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A Note on Fano Configurations in the Projective Space PG(5,2)

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Abstract

Let $n \geq 2$ and let $U_j \mid j \in J$, with $|J| = n^2 + n + 1$, be a set of disjoint subspaces (of the same dimension) of some finite projective space $PG(N,q)$ with the property that the number of such subspaces in the span of any two such subspaces is always $n+1$ and the intersection of any two different such spans is always a subspace U_j (so we have a projective plane of order n with point set $U_j \mid j \in J$.) In this work we search for Fano configurations in PG(5,2) whose lines are 3-spaces and points are lines.

Keywords: *Projective spaces; Fano Plane; Fano Axiom*

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1. Introduction

The smallest example of a projective plane is the Fano projective plane over the field $GF(2)$. It is denoted by $PG(2,2)$. It is known that it has seven points and seven lines, and every line has exactly three points. Hence the Fano plane consists of the points $(1,0,0),(0,1,0),(0,0,1),(1,1,0),$ $(1,0,1),(0,1,1)$ and $(1,1,1)$.

In finite geometry, $PG(3,2)$ is the smallest three-dimensional the projective space. It can be thought of as an extension of the Fano plane. It has 15 points, 35 lines, and 15 planes. It also has the following properties:

Each point is contained in 7 lines and 7 planes.

Each line is contained in 3 planes and contains 3 points.

Each plane contains 7 points and 7 lines.

Each plane is isomorphic to the Fano plane.

Every pair of distinct planes intersect in a line.

A line and a plane not containing the line intersect in exactly one point .

Some configurations in the projective spaces are worked in [1-8].

Let $n \geq 2$ and let $U_j \mid j \in J$, with $|J| = n^2 + n + 1$, be a set of disjoint subspaces (of the same dimension) of some finite the projective space $PG(N,q)$ with the property that the number of such subspaces in the span of any two such subspaces is always $n+1$ and the intersection of any two different such span is always a subspace U_j (so we have a projective plane of order n with point set $U_j \mid j \in J$.)

We construct a projective plane $PG(2,2)$ in the projective space $PG(5,2)$ with four skew lines in $PG(5,2)$. We start with three theorems and we give a main result.

Theorem 1.1. *Let $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4$ be four skew lines in the projective space $PG(5,2)$ as in the list:*

$$\begin{aligned}
\mathscr{L}_1 &= \{(1,0,0,0,0,0),(0,1,0,0,0,0),(1,1,0,0,0,0)\} \\
\mathscr{L}_2 &= \{(0,0,1,0,0,0),(0,0,0,1,0,0),(0,0,1,1,0,0)\} \\
\mathscr{L}_3 &= \{(0,0,0,0,1,0),(0,0,0,0,0,1),(0,0,0,0,1,1)\} \\
\mathscr{L}_4 &= \{(1,0,1,0,1,0),(0,1,0,1,1,1),(1,1,1,1,0,1)\}
\end{aligned}$$

It's possible to construct Fano plane in the projective space $PG(5,2)$ such that lines are projective 3-spaces and points are lines of the the projective space $PG(5,2)$.

Proof. Since $\mathscr{L}_1$ and $\mathscr{L}_2$ are disjoint lines in $PG(5,2)$ they span a 3-space in $PG(5,2)$. $\mathscr{L}_1$ and $\mathscr{L}_2$ span the projective 3-space $\mathscr{S}_1 = [a_1,b_1,c_1,d_1,e_1,f_1]$, $a_1 = 0, b_1 = 0, c_1 = 0, d_1 = 0$. We can write $\mathscr{S}_1 = [0,0,0,0,e_1,f_1]$. $\mathscr{L}_3$ and $\mathscr{L}_4$ are disjoint lines in $PG(5,2)$

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they span a 3-space in $PG(5,2)$. $\mathscr{L}_3$ and $\mathscr{L}_4$ span 3-space $\mathscr{S}_2 = [a_2,b_2,c_2,d_2,e_2,f_2]$, $a_2 = c_2, b_2 = d_2, e_2 = 0, f_2 = 0$. We can write $\mathscr{S}_2 = [a_2,b_2,a_2,b_2,0,0]$.

$\mathscr{L}_5 = \mathscr{S}_1 \cap \mathscr{S}_2 = \{(1,1,1,1,0,0),(1,0,1,0,0,0),(0,1,0,1,0,0)\}$

$\mathscr{L}_1$ and $\mathscr{L}_3$ are disjoint lines in $PG(5,2)$ they span a 3-space in $PG(5,2)$. $\mathscr{L}_1$ and $\mathscr{L}_3$ span projective 3-space $\mathscr{S}_3 = [a_3,b_3,c_3,d_3,e_3,f_3]$, $a_3 = 0, b_3 = 0, e_3 = 0, f_3 = 0$. We can write $\mathscr{S}_3 = [0,0,c_3,d_3,0,0]$. $\mathscr{L}_2,\mathscr{L}_4$ are disjoint lines in $PG(5,2)$ they span a 3-space in $PG(5,2)$. $\mathscr{L}_2$ and $\mathscr{L}_4$ span 3-space $\mathscr{S}_4 = [a_4,b_4,c_4,d_4,e_4,f_4]$, $b_4 = 0, c_4 = 0, a_4 = d_4, b_4 = f_4$. We can write $\mathscr{S}_4 = [a_4,b_4,0,0,a_4,b_4]$.

$\mathscr{L}_6 = \mathscr{S}_3 \cap \mathscr{S}_4 = \{(1,1,0,0,1,1),(1,0,0,0,1,0),(0,1,0,0,0,1)\}$

$\mathscr{L}_2$ and $\mathscr{L}_3$ are disjoint lines in $PG(5,2)$ they span a 3-space in $PG(5,2)$. $\mathscr{L}_2$ and $\mathscr{L}_3$ span projective 3-space $\mathscr{S}_5 = [a_5,b_5,c_5,d_5,e_5,f_5]$, $c = 0, d_5 = 0, e_5 = 0, f_5 = 0$. We can write $\mathscr{S}_5 = [a_5,b_5,0,0,0,0]$. $\mathscr{L}_1$ and $\mathscr{L}_4$ are disjoint lines in $PG(5,2)$ they span a 3-space in $PG(5,2)$. $\mathscr{L}_1$ and $\mathscr{L}_4$ span 3-space $\mathscr{S}_6 = [a_6,b_6,c_6,d_6,e_6,f_6]$, $b_6 = d_6, d_6 + e_6 + f_6 = 0$.

$\mathscr{L}_7 = \mathscr{S}_5 \cap \mathscr{S}_6 = \{(0,0,0,1,1,1),(0,0,1,1,0,1),(0,0,1,0,1,0)\}$

$\mathscr{L}_5$ and $\mathscr{L}_6$ are disjoint lines in $PG(5,2)$ they span a 3-space in $PG(5,2)$. $\mathscr{L}_5$ and $\mathscr{L}_6$ span projective 3-space $\mathscr{S}_7 = [a_7,b_7,c_7,d_7,e_7,f_7]$, $a_6 = c_7 = e_7, b_7 = d_7 = f_7$. We can write $\mathscr{S}_7 = [a_7,b_7,a_7,b_7,a_7,b_7]$. If $a_7 = 0, b_7 = 1$, $\mathscr{L}_7$ lies on $\mathscr{S}_7$, we have $\mathscr{S}_7 = [0,1,0,1,0,1]$.If we consider the set $\mathscr{L} = \{\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3..,\mathscr{L}_7\}$ as points and $\mathscr{S} = \{\mathscr{S}_1,\mathscr{S}_2,\mathscr{S}_3...\mathscr{S}_7\}$ as lines, o incidence relation, $U = (\mathscr{L},\mathscr{S},o)$ satisfies the projective plane axioms, hence it is a Fano plane in $PG(5,2)$. Incidence relation is as follows:

$$\{\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_5\}\, o \mathscr{S}_1, \{\mathscr{L}_3,\mathscr{L}_4,\mathscr{L}_5\}\, o \mathscr{S}_2$$
$$\{\mathscr{L}_1,\mathscr{L}_3,\mathscr{L}_6\}\, o \mathscr{S}_3, \{\mathscr{L}_2,\mathscr{L}_4,\mathscr{L}_6\}\, o \mathscr{S}_4$$
$$\{\mathscr{L}_2,\mathscr{L}_3,\mathscr{L}_7\}\, o \mathscr{S}_5, \{\mathscr{L}_1,\mathscr{L}_4,\mathscr{L}_7\}\, o \mathscr{S}_6$$
$$\{\mathscr{L}_5,\mathscr{L}_6,\mathscr{L}_7\}\, o \mathscr{S}_7$$

$U = (\mathscr{L},\mathscr{S},o)$ is isomorphic to the projective plane $PG(2,2)$. □

Theorem 1.2. *Let $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3,\mathscr{L}_4$ be skew lines in the projective space $PG(5,2)$ as in the list:*

$$\begin{aligned}
\mathscr{L}_1 &= \{(1,0,0,0,0,0),(0,1,0,0,0,0),(1,1,0,0,0,0)\}\\
\mathscr{L}_2 &= \{(0,0,1,0,0,0),(0,0,0,1,0,0),(0,0,1,1,0,0)\}\\
\mathscr{L}_3 &= \{(0,0,0,0,1,0),(0,0,0,0,0,1),(0,0,0,0,1,1)\}\\
\mathscr{L}_4 &= \{(0,1,1,0,1,1),(1,0,1,1,0,1),(1,1,0,1,1,0)\}
\end{aligned}$$

It's possible to construct Fano plane in the projective space $PG(5,2)$ such that lines are projective 3-spaces and points are lines of the the projective space $PG(5,2)$.

Proof. Similar arguments work in Theorem 1.1. □

Theorem 1.3. *Let $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3,\mathscr{L}_4$ be skew lines in the projective space $PG(5,2)$ as in the list:*

$$\begin{aligned}
\mathscr{L}_1 &= \{(0,0,1,0,1,0),(0,1,0,1,0,0),(0,1,1,1,1,0)\}\\
\mathscr{L}_2 &= \{(0,0,0,0,1,0),(0,1,0,1,1,1),(0,1,0,1,0,1)\}\\
\mathscr{L}_3 &= \{(1,0,0,1,1,0),(1,0,0,1,1,1),(0,0,0,0,0,1)\}\\
\mathscr{L}_4 &= \{(0,0,0,1,1,0),(0,0,0,1,0,1),(0,0,0,0,1,1)\}
\end{aligned}$$

It's not possible to construct Fano plane in the projective space $PG(5,2)$ such that lines are projective 3-spaces and points are lines of the projective space $PG(5,2)$.

Proof. Since $\mathscr{L}_1$ and $\mathscr{L}_2$ are disjoint lines in $PG(5,2)$ they span a 3-space in $PG(5,2)$. $\mathscr{L}_1$ and $\mathscr{L}_2$ spans 3-space $\mathscr{S}_1 = [a_1,b_1,c_1,d_1,e_1,f_1]$, $c_1 = 0, e_1 = 0, f_1 = 0, b_1 = d_1$. We can write 3-space $\mathscr{S}_1 = [a_1,b_1,0,b_1,0,0]$. $\mathscr{L}_3$ and $\mathscr{L}_4$ are disjoint lines in $PG(5,2)$ they span a 3-space in $PG(5,2)$. $\mathscr{L}_3$ and $\mathscr{L}_4$ spans 3-space $\mathscr{S}_2 = [a_2,b_2,c_2,d_2,e_2,f_2]$, $a_2 = 0, d_2 = 0, e_2 = 0, f_2 = 0$. We can write 3-space $\mathscr{S}_2 = [0,b_2,c_2,0,0,0]$.

$\mathscr{L}_5 = \mathscr{S}_1 \cap \mathscr{S}_2 = \{(0,0,0,0,0,1),(0,0,0,0,1,0),(0,0,0,0,1,1)\}$

But $\mathscr{L}_5$ and $\mathscr{L}_4$ are not skew lines. They have the intersection point $(0,0,0,0,1,1)$,hence $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3,\mathscr{L}_4$ in $PG(5,2)$ don't generate a projective plane in $PG(5,2)$. □

2. Conclusion

Theorem 1.3 shows that it's not always possible to construct Fano configuration in the projective space $PG(5,2)$ with four skew lines. To construct Fano configuration in the projective space $PG(5,2)$ we must have extra condition; Let $PG(5,2)$ be a 5-dimensional the projective space over $GF(2)$. Let $\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3,\mathscr{L}_4$ be the set of four skew lines in $PG(5,2)$. P_{1i}, $i = 1,2,3$ be the points of $\mathscr{L}_1$. Similarly P_{2j}, $j = 1,2,3$ be the points of $\mathscr{L}_2$, P_{3k}, $k = 1,2,3$ be the points of $\mathscr{L}_3$ and P_{4m}, $m = 1,2,3$ be the points of $\mathscr{L}_4$. Additionally, any four points $P_{1i},P_{2j},P_{3k},P_{4m}$ form a quadrilateral, i.e. any three of them are non-collinear. Let S_1 be 3-space of $PG(5,2)$ spanned by $\mathscr{L}_1$and $\mathscr{L}_2$. Let S_2 be 3-space of $PG(5,2)$ spanned by $\mathscr{L}_3$ and $\mathscr{L}_4$. $\mathscr{L}_5 = S_1 \cap S_2$. Let S_3 be 3-space of $PG(5,2)$ spanned by $\mathscr{L}_1$and $\mathscr{L}_3$. Let S_4 be 3-space of $PG(5,2)$ spanned by $\mathscr{L}_2$ and $\mathscr{L}_4$. $\mathscr{L}_6 = S_3 \cap S_4$. Let S_5 be 3-space of $PG(5,q)$ spanned by $\mathscr{L}_2$ and $\mathscr{L}_3$. S_6 be 3-space of $PG(5,q)$ spanned by $\mathscr{L}_1$ and $\mathscr{L}_4$. $\mathscr{L}_7 = S_5 \cap S_6$. Let S_7 be 3-space of $PG(5,2)$ spanned by $\mathscr{L}_5$ and $\mathscr{L}_6$. Now we can define the geometry U in $PG(5,2)$ as follows; the points of U are the elements of $\mathscr{L} = \{\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3,..,\mathscr{L}_7\}$ and the elements of $\mathscr{S} = \{\mathscr{S}_1,\mathscr{S}_2,\mathscr{S}_3...\mathscr{S}_7\}$ its lines such that the 3- dimensional subspaces spanned by any two distinct elements of $\mathscr{L}$. $U = (\mathscr{L},\mathscr{S},o)$ satisfies the projective plane axioms, hence it is a projective plane in $PG(5,2)$.

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