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Special Curves According to Extended Darboux Frame Field in $\mathbb{E}_1^4$

Esra ERDEM[1*], Mihriban ALYAMAÇ KÜLAHCI[2] ,Münevver YILDIRIM YILMAZ[3]
Deparment of Maths, Faculty of Science, Firat University, Elazig, Turkey
[*1] esra.6666.23@gmail.com, [2] mihribankulahci@gmail.com, [3] myildirim@firat.edu.tr

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Abstract: In this study, the involute of a curve is investigated in Minkowski space-time by using extended Darboux frame field and curves of the AW (k) type are studied in Minkowski space-time.

Key words: Involute curve, Minkowski space-time, Extended Darboux frame field.

$\mathbb{E}_1^4$ de Genişletilmiş Darboux Çatı Alanına Göre Özel Eğriler

Öz: Bu çalışma da Genişletilmiş Darboux çatı alanı kullanılarak Minkowski uzayında bir eğrinin involütü ve AW(k) tipli eğriler incelenmiştir.

Anahtar kelimeler: İnvolüt eğri, Minkowski uzayı, Genişletilmiş Darboux çatı alanı.

1. Introduction

In differential geometry, there are many significant results and properties of curves. Scientists follow studies about the curves. In the light of existing studies authors introduce new works by using frame fields. The Darboux frame field is known as one of the frame field of the differential geometry. Many mathematicians have presented studies using the Darboux frame field. One of them is the study of Şahin and Dirişen. Şahin and Dirişen gave position vectors of curves with respect to Darboux frame in the Galilean space G^3 [1] .In [2], Altunkaya and Aksoyak's obtained curves of constant breadth according to Darboux frame. In [3] , Düldül et all. introduced extended Darboux frame field.

Involutes of a curve is another attractive research subject among geometers. The idea of a string involute is due to C. Huygens (1658), who is also known for his work in optics. He discovered involutes trying to build a more accurate clock [4]. After, a characterization of space-like involute-evolute curve couple in Minkowski space-time were given in [5]. T. Soyfidan and M. A. Güngör studied a quaternionic curve Euclidean 4-space $\mathbb{E}^4$ and gave the quaternionic involute-evolute curves for quaternionic curve [6]. Another is As and Sarıoğlugil study's. They obtained the Bishop curvatures of involute-evolute curve couple in $\mathbb{E}^3$ [7].

Many studies related to curves of AW(k)-type have been done by several authors and many interesting results on AW(k)-type curves have been obtained by many mathematicians (see [8-11]). In [12-13] the authors obtained some characterizations related to these curves in $\mathbb{E}^m$. Kılıç and Arslan considered curves and surfaces of AW(k)-type and they also gave related examples of curves and surfaces sufficient AW(k)-type conditions [14].
In this paper, involute curve is given in Minkowski space-time $\mathbb{E}_1^4$ by using extended Darboux frame field. In addition, considering extended Darboux frame field, AW(k)-type curves are studied in $\mathbb{E}_1^4$.

2. Preliminaries

The Minkowski space-time $\mathbb{E}_1^4$ is a Euclidean space provided with the indefinite flat metric given by

$$g=-dx_1^2+dx_2^2+dx_3^2+dx_4^2$$

where (x_1,x_2,x_3,x_4) is a rectangular coordinate system of $\mathbb{E}_1^4$. Recall that an arbitrary vector $\upsilon\in\mathbb{E}_1^4-\{0\}$ can be spacelike, timelike or null (lightlike vector), if holds $g\langle\upsilon,\upsilon\rangle\rangle 0, g\langle\upsilon,\upsilon\rangle\langle 0$ or $g\langle\upsilon,\upsilon\rangle=0$ respectively.

* Corresponding author: esra.6666.23@gmail.com. ORCID Number of authors: [1] 0000-0001-8656-0196, [2] 0000-0002-8621-5779, [3] 0000-0003-1278-3981

Especially, the vector $\upsilon = 0$ is spacelike. Besides an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^4$ can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null respectively. The norm of a vector υ is given by $\|\upsilon\| = \sqrt{|g\langle \upsilon, \upsilon\rangle|}$. Thus $\bar{\upsilon}$ is a unit vector if $g\langle \upsilon, \upsilon\rangle = \pm 1$. For an arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^4$ the curve is named a spacelike, a timelike and a null (lightlike) curve, if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike, and null (lightlike), respectively [15].

A hypersurface in the Minkowski space-time $\mathbb{E}_1^4$ is called a spacelike or a timelike hypersurface if the induced metric on the hypersurface is a positive definite Riemannian metric or a Lorentzian metric, respectively. The normal vector on the spacelike or the timelike hypersurface is, respectively, a timelike or a spacelike vector.Let M be an orientable non-null hypersurface

$$\alpha : I \subset R \to M$$

be a unit speed non-null curve in $\mathbb{E}_1^4$. Let's indicate by $\{t, n, b_1, b_2\}$ the acting Frenet frame along the curve $\alpha(s)$ in the space $\mathbb{E}_1^4$. Then $\{t, n, b_1, b_2\}$ are, respectively, the unit tangent, the principal normal, the first binormal and the second binormal vector fields. If k_1, k_2, k_3 are named the curvature functions of the unit speed non-null curve α, then for the non-null frame vectors, we obtain Frenet equations as follows:

$$\begin{aligned}
t' &= \varepsilon_n k_1 n \\
n' &= -\varepsilon_t k_1 t + \varepsilon_{b_1} k_2 b_1 \\
b_1' &= -\varepsilon_n k_2 n - \varepsilon_t \varepsilon_n \varepsilon_{b_1} k_3 b_2 \varepsilon \\
b_2' &= -\varepsilon_{b_1} k_3 b_1
\end{aligned}$$

where $\varepsilon_t = \langle t, t\rangle$, $\varepsilon_n = \langle n, n\rangle$, $\varepsilon_{b_1} = \langle b_1, b_1\rangle$, $\varepsilon_{b_2} = \langle b_2, b_2\rangle$ whereby $\varepsilon_t, \varepsilon_n, \varepsilon_{b_1}, \varepsilon_{b_2} \in \{-1, 1\}$, $1 \le i \le 4$ and $\varepsilon_t \varepsilon_n \varepsilon_{b_1} \varepsilon_{b_2} = -1$, [16].

3. Extended Darboux Frame Field in $\mathbb{E}_1^4$

Let M be an orientable non-null hypersurface, N be its non-null unit normal vector field in $\mathbb{E}_1^4$ and $x(s)$ be a non-null Frenet curve parametrized by arc-lenght parameter s lying on M. If the non-null unit tangent vector field of x is indicated by T, and the non-null unit normal vector field of M restricted to x is indicated by N, we attain

$$x'(s) = T(s) \tag{1}$$

and

$$N(x(s)) = N(s) \tag{2}$$

[17].

As in Euclidean 4- space $\mathbb{E}^4$ [3], the extended Darboux frame can be construct in two different cases in Minkowski space-time $\mathbb{E}_1^4$ with respect to whether the set $\{N, T, x''\}$ is linearly independent or linearly dependent.

Let us show the ED-frame field is first kind and the second kind if the set $\{N, T, x''\}$ is linearly independent and linearly dependent, respectively , [17].

Now, let us construct the ED-frame fields for the non-null Frenet curve α in $\mathbb{E}_1^4$. As explained in [3], using the Gram-Schmidt orthonormalization method, we obtain

$$E = \frac{x''' - \langle x''', N\rangle N - \langle x''', T\rangle T}{\left\| x''' - \langle x''', N\rangle N - \langle x''', T\rangle T \right\|} \tag{3}$$

[17]. If we detect

$$-D = N \otimes T \otimes E \tag{4}$$

for both cases, we obtain the orthonormal frame field $\{T, E, D, N\}$ another from Frenet frame field $\{T, n, b_1, b_2\}$ along the curve x. According to the orthonormal frame $\{T, E, D, N\}$, the vector fields $\{T', E', D', N'\}$ have the following decompositions:

$$\begin{aligned}
T' &= \varepsilon_1\langle T', T\rangle T + \varepsilon_2\langle T', E\rangle E + \varepsilon_3\langle T', D\rangle D + \varepsilon_4\langle T', N\rangle N \\
E' &= \varepsilon_1\langle E', T\rangle T + \varepsilon_2\langle E', E\rangle E + \varepsilon_3\langle E', D\rangle D + \varepsilon_4\langle E', N\rangle N \\
D' &= \varepsilon_1\langle D', T\rangle T + \varepsilon_2\langle D', E\rangle E + \varepsilon_3\langle D', D\rangle D + \varepsilon_4\langle D', N\rangle N \\
N' &= \varepsilon_1\langle N', T\rangle T + \varepsilon_2\langle N', E\rangle E + \varepsilon_3\langle N', D\rangle D + \varepsilon_4\langle N', N\rangle N
\end{aligned}$$

where $\varepsilon_1 = \langle T, T\rangle, \varepsilon_2 = \langle E, E\rangle, \varepsilon_3 = \langle D, D\rangle, \varepsilon_4 = \langle N, N\rangle$ whereby $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{-1, 1\}$. Moreover, when $\varepsilon_i = -1$, then $\varepsilon_j = 1$ for all $j \neq i, 1 \leq i, j \leq 4$ and $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = -1$, [17].

If

$$\langle D', N\rangle = \tau_g^2, \langle T', E\rangle = \kappa_g^1, \langle E', D\rangle = \kappa_g^2, \tag{5}$$

where κ_g^i and τ_g^i are the geodesic curvature and the geodesic torsion of order $i, (i = 1, 2)$ respectively, then the differential equations for ED-frame field is obtained ,if

$$\left(\begin{vmatrix} T' \\ E' \\ D' \\ N' \end{vmatrix}\right) = \left(\begin{vmatrix} 0 & 0 & 0 & \varepsilon_4\kappa_n \\ 0 & 0 & \varepsilon_3\kappa_g^2 & \varepsilon_4\tau_g^1 \\ 0 & -\varepsilon_3\kappa_g^2 & 0 & 0 \\ -\varepsilon_1\kappa_n & -\varepsilon_2\tau_g^1 & 0 & 0 \end{vmatrix}\right) \tag{6}$$

[17].

Theorem 3.1: Let $x : I \to \mathbb{E}_1^4$ be a regular unit speed non-null Frenet curve and any real non-null Frenet curve $\bar{x} : I \to \mathbb{E}_1^4$ be the involute of x . The Serret-Frenet apparatus of non-null Frenet curve $\bar{x}$ can be formed by apparatus of x .

Proof: Let $x = x(s)$ be a regular unit speed non-null Frenet curve.Without loss of generality, suppose that $\bar{x} = \bar{x}(s)$ is the involute of x. Therefore, we obtain

$$\bar{x}(s) = x(s) + \lambda_1(s) T_x(s) \tag{7}$$

By using Equation (6) and differentiating Equation (7) with respect to s, we get

$$\bar{x}'(s) = \left[1 + \lambda_1'(s)\right] T_x(s) + \lambda_1(s) \varepsilon_4 \kappa_n N_x(s).$$

Moreover, we have

$$\langle T_{\bar{x}}(s), T_x(s)\rangle = 0, \quad \langle T_x(s), N_x(s)\rangle = 0, \quad \langle T_x(s), T_x(s)\rangle = \varepsilon_1.$$

Thus, we get

$$\left(1 + \lambda_1'(s)\right)\varepsilon_1 = 0$$
$$\lambda_1(s) = c - s$$

Hence, we attain

$$\bar{x}'(s) = (c-s)\varepsilon_4 \kappa_n N_x(s). \tag{8}$$

$$T_{\bar{x}}(s) = (c-s)\varepsilon_4 \kappa_n N_x(s). \tag{9}$$

By using equation Equation (6) and differentiating Equation (8) twice with respect to s, we have

$$\begin{aligned}\bar{x}''(s) = &-\left[(c-s)\varepsilon_1\varepsilon_4(\kappa_n)^2\right] T_x(s) - (c-s)\varepsilon_2\varepsilon_4 \tau_g^1 \kappa_n E_x(s) \\ &+ \left[(c-s)\kappa_n'\varepsilon_4 - \kappa_n\varepsilon_4\right] N_x(s),\end{aligned} \tag{10}$$

$$\begin{aligned}\bar{x}'''(s) &= \left[2\varepsilon_1\varepsilon_4(\kappa_n)^2 - 3(c-s)\varepsilon_1\varepsilon_4(\kappa_n)\kappa_n'\right]T_x(s)\\ &+\begin{bmatrix}2\varepsilon_2\varepsilon_4\tau_g^1-(c-s)\varepsilon_2\varepsilon_4(\tau_g^1)'\kappa_n\\ -2(c-s)\varepsilon_2\varepsilon_4\tau_g^1(\kappa_n')\end{bmatrix}E_x(s)\\ &-\left[(c-s)\varepsilon_2\varepsilon_3\varepsilon_4\tau_g^1\kappa_n(\kappa_g)^2\right]D_x(s)\\ &+\begin{bmatrix}-2\varepsilon_4\kappa_n'+(c-s)\kappa_n''\varepsilon_4-(c-s)\varepsilon_1(\varepsilon_4)^2(\kappa_n)^3\\ -(c-s)\varepsilon_2(\varepsilon_4)^2(\kappa_n)(\tau_g^1)^2\end{bmatrix}N_x(s).\end{aligned}$$

Consequently, substituting

$$\bar{x}'''(s) = AT_x(s)+BE_x(s)-CD_x(s)+FN_x(s). \tag{11}$$

then, by using the Equation (2) ,we obtain

$$N_{\bar{x}}(s)=\frac{\begin{array}{c}\left[-(c-s)(\varepsilon_1)(\kappa_n)^2\right]T_x(s)-\left[(c-s)(\varepsilon_2)(\tau_g^1)(\kappa_n)\right]E_x(s)\\ +\left[(c-s)(\kappa_n')-(\kappa_n)\right]N_x(s)\end{array}}{W=\left\|\bar{x}''(s)\right\|}. \tag{12}$$

From the Equation (11) and Equation (12), we get

$$\begin{aligned}\langle\bar{x}'''(s),N_{\bar{x}}(s)\rangle &= -A(c-s)(\varepsilon_1)^2(\kappa_n)^2-B(c-s)(\varepsilon_2)^2(\tau_g^1)(\kappa_n)\\ &+F\left[(c-s)(\kappa_n')-(\kappa_n)\right]\varepsilon_4.\end{aligned}$$

Consequently, substituting

$$\begin{aligned}P &= -A(c-s)(\varepsilon_1)^2(\kappa_n)^2-B(c-s)(\varepsilon_2)^2(\tau_g^1)(\kappa_n)\\ &+F\left[(c-s)(\kappa_n')-(\kappa_n)\right]\varepsilon_4.\end{aligned}$$

we obtain

$$\langle\bar{x}'''(s),N_{\bar{x}}(s)\rangle N_{\bar{x}}(s)=\frac{P}{W}\begin{bmatrix}\left[-(c-s)(\varepsilon_1)(\kappa_n)^2\right]T_x(s)\\ -\begin{bmatrix}(c-s)(\varepsilon_2)\\(\tau_g^1)(\kappa_n)\end{bmatrix}E_x(s)\\ +\left[(c-s)(\kappa_n')-(\kappa_n)\right]N_x(s)\end{bmatrix}. \tag{13}$$

By using Equation (9) and Equation (11), we have

$$\langle \overline{x}'''(s), T_{\overline{x}}(s)\rangle T_{\overline{x}}(s) = F(c-s)^2(\varepsilon_4)^3(\kappa_n)^2 N_x(s)\,, \tag{14}$$

and if we write the Equations (11),(13) and Equation (14) into Equation (3), we get

$$E_{\overline{x}}(s) = \frac{\begin{array}{c}\left[A+\frac{P}{W}\left[(c-s)(\varepsilon_1)(\kappa_n)^2\right]\right]T_x(s)\\ \left[B+\frac{P}{W}\left[(c-s)(\varepsilon_2)(\tau_g^1)(\kappa_n)\right]\right]E_x(s)-CD_x(s)\\ \left[F-\frac{P}{W}\left[(c-s)(\kappa_n')-(\kappa_n)\right]-F(c-s)^2(\varepsilon_4)^3(\kappa_n)^2\right]N_x(s)\end{array}}{L=\left\|x'''-\langle x''',N\rangle N-\langle x''',T\rangle T\right\|}. \tag{15}$$

Then, we have

$$E_{\overline{x}}'(s) = UT_x(s)+VE_x(s)+YD_x(s)+ZN_x(s). \tag{16}$$

Similarly, using the Equations (4) , (9), (12), and Equation (15) and essential arrengements , $D_{\overline{x}}(s)$ is obtained as follows that

$$D_{\overline{x}}(s) = -\frac{1}{WL}\begin{bmatrix}\left[-(c-s)^2\varepsilon_2\varepsilon_4 C(\tau_g^1)(\kappa_n)^2\right]T_x(s)\\ \left[(c-s)^2\varepsilon_1\varepsilon_4 C(\kappa_n)^3\right]E_x(s)\\ \left[\left[(c-s)^2\varepsilon_4(\kappa_n)^2\left(\varepsilon_1 B-\varepsilon_2(\tau_g^1)A\right)\right]D_x(s)\right]\end{bmatrix}. \tag{17}$$

By using Equation (6) and differentiating Equation (17) with respect to s, we obtain

$$D_{\bar{x}}^{'}(s)=\left[-\frac{1}{WL}\right]^{'}\begin{matrix}\left[-(c-s)^2\varepsilon_2\varepsilon_4C\left(\tau_g^1\right)(\kappa_n)^2\right]T_x(s)\\ \left[+(c-s)^2\varepsilon_1\varepsilon_4C(\kappa_n)^3\right]E_x(s)\\ \left[+(c-s)^2\varepsilon_4(\kappa_n)^2\left(\varepsilon_1B-\varepsilon_2\left(\tau_g^1\right)A\right)\right]D_x(s)\end{matrix}$$

$$-\frac{1}{WL}\left[\begin{matrix}\left[\begin{matrix}2(c-s)\varepsilon_2\varepsilon_4C\left(\tau_g^1\right)(\kappa_n)^2\\ -(c-s)^2\varepsilon_2\varepsilon_4C^{'}\left(\tau_g^1\right)(\kappa_n)^2\\ -(c-s)^2\varepsilon_2\varepsilon_4C\left(\tau_g^1\right)^{'}(\kappa_n)^2\\ -2(c-s)^2\varepsilon_2\varepsilon_4C\left(\tau_g^1\right)(\kappa_n)(\kappa_n)^{'}\end{matrix}\right]T_x(s)\\ \left[\begin{matrix}-2(c-s)\varepsilon_1\varepsilon_4C(\kappa_n)^3\\ +3(c-s)^2\varepsilon_1\varepsilon_4C(\kappa_n)^2(\kappa_n)^{'}\\ -(c-s)^2\varepsilon_2\varepsilon_4\left(\tau_g^1\right)(\kappa_n)^2\\ \kappa_g^2\left(\varepsilon_1B-\varepsilon_2\left(\tau_g^1\right)A\right)\\ (c-s)^2\varepsilon_1\varepsilon_4C^{'}(\kappa_n)^3\end{matrix}\right]E_x(s)\\ \left[\begin{matrix}(c-s)^2\varepsilon_1\varepsilon_3\varepsilon_4C\kappa_g^2(\kappa_n)^3\\ -2(c-s)\varepsilon_4(\kappa_n)^2\left(\varepsilon_1B-\varepsilon_2\left(\tau_g^1\right)A\right)\\ +2(c-s)^2(\varepsilon_4)(\kappa_n)\\ (\kappa_n)^{'}\left(\varepsilon_1B-\varepsilon_2\left(\tau_g^1\right)A\right)\\ (c-s)^2\varepsilon_4(\kappa_n)^2\\ \left(\begin{matrix}\varepsilon_1B^{'}\\ -\left(\varepsilon_2\left(\tau_g^1\right)^{'}A+\varepsilon_2\left(\tau_g^1\right)A^{'}\right)\end{matrix}\right)\end{matrix}\right]D_x(s)\end{matrix}\right]. \tag{18}$$

and if we write the Equation (18), we have

$$D_{\bar{x}}^{'}(s)=-\left[\left[\frac{1}{WL}\right]^{'}-\frac{1}{WL}\right]\left[MT_x(s)+NE_x(s)+RD_x(s)\right].$$

Finally, taking into consideration of the Equation (6) and mading essential arrangement, we get

$$\tau_g^2=-\left[\frac{1}{WL}\right]^{'}\frac{1}{W}\left[-M(c-s)(\varepsilon_1)^2(\kappa_n)^2-N(c-s)(\varepsilon_2)^2\left(\tau_g^1\right)(\kappa_n)\right]$$

$$\kappa_g^1 = \frac{1}{L}\begin{bmatrix} -(c-s)(\varepsilon_1)^2\varepsilon_4(\kappa_n)^2\left[A+\frac{P}{W}(c-s)\varepsilon_1(\kappa_n)^2\right] \\ -(c-s)(\varepsilon_2)^2\varepsilon_4(\tau_g^1)(\kappa_n)\left[B+\frac{P}{W}(c-s)\varepsilon_2(\tau_g^1)(\kappa_n)\right] \\ \left(+\varepsilon_4\left((c-s)(\kappa_n)^{'}-\varepsilon_4\kappa_n\right)\right)\begin{bmatrix} F-\frac{P}{W}\left((c-s)(\kappa_n)^{'}-\kappa_n\right) \\ -F(c-s)^2(\varepsilon_4)^3(\kappa_n)^2 \end{bmatrix} \end{bmatrix}$$

$$\kappa_g^2 = -\frac{1}{WL}\begin{bmatrix} -U(c-s)^2\varepsilon_2\varepsilon_4 C(\tau_g^1)(\kappa_n)^2 \\ +V(c-s)^2\varepsilon_1\varepsilon_4 C(\kappa_n)^3 \\ +Z\left[(c-s)^2\varepsilon_4(\kappa_n)^2\left(\varepsilon_1 B-\varepsilon_2(\tau_g^1)A\right)\right] \end{bmatrix}$$

This completes the proof. (Remark 3.1 : $\langle E^{'}, D\rangle$ is left to reader.)

4. AW (k)-Type Curves

Let $x : I \to \mathbb{E}_1^4$ be a unit speed curve in $\mathbb{E}_1^4$. The curve $x(s)$ is a Frenet curve of osculating order d when its higher order derivatives $x^{'}(s), x^{''}(s), \ldots, x^{d}(s)$ are linearly independent, and $x^{'}(s), x^{''}(s), \ldots, x^{d+1}(s)$ are no longer independent for all $s \in I$. Each frenet curve of osculating order d is associated with an orthonormal $d-$ frame $v_1, v_2, \ldots, v_d$ along $x(s)$ (such that $x^{'}(s) = T$) known as the Frenet frame as well as the functions $\kappa_1(s), \kappa_2(s), \ldots, \kappa_{d-1}(s) : I \to \mathbb{R}$ known as Frenet curvatures [12].

In this section we consider AW (k)-type curves of order 3 in Minkowski space. Let $x : I \to \mathbb{E}_1^4$ be a curve with an extended Darboux frame $\{T(s), E(s), D(s), N(s)\}$,{as it is given in (6)}.

Proposition 4.1. Let $x : I \to \mathbb{E}_1^4$ be a unit speed curve, therefore, we have

$$\begin{aligned}
x^{'}(s) &= T(s) \\
x^{''}(s) &= \varepsilon_4(\kappa_n)N \\
x^{'''}(s) &= -\varepsilon_1\varepsilon_4(\kappa_n)^2 T - \varepsilon_2\varepsilon_4\kappa_n\tau_g^1 E + \varepsilon_4(\kappa_n)^{'} \\
x^{''''}(s) &= -3\varepsilon_1\varepsilon_4\kappa_n(\kappa_n)^{'}T + \left[-2\varepsilon_2\varepsilon_4\tau_g^1(\kappa_n)^{'} - \varepsilon_2\varepsilon_4\kappa_n(\tau_g^1)^{'}\right]E \\
&\quad -\varepsilon_2\varepsilon_3\varepsilon_4\kappa_n\tau_g^1\kappa_g^2 D + \begin{bmatrix} -\varepsilon_1(\varepsilon_4)^2(\kappa_n)^3 - \varepsilon_2(\varepsilon_4)^2(\kappa_n) \\ +(\varepsilon_4)(\kappa_n)^{''} \end{bmatrix}N
\end{aligned}$$

Notation. Let us write

$$N_1(s) = \varepsilon_4(\kappa_n)N \tag{19}$$

$$N_2(s) = -\varepsilon_2\varepsilon_4\kappa_n\tau_g^1 E + \varepsilon_4(\kappa_n)' \tag{20}$$

$$N_3(s) = -\varepsilon_2\varepsilon_3\varepsilon_4\kappa_n\tau_g^1\kappa_g^2 D + \begin{bmatrix} -\varepsilon_1(\varepsilon_4)^2(\kappa_n)^3 - \varepsilon_2(\varepsilon_4)^2(\kappa_n) \\ +(\varepsilon_4)(\kappa_n)'' \end{bmatrix} N \tag{21}$$

Remark 4.1. $x'(s), x''(s), x'''(s)$ and $x''''(s)$ are linearly dependent if and only if $N_1(s), N_2(s)$ and $N_3(s)$ are linearly dependent.

Definition 4.1. Frenet curves are

(i) of type AW (1) if they satisfy

$$N_3(s) = 0,$$

(ii) of type AW (2) if they satisfy

$$\|N_2(s)\|^2 N_3(s) = \langle N_3(s), N_2(s)\rangle N_2(s). \tag{22}$$

(iii) of type AW (3) if they satisfy

$$\|N_1(s)\|^2 N_3(s) = \langle N_3(s), N_1(s)\rangle N_1(s). \tag{23}$$

Theorem 4.1. Let $x(s)$ be a Frenet curve of order 3 according to extended Darboux frame in $\mathbb{E}_1^4$. Therefore, $x(s)$ is AW (1) -type curve if and only if

$$-\varepsilon_2\varepsilon_3\varepsilon_4\kappa_n\tau_g^1\kappa_g^2 = 0, \tag{24}$$

and

$$\begin{matrix} -\varepsilon_1(\varepsilon_4)^2(\kappa_n)^3 - \varepsilon_2(\varepsilon_4)^2(\kappa_n) \\ +(\varepsilon_4)(\kappa_n)'' \end{matrix} = 0. \tag{25}$$

Proof: Let $x(s)$ be a Frenet curve of type AW (1). From Definition $(4.1.i)$, $N_3(s) = 0$.Then from Equation (21) equality, we get

$$-\varepsilon_2\varepsilon_3\varepsilon_4\kappa_n\tau_g^1\kappa_g^2 D + \begin{bmatrix} -\varepsilon_1(\varepsilon_4)^2(\kappa_n)^3 - \varepsilon_2(\varepsilon_4)^2(\kappa_n) \\ +(\varepsilon_4)(\kappa_n)'' \end{bmatrix} N = 0.$$

Besides, since D and N are linearly independent, one can attain Equation (24) and Equation (25) .This completes the proof of the theorem.

Theorem 4.2 Let $x(s)$ be a Frenet curve of order 3 according to extended Darboux frame in $\mathbb{E}_1^4$. Therefore,

$x(s)$ is AW(2) -type curve if and only if

$$-\varepsilon_2\varepsilon_3(\varepsilon_4)^4\kappa_n\left(\kappa_n^{'}\right)^2\tau_g^1\kappa_g^2=0. \tag{26}$$

Proof: Since $x(s)$ is type of AW (2) , Equation (22) holds on x. Substituting Equation (20) and Equation (21) into Equation (22) , one can attain Equation (26) . Thus, the proof of the theorem completes.

Theorem 4.3 Let $x(s)$ be a Frenet curve of order 3 according to extended Darboux frame in $\mathbb{E}_1^4$. Therefore, $x(s)$ is AW (3) -type curve if and only if

$$-\varepsilon_2\varepsilon_3(\varepsilon_4)^4(\kappa_n)^3\tau_g^1\kappa_g^2=0. \tag{27}$$

Proof: If $x(s)$ is type of AW (3), Equation (23) holds on x. So, substituting Equation (19) and Equation (21) into Equation (23), we get Equation (27) . This proves statement (iii) of the definition (4.1).

5. Conclusion

This study contains **s**pecial curves according to extended Darboux frame field in $\mathbb{E}_1^4$. That is, are investigated Serret-Frenet apparatus of involute of a curve and AW (k) Type curves using extended Darboux frame field in $\mathbb{E}_1^4$. As a result of these, this study fills an important gap in literature and some findings in this work can be considered as an attractive by researchers, who are interested in special curve. Moreover, it can be a light for those who want to work in the future.

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