(\psi,\eta,e_2)| \le l_g |e_1 - e_2|,$$

*for all* $\psi, \eta \in \Re_+$ *and* $e_1, e_2 \in \Omega$;

*(V₄)*

$$\exists l_K(\psi,\eta,m,n) : |K(\psi,\eta,m,n,e_1) - K(\psi,\eta,m,n,e_1)| \le l_K(\psi,\eta,m,n)|e_1 - e_2|,$$

*for all* $\psi, \eta, m, n \in \Re_+$ *and* $e_1, e_2 \in \Omega$;

*(V₅)* $l_K \in C(\Re_+^4, \Re_+)$ *and*

$$\int_0^\psi \int_0^\eta l_K(\psi,\eta,m,n) e^{\tau(m+n)} dmdn \le l e^{\tau(\psi+\eta)},$$

*for all* $\psi, \eta \in \Re_+$;

*(V₆)* $l_g l_h + l < 1$.

*Then, the equation* (57) *has a unique solution* $z \in X_\tau$ *and the sequence of successive approximations*

$$u_{s+1}(\psi,\eta) = g(\psi,\eta,h(u_s(\psi,\eta))) + \int_0^\psi \int_0^\eta K(\psi,\eta,m,n,u_s(m,n))dmdn, \tag{58}$$

*for all* $s \in \mathbb{N}$ *converges uniformly to* $z$ *.*

We now give our main result in this section.

**Theorem 8.2.** *Let* $\{\psi_s\}$ *be* $A^*$ *iterative method defined by* (6) *with sequences* $\{r_s\}$ *and* $\{p_s\}$ *in [0,1] such that* $\sum_{s=0}^{\infty} r_s = \infty$. *If all the conditions* $(V_1) - (V_6)$ *in theorem 8.1 are satisfied, then the equation* (57) *has a unique solution* $z$ *in* $X_\tau$ *and the* $A^*$ *iterative sequence* (6) *converges strongly to* $z$.

*Proof.* Let $\{\psi_s\}$ be an iterative sequence generated by $A^*$ iterative method (6) for the operator $A : X_\tau \to X_\tau$ defined by

$$A(u(\psi,\eta)) = g(\psi,\eta,h(u(\psi,\eta))) + \int_0^\psi \int_0^\eta K(\psi,\eta,m,n,u(m,n))dmdn. \tag{59}$$

We will prove that $\psi_s \to 0$ as $s \to \infty$. Using (6), we obtain

$$\|\psi_{s+1} - z\|_\tau = \sup_{\psi,\eta \in \Re_+} (|A(\eta_s(\psi,\eta)) - A(z(\psi,\eta))|e^{-\tau(\psi+\eta)}).$$

Now,

$$\begin{aligned}|A(\eta_s(\psi,\eta))) &- A(z(\psi,\eta))|\\ &\leq |g(\psi,\eta,h(\eta_s(\psi,\eta))) - g(\psi,\eta,h(z(\psi,\eta)))|\\ &\quad +\Big|\int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,\eta_s(m,n))dmdn\\ &\quad -\int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,z(m,n))dmdn\Big|\\ &\leq l_g|h(\eta_s(\psi,\eta)) - h(z(\psi,\eta))|\\ &\quad +\int_0^{\psi}\int_0^{\eta} |K(\psi,\eta,m,n,\eta_s(m,n))\\ &\quad -K(\psi,\eta,m,n,z(m,n))|dmdn\\ &\leq l_g l_h\|\eta_s - z\|_\tau e^{\tau(\psi+\eta)}\\ &\quad +\int_0^{\psi}\int_0^{\eta} l_K(\psi,\eta,m,n)|\eta_s(m,n) - z(m,n)|dmds\\ &\leq l_g l_h\|\eta_s - z\|_\tau e^{\tau(\psi+\eta)} + l\|\eta_s - z\|_\tau e^{\tau(\psi+\eta)}\\ &= (l_g l_h + l)\|\eta_s - z\|_\tau e^{\tau(\psi+\eta)}.\end{aligned}$$

Hence,

$$\|\psi_{s+1} - z\|_\tau \leq (l_g l_h + l)\|\eta_s - z\|_\tau. \tag{60}$$

Similarly,

$$\|\eta_s - z\|_\tau \leq (l_g l_h + l)\|k_s - z\|_\tau. \tag{61}$$

Putting (61) into (60), we get

$$\|\psi_{s+1} - z\|_\tau \leq (l_g l_h + l)^2\|k_s - z\|_\tau. \tag{62}$$

Again,

$$\|k_s - z\|_\tau = \sup_{\psi,\eta\in\Re_+}(|A((1-r_s)g_s + r_sGg_s)(\psi,\eta) - A(z(\psi,\eta))|e^{-\tau(\psi+\eta)}),$$

and

$$\begin{aligned}|A((1-r_s)g_s &+ r_sAg_s)(\psi,\eta)) - A(z(\psi,\eta))|\\ &\leq |g(\psi,\eta,h(((1-r_s)g_s + r_sAg_s)(\psi,\eta))) - g(\psi,\eta,h(z(\psi,\eta)))|\\ &\quad +\Big|\int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,((1-r_s)g_s + r_sAg_s)(m,n))dmdn\\ &\quad -\int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,z(m,n))dmdn\Big|\end{aligned}$$

$$\begin{aligned}
&\leq l_g|h(((1-r_s)g_s+r_sAg_s)(\psi,\eta))-h(z(\psi,\eta))| \qquad (63)\\
&\quad+\int_0^{\psi}\int_0^{\eta}|K(\psi,\eta,m,n,((1-r_s)g_s+r_sAg_s)(m,n))\\
&\quad-K(\psi,\eta,m,n,z(m,n))|dmdn\\
&\leq l_gl_h\|((1-r_s)g_s+r_sAg_s)-z\|_\tau e^{\tau(\psi+\eta)}\\
&\quad+\int_0^{\psi}\int_0^{\eta}l_K(\psi,\eta,m,n)|((1-r_s)g_s+r_sAg_s)(m,n)-z(m,n)|dmds\\
&\leq l_gl_h\|((1-r_s)g_s+r_sAg_s)-z\|_\tau e^{\tau(\psi+\eta)}\\
&\quad+L\|((1-r_s)g_s+r_sAg_s)-z\|_\tau e^{\tau(\psi+\eta)}\\
&= (l_gl_h+l)\|((1-r_s)g_s+r_sAg_s)-z\|_\tau e^{\tau(\psi+\eta)}\\
&\leq (l_gl_h+l)\|((1-r_s)g_s+r_sAg_s)-z\|_\tau. \qquad (64)
\end{aligned}$$

So

$$\begin{aligned}
\|((1-r_s)g_s+r_sAg_s)-z\|_\tau &= \|((1-r_s)(g_s-z)+r_s(Ag_s-z)\|_\tau\\
&\leq (1-r_s)\|g_s-z\|_\tau+r_s\|Ag_s-z\|_\tau. \qquad (65)
\end{aligned}$$

Now

$$\|Ag_s-Az\|_\tau=\sup_{\psi,\eta\in\Re_+}(|A(g_s(\psi,\eta))-A(z(\psi,\eta))|e^{-\tau(\psi+\eta)}),$$

and

$$\begin{aligned}
|A(g_s(\psi,\eta)))-A(z(\psi,\eta))| &\leq |g(\psi,\eta,h(g_s(\psi,\eta)))-g(\psi,\eta,h(z(\psi,\eta)))|\\
&\quad+\Big|\int_0^{\psi}\int_0^{\eta}K(\psi,\eta,m,n,g_s(m,n))dmdn\\
&\quad-\int_0^{\psi}\int_0^{\eta}K(\psi,\eta,m,n,z(m,n))dmdn\Big|\\
&\leq l_g|h(g_s(\psi,\eta))-h(z(\psi,\eta))|\\
&\quad+\int_0^{\psi}\int_0^{\eta}|K(\psi,\eta,m,n,g_s(m,n))\\
&\quad-K(\psi,\eta,m,n,z(m,n))|dmdn\\
&\leq l_gl_h\|g_s-z\|_\tau e^{\tau(\psi+\eta)}\\
&\quad+\int_0^{\psi}\int_0^{\eta}l_K(\psi,\eta,m,n)|g_s(m,n)-z(m,n)|dmds\\
&\leq l_gl_h\|g_s-z\|_\tau e^{\tau(\psi+\eta)}+l\|g_s-z\|_\tau e^{\tau(\psi+\eta)}\\
&= (l_gl_h+l)\|g_s-z\|_\tau e^{\tau(\psi+\eta)}.
\end{aligned}$$

Thus,

$$\|Ag_s-Az\|_\tau\leq(l_gl_h+l)\|g_s-z\|_\tau. \qquad (66)$$

From (65) and (66), we obtain

$$\begin{aligned}
\|((1-r_s)g_s+r_sAg_s)-z\| &\leq (1-r_s)\|g_s-z\|+r_s(l_gl_h+l)\|g_s-z\|_\tau\\
&= [1-r_s\{1-(l_gl_h+l)\}]\|g_s-z\|_\tau. \qquad (67)
\end{aligned}$$

Using (64) and (67), we have

$$\|k_s-z\|_\tau\leq(l_gl_h+l)[1-r_s\{1-(l_gl_h+l)\}]\|g_s-z\|_\tau. \qquad (68)$$

Putting (68) into (62), we obtain

$$\|\psi_{s+1} - z\|_\tau \leq (l_h l_h + l)^3[1 - r_s\{1 - (l_g l_h + l)\}]\|g_s - z\|_\tau. \tag{69}$$

Similarly, using (6), we have that

$$\|g_s - z\|_\tau \leq (l_g l_h + l)[1 - p_s\{1 - (l_g l_h + l)\}]\|\psi_s - z\|_\tau. \tag{70}$$

From (69) and (70), we get

$$\begin{aligned} \|\psi_{s+1} - z\|_\tau &\leq (l_g l_h + l)^4[1 - r_s\{1 - (l_g l_h + l)\}] \\ &\quad \times[1 - p_s\{1 - (l_g l_h + l)\}]\|\psi_s - z\|_\tau. \end{aligned} \tag{71}$$

Recalling from assumption $(C_6)$ that $l_g l_h + l < 1$ and since $p_s \in [0,1]$, then it follows that $1 - p_s\{1-(l_g l_h + l)\} < 1$. Thus, from (71), we obtain

$$\|\psi_{s+1} - z\|_\tau \leq [1 - r_s\{1 - (l_g l_h + l)\}]\|\psi_s - z\|_\tau.$$

Inductively, from (72), we have

$$\|\psi_{s+1} - z\|_\tau \leq \|\psi_0 - z\|_\tau \prod_{k}^{s}[1 - r_k\{1 - (l_g l_h + l)\}]. \tag{72}$$

Since $r_k \in [0,1]$ for all $k \in \mathbb{N}$ and assumption $(C_6)$ gives

$$1 - r_k\{1 - (l_g l_h + l)\} < 1.$$

From classical analysis, we know that $1 - \psi \leq e^{-\psi}$ for all $\psi \in [0,1]$. Thus, (72) becomes

$$\|\psi_{s+1} - z\|_\tau \leq \|\psi_0 - z\|_\tau e^{-[1-r_k\{1-(l_g l_h + l)\}]\sum_{k=0}^{s} r_k}$$

which yields $\lim_{s\to\infty} \|\psi_s - z\|_\tau = 0$. This completes the proof. □

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