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Spectral Theorem for Compact Self-adjoint Operator in Γ-Hilbert space

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Abstract

In this article, we investigate some basic results of self-adjoint operator in Γ-Hilbert space. We proof some similar results on self-adjoint operator in this space with some specific norm. Finally we will prove that the spectral theorem for compact self-adjoint operator in Γ-Hilbert space and the converse is also true.

Keywords: compact operator, self-adjoint operator, Spectral theorem, Γ-Hilbert space.
2010 MSC: 46C50 , 47B07.

1. Introduction

The theory of Hilbert spaces was first introduced and studied by David Hilbert (1862-1943). Hilbert space operators represent the physical quantities in real-life applications, that's why physicist and mathematicians are interested to classify the operators in Hilbert space. The concepts of spectral theory play a central role in Hilbert spaces, trying to classify linear operators.

The concepts of Γ-Hilbert space was first commenced by D.K Bhattacharya and T.E. Aman in their research paper [1]. This definition gives the generalization of Hilbert spaces in real or complex field. Further development of this literature was found in 2017 by A. Ghosh, A. Das, and T.E. Aman in their paper [2]. They defined γ-orthogonality, proved the Unique decomposition theorem and Representation theorem in Γ-Hilbert space. In [3] A. Das and S. Islam introduce self-adjoint operator and characterize these operator in Γ-Hilbert space.

Spectral theorem for self-adjoint operator is a generalization of the conventional theorem from Linear algebra, which states that "every Hermitian matrix is unitarily similar to a real diagonal matrix". This is

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our motivation to study Spectral theory for a self-adjoint operator in Γ -Hilbert space. Before studying the spectral theory for compact self-adjoint operator, we have to prove some basic results in this space. Also, we present the spectral theorem in functional calculus form. Here we consider both finite-dimensional and infinite-dimensional complex Hilbert spaces. We have made some small changes in the definition of Γ-Hilbert space after consulting with the main authors [1].

2. Preliminaries

In this paper, we shall denote the set of all bounded linear operator on a Γ-Hilbert space H_Γ is $\mathcal{B}(H_\Gamma)$. An operator is called finite rank (dimensional) if its range is finite dimension. A complex number η is an eigenvalue of T if $Tx = \eta x$ holds for any non zero vector $x \in H_\Gamma$. The set of all eigen values of T denoted by $\sigma_p(T)$.

At first, we recall some important definitions :

Definition 2.1. *[1] Let H be a linear space over the field $\mathbb{C}$ and Γ be a semi group with respect to addition. A mapping $\langle .,.,.\rangle$: $H \times \Gamma \times H \longrightarrow \mathbb{C}$ is called a Γ-inner product on H if the following conditions are satisfied:*
(i) $\langle x, \gamma, y\rangle$ is linear in each variable.
(ii) $\langle x, \gamma, y\rangle = \overline{\langle y, \gamma, x\rangle}$ for all $x, y \in H$ and for all $\gamma \in \Gamma$.
(iii) $\langle x, \gamma, x\rangle > 0$ for all $x \neq \theta$ and for all $\gamma \in \Gamma$.
(iv) $\langle x, \gamma, x\rangle = 0$ if $x = \theta$.

Definition 2.2. $(H, \Gamma, \langle .,.,.\rangle)$ *is called a Γ-inner product space over $\mathbb{C}$. A complete Γ-inner product space is called Γ-Hilbert space and is denoted by H_Γ.*

Now consider as

$$\|x\|_\gamma = \langle x, \gamma, x\rangle^{\frac{1}{2}}$$

then it is easy to prove that $\|x\|_\gamma$ satisfy all the conditions of norm. The author in [2] has all ready proved the following results.

Definition 2.3. *[2] Let $x, y \in H_\Gamma$, then x, y are γ-orthogonal if and only if $\langle x, \gamma, y\rangle = 0$. In symbol we write $x \perp_\gamma y$.*

Let M is a subset of H_Γ then define γ-orthogonal compliment of M as
$M^{\perp_\gamma} = \{x \in H_\Gamma : x \perp_\gamma y \ \ for \ \ all \ \ y \in H_\Gamma\}$.

Definition 2.4. *A sequence (x_n) in H_Γ is said to be γ-orthonormal if for any $\gamma \in \Gamma$ it follows two conditions*

1. *$\langle x_n, \gamma, x_m\rangle = 1$ when $n = m$;*
2. *$\langle x_n, \gamma, x_m\rangle = 0$ when $n \neq m$.*

Theorem 2.5. *[2] For all $x, y \in H_\Gamma$ and any $\gamma \in \Gamma$*

1. *Cauchy-Schwartz's inequality :* $|\langle x, \gamma, y\rangle| \leq ||x||_\gamma ||y||_\gamma$.
2. *Parallelogram Law :* $||x + y||_\gamma^2 + ||x - y||_\gamma^2 = 2||x||_\gamma^2 + 2||y||_\gamma^2$.
3. *Pythagorean Theorem : $x \perp_\gamma y$ if and only if*

$$||x + y||_\gamma^2 = ||x||_\gamma^2 + ||y||_\gamma^2.$$

Theorem 2.6. *[2] (Projection Theorem). Let M be a closed subspace of H_Γ. Then $M^{\perp_\gamma}$ is a closed subspace and $H_\Gamma = M \oplus M^{\perp_\gamma}$.*

2.1. A bounded linear Operator on Γ-Hilbert space

Definition 2.7. *A linear operator T on a Γ-Hilbert space H_Γ is said to be bounded if there exists $C > 0$ such that $||Tx||_\gamma \leq C||x||_\gamma$ for all $x \in H_\Gamma$ and for any $\gamma \in \Gamma$, $||T||_\gamma$ is defined by*

$$||T||_\gamma = inf\{C > 0 : ||Tx||_\gamma \leq C||x||_\gamma \quad for\ all \quad x \in H_\Gamma\}$$

then $||T||_\gamma$ is said to be the operator γ norm of T.

Definition 2.8. *A linear operator $T : H_\Gamma \longrightarrow H_\Gamma$ is compact if for every bounded sequence (x_n) in H_Γ, there exists a convergent sub sequence of (Tx_n) in H_Γ.*

Lemma 2.9. *If T is compact operator and M is a closed subspace of H_Γ, then the restriction operator $T|_M$ is also compact.*

Theorem 2.10. *[5] Every finite-dimensional bounded operator is compact.*

2.2. Self-adjoint Operator on Γ-Hilbert space

Definition 2.11. *[3] A bounded linear operator $T : H_\Gamma \longrightarrow H_\Gamma$ on Γ-Hilbert space is called self-adjoint if and only if for all $x, y \in H_\Gamma$, we have $\langle Tx, \gamma, y\rangle = \langle x, \gamma, Ty\rangle$.*
i.e. T is self-adjoint if and only if $T = T^*$.

We start with some basic properties of self-adjoint operator on Γ-Hilbert space.

Theorem 2.12. *[3] Let T be a self-adjoint operator on H_Γ and for any γ, then*

$$\left\|T\right\|_\gamma = sup\{|\langle Tx, \gamma, x\rangle| : ||x||_\gamma = 1\}.$$

3. Main Results

Proposition 3.1. *Let A, B be an operator on a Γ- Hilbert space H_Γ. Then A^* and B^* is also an operator on Γ- Hilbert space H_Γ and the following properties hold:*

1. $(A + B)^* = A^* + B^*$
2. $(AB)^* = B^*A^*$
3. $(\lambda A)^* = \bar{\lambda}A^*$
4. $(B^*)^* = B$
5. $\left\|B^*\right\|_\gamma = \left\|B\right\|_\gamma$

Proof. all the proofs are straight forward. □

Theorem 3.2. *Let $T \in \mathcal{B}(H_\Gamma)$, then $||T||_\gamma = sup\ \{||Tx||_\gamma : ||x||_\gamma = 1\}$ for any $\gamma \in \Gamma$.*

Proof. Let us consider $M = sup\ \{||Tx||_\gamma : ||x||_\gamma = 1\}$. Since T is bounded so $||Tx||_\gamma \leq ||T||_\gamma ||x||_\gamma$. Now $||x||_\gamma = 1$ implies that $||Tx||_\gamma \leq ||T||_\gamma$. By previous definition, we have $M \leq ||T||_\gamma$. On the other hand for any vector $x \in H_\Gamma$, we have

$$\left\|Tx\right\|_\gamma = \left\|T(||x||\frac{x}{||x||})\right\|_\gamma = \left\|T(\frac{x}{||x||})\right\|_\gamma ||x||_\gamma \leq M||x||_\gamma.$$

So $\left\|T\right\|_\gamma \leq M$. Therefore $\left\|T\right\|_\gamma = M$.

□

Theorem 3.3. *Let T be a Self-adjoint Operator on H_Γ, then $\sigma_p(T) \subseteq \mathbb{R}$ for any $\gamma \in \Gamma$.*

Proof. Let $\xi \in \sigma_p(T)$ and $Tx = \xi x \quad (x \neq \theta)$. Then

$$\xi ||x||_\gamma^2 = \langle Tx, \gamma, x\rangle = \langle x, \gamma, Tx\rangle = \bar{\xi}||x||_\gamma^2.$$

Which gives us $\xi = \bar{\xi}$ and the proof is complete.

□

Theorem 3.4. *Suppose x and y are eigenvectors of a Self-adjoint operator T on H_Γ corresponding to distinct eigenvalues, then $x \perp_\gamma y$.*

Proof. Let $\eta, \mu \in \sigma_p(T)$ such that $Tx = \eta x (x \neq \theta)$ and $Ty = \mu y (y \neq \theta)$. Then

$$\eta \langle x, \gamma, y\rangle = \langle Tx, \gamma, y\rangle = \langle x, \gamma, Ty\rangle = \mu \langle x, \gamma, y\rangle,$$

which implies $(\eta - \mu)\langle x, \gamma, y\rangle = 0$. Since η and μ are distinct real numbers, then $\langle x, \gamma, y\rangle = 0$. Hence $x \perp_\gamma y$.
□

Theorem 3.5. *Let H_Γ be a complex Hilbert space and T be a bounded operator on H_Γ, then T is self-adjoint if and only if $\langle Tx, \gamma, x\rangle \in \mathbb{R}$ for all $x \in H_\Gamma$ for all $\gamma \in \Gamma$.*

Proof. Suppose T is self-adjoint then we have,

$$\langle Tx, \gamma, x\rangle = \langle x, \gamma, Tx\rangle = \overline{\langle Tx, \gamma, x\rangle}.$$

Hence $\langle Tx, \gamma, x\rangle \in \mathbb{R}$.
Conversely, let $\langle Tx, \gamma, x\rangle \in \mathbb{R}$ for all $x \in H_\Gamma$ and for all $\gamma \in \Gamma$.
If $c \in \mathbb{C}$ and $x, y \in H_\Gamma$ then

$$\langle T(x + cy), \gamma, (x + cy)\rangle \in \mathbb{R},$$

implies that

$$\langle T(x + cy), \gamma, x\rangle = \overline{\langle T(x + cy), \gamma, x\rangle},$$

that is

$$\langle T(x + cy), \gamma, x\rangle = \langle x, \gamma, T(x + cy)\rangle.$$

simplify the above equation gives us

$$c\langle Ty, \gamma, x\rangle + \bar{c}\langle Tx, \gamma, y\rangle = \bar{c}\langle x, \gamma, Ty\rangle + c\langle y, \gamma, Tx\rangle,$$

that is

$$c\langle Ty, \gamma, x\rangle + \bar{c}\langle Tx, \gamma, y\rangle = \bar{c}\langle T^*x, \gamma, y\rangle + c\langle T^*y, \gamma, x\rangle.$$

By first taking $c = 1$ and then $c = i$ we obtain as two equations

$$\langle Ty, \gamma, x\rangle + \langle Tx, \gamma, y\rangle = \langle T^*x, \gamma, y\rangle + \langle T^*y, \gamma, x\rangle$$

and

$$i\,\langle Ty, \gamma, x\rangle - i\,\langle Tx, \gamma, y\rangle = -i\,\langle T^*x, \gamma, y\rangle + i\,\langle T^*y, \gamma, x\rangle\,.$$

By cancel i from last equation and add those two equation implies

$$\langle Ty, \gamma, x\rangle = \langle T^*y, \gamma, x\rangle\,.$$

Hence

$$\langle Ty, \gamma, x\rangle = \langle x, \gamma, Ty\rangle$$

for all $x, y \in H_\Gamma$ for all $\gamma \in \Gamma$. Thus $T = T^*$.

□

Theorem 3.6. *Let T be a bounded Self-adjoint operator on H_Γ, then either $\|T\|_\gamma$ or $-\|T\|_\gamma$ is an eigenvalue of T for any $\gamma \in \Gamma$.*

Proof. By theorem (2.12) there exist a sequence (y_n) in H_Γ with $||y_n||_\gamma = 1$
for all n such that $\langle Ty_n, \gamma, y_n\rangle \longrightarrow \xi$, where $\xi = \|T\|_\gamma$ or $\xi = -\|T\|_\gamma$.
Now

$$\begin{aligned} \|Ty_n - \xi y_n\|_\gamma^2 &= \|Ty_n\|_\gamma^2 - 2\xi\,\langle Ty_n, \gamma, y_n\rangle + \xi^2 \\ &\leq 2\xi^2 - 2\xi\,\langle Ty_n, \gamma, y_n\rangle \quad \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Since T is compact, there exists a sub sequence (Ty_{n_j}) of (Ty_n) such that $Ty_{n_j} \to y$. So $(Ty_n - \xi y_n) \longrightarrow 0$ as $n \to \infty$. That is $y_n \longrightarrow \frac{1}{\xi}Ty$ where y = lim $Ty_{n_j} = \frac{1}{\xi}Ty$. Hence ξ is an eigen value of T.

□

Corollary 3.7. *If T be a compact self-adjoint operator on H_Γ, then*

$$\|T\|_\gamma = max\{|\,\langle Tx, \gamma, x\rangle\,| : ||x||_\gamma = 1\}.$$

Theorem 3.8. *(The Spectral theorem) Let T be a bounded compact self-adjoint operator on H_Γ. Then for any $\gamma \in \Gamma$*

1. *there exists a system of orthonormal eigenvectors ψ_1, ψ_2, ψ_3,... of T and corresponding eigenvalues ξ_1, ξ_2, ξ_3,... such that $|\xi_1| \geq |\xi_2| \geq |\xi_3| \geq$.....*
2. *If ξ_n is infinite, then $\xi_n \longrightarrow 0$ as $n \to \infty$.*
3. *If $Tx = \sum_{k=1}^{\infty} \xi_k\,\langle x, \gamma, \psi_k\rangle\,\psi_k$ for all $x \in H_\Gamma$, then the series on the right hand side converges in the operator norm of $\mathcal{B}(H_\Gamma)$.*

Proof. 1. Let $H_1 = H$ and $T_1 = T$. Theorem (3.6) guarantees that, there exists an eigenvalue ξ_1 of T_1 and an eigenvector ψ_1 such that $\|\psi_1\|_\gamma = 1$ and $|\xi_1| = \|T_1\|_\gamma$. Since span $\{\psi_1\}$ is a closed subspace of H_1, then theorem (2.6) gives us $H_1 = span\{\psi_1\} \oplus span\{\psi_1\}^{\perp_\gamma}$.

Again let $H_1 = span\{\psi_1\}^{\perp_\gamma}$. Clearly H_2 is a closed subspace of H_1 and $T(H_2) \subseteq H_2$. Now let $T_2 = T_1|_{H_2}$.
Then T_2 is self-adjoint operator and also compact in $\mathcal{B}(H_2)$. If $T_2 = 0$, then it is trivially true. Assume that $T_2 \neq 0$. Then by Theorem (3.6), there exists an eigenvalue ξ_2 of T_2 with $|\xi_2| = \|T_2\|_\gamma$ and a corresponding eigenvector ψ_2 such that $\|\psi_2\|_\gamma = 1$.
Since T_2 is a restriction of T_1, $|\xi_2| = \|T_2\|_\gamma \leq \|T_1\|_\gamma = |\xi_1|$.
By the construction, ξ_1 and ξ_2 are orthonormal. Now let $H_3 = span\{\psi_1, \psi_2\}^{\perp_\gamma}$. Clearly $H_3 \subseteq H_2$. It is easy to show that $T(H_3) \subseteq H_3$. The operator $T_3 = T|_{H_3}$ is compact and self-adjoint. Hence by theorem (3.6) there exits an eigen value ξ_3 of T_3 and a corresponding eigen vector ψ_3 with $\|\psi_3\|_\gamma = 1$. Here $|\xi_3| = \|T_3\|_\gamma$ and hence $|\xi_3| \leq |\xi_2| \leq |\xi_1|$.
Repeating the above process in the same manner, in some stage either n, $T_n = 0$ or there exists a sequence ξ_n of eigenvalues of T and corresponding vector ψ_n with $\|\psi_n\|_\gamma = 1$ and $|\xi_n| = \|T_n\|_\gamma$. Also $|\xi_n| \geq |\xi_{n+1}|$ for each n.

2. If possible let ξ_n does not converges to 0, there exists $\epsilon > 0$ such that $|\xi_n| \geq \epsilon$ for infinitely many n. If $m \neq n$ then
$$\|T\psi_n - T\psi_m\|_\gamma^2 = \|\xi\psi_n - \xi\psi_m\|_\gamma^2 = \xi_n^2 + \xi_m^2 > \epsilon^2.$$
This implies that $T\psi_n$ has no convergent sub sequence, which contradict the fact that T is compact. Hence $\xi_n \longrightarrow 0$ as $n \to \infty$.

3. Here we consider two cases.
Case 1 : If $T_n = 0$ for some n.
$$Let \quad x_n = x - \sum_{k=1}^{n} \langle x, \gamma, \psi_k \rangle \psi_k.$$
Then x_n and ψ_i are γ-orthogonal for $1 \leq i \leq n$. Hence
$$0 = T_n x_n = Tx - \sum_{k=1}^{n} \xi_k \langle x, \gamma, \psi_k \rangle \psi_k.$$
That is
$$Tx = \sum_{k=1}^{n} \xi_k \langle x, \gamma, \psi_k \rangle \psi_k.$$
Case 2: If $T_n \neq 0$ for infinitely many n. For $x \in H_\gamma$ by case 1,
we have
$$\left\| Tx - \sum_{k=1}^{n} \xi_k \langle x, \gamma, \psi_k \rangle \psi_k \right\|_\gamma = \|T_n x_n\|_\gamma \leq \|T_n\|_\gamma \|x_n\|_\gamma$$
$$= |\xi_n| \|x_n\|_\gamma \leq |\xi_n| \|x\|_\gamma$$
$$\to 0 \text{ as } n \to \infty.$$
Finally we get the result
$$Tx = \sum_{k=1}^{\infty} \xi_k \langle x, \gamma, \psi_k \rangle \psi_k.$$
□

It is natural to ask that the converse of the above theorem is true or not, the answer is given in the next theorem. Before that we give an example.

Example 3.9. *Let us consider $H_\Gamma = \ell^2$ and $\Gamma = (0, \infty)$ with Γ inner product $\langle x, \gamma, y\rangle = \sum_{k=0}^{\infty} x_k \gamma y_k$, where $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ are elements of ℓ^2. Define $T : \ell^2 \to \ell^2$ by*

$$T(x_1, x_2, ..) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...).$$

Then T is compact self-adjoint. Also $Te_n = \frac{1}{n}e_n$, thus $\frac{1}{n}$ is an eigenvalue with an eigen vector e_n. Since $\frac{1}{n} \longrightarrow 0$ as $n \to \infty$. Then by spectral theorem (3.8) T can be represented as for all $x \in \ell^2$.

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{n} \langle x, \gamma, e_n\rangle e_n.$$

Theorem 3.10. *Suppose that ψ_1, ψ_2, ψ_3,... sequence of orthonormal vectors in H_Γ and ξ_1, ξ_2, ξ_3,... be a sequence of real number such that $\{\xi_n\}$ is finite or converges to 0. Then for any $\gamma \in \Gamma$ and $x \in H_\Gamma$ the operator defined by*

$$Tx = \sum_{k=1}^{\infty} \xi_k \langle x, \gamma, \psi_k\rangle \psi_k$$

is compact and self-adjoint.

Proof. We prove it in two cases.
Case 1: $\{\xi_n\}$ is finite Consider

$$Tx = \sum_{k=1}^{\infty} \xi_k \langle x, \gamma, \psi_k\rangle \psi_k.$$

Then

$$\|Tx\|_\gamma^2 = \langle Tx, \gamma, Tx\rangle = \sum_{k=1}^{\infty} |\xi_k|^2 | \langle x, \gamma, \psi_k\rangle |^2 \leq \max_k |\xi_n| ||x||_\gamma^2.$$

Therefore, T is bounded and T is a finite rank operator. Hence T is compact.

Case 2: $\{\xi_n\}$ is infinite and $\xi_n \longrightarrow 0$ as $n \to \infty$. Then by the spectral theorem (3.8), we have

$$\|Tx\|_\gamma^2 = \sum_{k=1}^{\infty} |\xi_k|^2 | \langle x, \gamma, \psi_k\rangle |^2 \leq \max_{k \geq n} |\xi_n| ||x||_\gamma^2.$$

Thus T is bounded operator. Now define $T_n x = \sum_{k=1}^{n} \xi_k \langle x, \gamma, \psi_k\rangle \psi_k$.
Then

$$\|T - T_n\|_\gamma^2 = \sup_{||x||=1} \{\Big\| \sum_{k=1}^{\infty} |\xi_k| \langle x, \gamma, \psi_k\rangle \psi_k \Big\|_\gamma^2$$

$$\leq \sup_{k>n} |\xi_n|^2 \quad \longrightarrow 0 \text{ as } n \to \infty.$$

Since each T_n is finite rank and hence compact, so T is compact. Now

$$\begin{aligned}\langle Tx, \gamma, y\rangle &= \left\langle \ \textstyle\sum_{k=1}^{\infty} \xi_k \langle x, \gamma, \psi_k\rangle \psi_k, \gamma, y\right\rangle \\ &= \textstyle\sum_{k=1}^{\infty}\ \xi_k\ \langle x, \gamma, \psi_k\rangle\ \langle \psi_k, \gamma, y\rangle \\ &= \textstyle\sum_{k=1}^{\infty}\ \xi_k\ \langle \psi_k, \gamma, y\rangle\ \langle x, \gamma, \psi_k\rangle \\ &= \textstyle\sum_{k=1}^{\infty}\ \overline{\xi_k}\ \overline{\langle y, \gamma, \psi_k\rangle}\ \langle x, \gamma, \psi_k\rangle \ (\text{ as } \xi_k \text{ real number for all k}) \\ &= \left\langle x, \gamma,\ \textstyle\sum_{k=1}^{\infty} \xi_k \langle y, \gamma, \psi_k\rangle \psi_k\right\rangle = \langle x, \gamma, Ty\rangle.\end{aligned}$$

Hence T is self-adjoint.

□

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