Advances in the Theory of Nonlinear Analysis and its Applications **5** (2021) No. 3, 368–381.
https://doi.org/10.31197/atnaa.878951
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Fixed point results for a class of nonexpansive type mappings in Banach spaces

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Abstract

In this paper, we present some new fixed point results for a well-known class of generalized nonexpansive type mappings and associated Krasnosel'skiĭ type mappings in Banach spaces. Further, we consider Mann type iteration procedure for finding a common fixed point of a nonexpansive type semigroup. We also present a couple of nontrivial examples to illustrate facts and show numerical convergence.

Keywords: Nonexpnasive mapping; condition (E); nonexpansive semigroup; Banach space.
2010 MSC: 47H10; 47H09.

1. Introduction and Preliminaries

The classical Banach contraction principle is an important result in metric fixed point theory because of its simplicity and applicability to various domains [11, 24]. Nonexpansive mappings are natural generalization of contraction mappings. These mappings are important due to their connection with the monotonicity methods and also appear in applications for initial value, variational inequality, optimization, equilibrium and many other problems in nonlinear analysis [24]. It is well-known that a nonexpansive self-mapping of a complete metric space need not have a fixed point. Also, even though a nonexpansive mapping has a fixed point, it is possible that the sequence of iterates (the Picard sequence) may not converge to a fixed point of the mapping, unlike the contraction mappings. Therefore the study of existence and convergence of fixed points of nonexpansive mappings is an important subject. In 1965, using the geometric properties of Banach spaces, first existence results for nonexpansive mappings were obtained by Browder [2], Göhde [9] and

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Received December 12, 2020; Accepted: May 24, 2021; Online: May 25, 2021.

Kirk [10], independently. A number of extensions and generalizations of their theorems and nonexpansive mappings, appeared in [1, 6, 29, 28, 27, 7, 8, 12, 14, 13, 16, 19, 20, 22, 23] and elsewhere.

In 2008, Suzuki [23] introduced a new class of nonexpansive type mappings known as mappings satisfying condition (C) and obtained some important fixed point results for these mappings. He showed that this class of mappings need not be continuous on their domains, unlike the nonexpansive mappings. In 2011, Garcïa-Falset *et al.* [6] considered a generalization of Suzuki's type nonexpansive mappings. These mappings are known as mappings satisfying condition (E). We also studied some fixed points results for mappings satisfying condition (E) in [19, 21]. In this paper, we continue our study and present certain new fixed point results for mappings satisfying condition (E) and associated Krasnosel'skiĭ type mappings. One can find some convergence results of Krasnosel'skiĭ- Mann iteration procedure of nonexpansive mappings in [25, 26] and references therein. We also consider Mann type iteration procedure for finding common fixed points of nonexpansive type semigroups and obtain a strong convergence theorem. To illustrate our results, we present a couple of nontrivial examples. Finally, we present numerical convergence analysis for different choices of coefficients and initial guesses.

Now, we recall some useful notations, definitions and results from the literature.

Definition 1.1. *[11]. A Banach space X is said to be uniformly convex if for each $\varepsilon \in (0,2]$ $\exists$ $\delta > 0$ such that $\left\|\frac{u+v}{2}\right\| \leq 1-\delta$ for all $u, v \in X$ with $\|u\| = \|v\| = 1$ and $\|u-v\| > \varepsilon$. The Banach space X is strictly convex if*

$$\left\|\frac{u+v}{2}\right\| < 1,$$

whenever $u, v \in X$ with $\|u\| = \|v\| = 1$, $u \neq v$.

Lemma 1.2. ([24]p.484). *Let X be a uniformly convex Banach space, and two sequences (u_n) and (v_n) in X such that*

$$\lim_{n\to\infty} \|u_n\| \leq d, \quad \lim_{n\to\infty} \sup \|v_n\| \leq d, \quad and \quad \lim_{n\to\infty} \|\alpha_n u_n + (1-\alpha_n)v_n\| = d,$$

where $0 < \eta_1, \eta_2 < 1$, $\{\alpha_n\} \subset [\eta_1, \eta_2]$ and $d \geq 0$. Then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

Definition 1.3. *[18]. A Banach space X satisfies Opial property if, for every weakly convergent sequence (u_n) with weak limit $u \in X$ it holds:*

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

for all $v \in X$ with $u \neq v$.

All finite dimensional Banach spaces, all Hilbert spaces and ℓ^p $(1 \leq p < \infty)$ satisfy the Opial property. A Banach space having a weakly sequentially continuous duality mapping also satisfies the Opial condition. But L_p $(0 < p < \infty,\ p \neq 2)$ do not have the Opial property [5] .

Definition 1.4. *[11]. Let $(X, \|\cdot\|)$ be a Banach space and K a nonempty subset of X. A mapping $T : K \to K$ is said to be nonexpansive if for all $u, v \in K$,*

$$\|T(u) - T(v)\| \leq \|u - v\|.$$

A point $w \in K$ is said to be a fixed point of T if $T(w) = w$. We denote the set of all fixed points of T by $F(T)$.

Definition 1.5. *[6]. The mapping $T : K \to K$ is said to satisfy condition (E_μ) on K if there exists $\mu \geq 1$ such that for all $u, v \in K$,*

$$\|u - T(v)\| \leq \mu\|u - T(u)\| + \|u - v\|.$$

We say that T satisfies condition (E) on K whenever T satisfies (E_μ) for some $\mu \geq 1$.

Definition 1.6. *[11]. The mapping $T : K \to K$ is said to be a quasi-nonexpansive if*

$$\|T(u) - w\| \leq \|u - w\|$$

for all $u \in K$ and $w \in \mathrm{F}(T)$.

It is well-known that a nonexpansive mapping with a fixed point is quasi-nonexpansive. However the converse need not be true.

Definition 1.7. *[3]. The mapping $T : K \to K$ is called asymptotically regular if for all $u \in K$*

$$\lim_{n\to\infty} \|T^n(u) - T^{n+1}(u)\| = 0.$$

Proposition 1.8. [6]. *Let K be a nonempty subset of a Banach space X. If $T : K \to K$ is a mapping satisfying condition (E) with $F(T) \neq \emptyset$ then T is quasi-nonexpansive.*

Definition 1.9. *[15]. Let K be a nonempty convex subset of a Banach space X and $T : K \to K$ a mapping. A mapping $T_\alpha : K \to K$ is said to be an α-Krasnosel'skiĭ mapping associated with T if there exists $\alpha \in (0, 1)$ such that*

$$T_\alpha(u) = (1 - \alpha)u + \alpha T(u)$$

for all $u \in K$.

Lemma 1.10. *Let K be a nonempty convex subset of a Banach space X and $T, T_\alpha : K \to K$ are mappings with $\alpha \in (0, 1)$. Then $F(T) = F(T_\alpha)$.*

Proof. From the definition of mapping T_α, it evident that a fixed point of T is also a fixed point of T_α, So, $F(T) \subseteq F(T_\alpha)$. Conversely, let $w \in F(T_\alpha)$. Then $T_\alpha(w) = w$. Now

$$w = T_\alpha(w) = (1 - \alpha)w + \alpha T(w)$$
$$w = w - \alpha w + \alpha T(w),$$

which implies $T(w) = w$. Hence $F(T_\alpha) \subseteq F(T)$. This completes the proof. □

Lemma 1.11. (Demiclosedness principle [4]). *Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $T : K \to X$ be a mapping with $F(T) \neq \emptyset$. Suppose (u_n) is a sequence in X such that (u_n) converges weakly to u and $\lim\limits_{n\to\infty} \|u_n - T(u_n)\| = 0$. Then $T(u) = u$. That is, $I - T$ is demiclosed at zero.*

Theorem 1.12. [21]. *Let K be a nonempty convex subset of a uniformly convex Banach space X. If $T : K \to K$ is a mapping satisfying condition (E) with $F(T) \neq \emptyset$, then the α-Krasnosel'skiĭ mapping T_α for $\alpha \in (0, 1)$ is asymptotically regular.*

Definition 1.13. *[11]. Let K be a nonempty bounded subset of Banach space X. Then the asymptotic radius r and the asymptotic centre c of a sequence (u_n) relative to K are respectively:*

$$r = \inf\{\lim_{n\to\infty} \sup \|u_n - u\| : u \in K\},$$
$$c = \{u \in K : \lim_{n\to\infty} \sup \|u_n - u\| = r\}.$$

Definition 1.14. *[21]. Let K be a nonempty closed convex subset of a Banach space X. Let $\mathcal{S} = \{S(\zeta) : \zeta > 0\}$ be a family of mappings with domain $D(\mathcal{S}) = \bigcap\limits_{\zeta>0} D(S(\zeta))$ and range $R(\mathcal{S})$, where $D(S(\zeta)), R(\mathcal{S}) \subseteq K$. A one parameter E_μ nonexpansive semigroup is a family $\mathcal{S} = \{S(\zeta) : \zeta > 0\}$ of mappings satisfying the following conditions:*

1. *For each $\zeta > 0$, $S(\zeta)$ is a mapping satisfying condition (E), i.e. there exists $\mu \geq 1$ and for all $u, v \in D(\mathcal{S})$*
$$\|u - S(\zeta)(v)\| \leq \mu\|u - S(\zeta)(u)\| + \|u - v\|; \tag{1}$$
2. *$S(0)(u) = u$ for all $u \in D(\mathcal{S})$;*
3. *$S(\zeta + \xi)(u) = S(\zeta) \cdot S(\xi)(u)$ for all $\zeta, \xi > 0$ and $u \in D(\mathcal{S})$.*

Definition 1.15. *[17]. Let K be a nonempty subset of a Banach space X. Let $\mathcal{S}$ and $\{S_n\}$ be two families of mappings of K with $\cap_{n=1}^{\infty} F(S_n) = F(\mathcal{S}) \neq \emptyset$, where $F(\mathcal{S})$ is the set of all common fixed points of all mappings in $\mathcal{S}$. The family of mappings $\{S_n\}$ is said to satisfy NST*- condition with $\mathcal{S}$ if for every bounded sequence (u_n) in K*
$$\lim_{n\to\infty} \|u_n - S_n(u_n)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n - u_{n+1}\| = 0,$$
$$\text{imply that} \lim_{n\to\infty} \|u_n - S(u_n)\| = 0 \quad \text{for all} \quad S \in \mathcal{S}.$$

Example 1.16. *[17] Let $H = \mathbb{R}^2$ and $K = [0,1] \times [0,1]$. Define $T_1, T_2 : K \to K$ as follows*
$$T_1(u, v) = (u, 1 - v), \quad T_2(u, v) = (1 - u, v)$$
for all $(u, v) \in K$. Hence, T_1, T_2 are nonexpansive mappings with
$$F(T_1) \cap F(T_2) = \left([0,1] \times \left\{\frac{1}{2}\right\}\right) \cap \left(\left\{\frac{1}{2}\right\} \times [0,1]\right) = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\} \neq \emptyset.$$
Here T_n for $n = 1, 2$ satisfies NST-condition but fails to satisfy NST-condition (I) and NST-condition (II).*

Definition 1.17. *Let K be a nonempty subset of a Banach space X. Let $\mathcal{S} = \{S(\zeta) : \zeta > 0\}$ be a family of mappings from K into itself with $F(\mathcal{S}) \neq \emptyset$, where $F(\mathcal{S})$ is the set of all common fixed points of mappings in $\mathcal{S}$. Let $\{S(\zeta_n)\}$ be a subclass of $\mathcal{S}$. The family of mappings $\{S(\zeta_n)\}$ is said to satisfy NST***- condition with $\mathcal{S}$ if for every bounded sequence (u_n) in K*
$$\lim_{n\to\infty} \|u_n - S(\zeta_n)(u_n)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n - u_{n+1}\| = 0,$$
$$\text{imply that} \lim_{n\to\infty} \|u_n - S(\zeta)(u_n)\| = 0 \quad \text{for all} \quad \zeta > 0.$$

2. Main Results

We begin with the following strong convergence theorem.

Theorem 2.1. *Let X be a Banach space and $T : X \to X$ be a mapping satisfying condition (E). For a given $u_0 \in X$ and $\alpha \in (0, 1)$, if the sequence of iterates $(T_\alpha^n(u_0))$ converges strongly to $u^\dagger$, then $u^\dagger \in F(T)$.*

Proof. Define $u_n = T_\alpha^n(u_0)$, $n \in \mathbb{N} \cup \{0\}$. Then
$$\begin{aligned}\|u_{n+1} - u_n\| &= \|T_\alpha^{n+1}(u_0) - u_n\| \\ &= \|T_\alpha(T_\alpha^n(u_0)) - u_n\| \\ &= \|T_\alpha(u_n) - u_n\| \\ &= \|(1 - \alpha)u_n + \alpha T(u_n) - u_n\| \\ &= \alpha\|u_n - T(u_n)\|.\end{aligned}$$

Now,
$$\begin{aligned}\|u_{n+1} - T(u^\dagger)\| &= \|(1 - \alpha)u_n + \alpha T(u_n) - T(u^\dagger)\| \\ &\leq \|u_n - T(u^\dagger)\| + \alpha\|u_n - T(u_n)\|.\end{aligned}$$

From the definition of T and above inequalities, we get

$$\begin{aligned}\|u_{n+1} - T(u^\dagger)\| &\le \mu\|u_n - T(u_n)\| + \|u_n - u^\dagger\| + \alpha\|u_n - T(u_n)\| \\ &= (\mu+\alpha)\|u_n - T(u_n)\| + \|u_n - u^\dagger\| \\ &= \frac{(\mu+\alpha)}{\alpha}\|u_n - u_{n+1}\| + \|u_n - u^\dagger\| \\ &\le \frac{(\mu+\alpha)}{\alpha}\|u_n - u^\dagger\| + \frac{(\mu+\alpha)}{\alpha}\|u^\dagger - u_{n+1}\| + \|u_n - u^\dagger\| \\ &= \frac{(\mu+2\alpha)}{\alpha}\|u_n - u^\dagger\| + \frac{(\mu+\alpha)}{\alpha}\|u^\dagger - u_{n+1}\|.\end{aligned}$$

Since $\lim_{n\to\infty} u_n \to u^\dagger$, we have $\lim_{n\to\infty} \|u_{n+1} - T(u^\dagger)\| = 0$. Therefore $u^\dagger = T(u^\dagger)$. □

Theorem 2.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $T : K \to K$ be a mapping satisfying condition (E) with $F(T) = \{u^\dagger\}$. Assume that the mapping $I - T_\alpha$ is demiclosed at zero, where T_α is the α-Krasnosel'skiĭ mapping associated with T, and $\alpha \in (0,1)$. Then for each $u_0 \in K$ the sequence of iterates $(T_\alpha^n(u_0))$ converges weakly to $u^\dagger$.*

Proof. Define $u_n = T_\alpha^n(u_0)$, $n \in \mathbb{N}\cup\{0\}$. Let $u^\dagger \in K$ be a fixed point of T. From Proposition 1.8, we have

$$\begin{aligned}\|u_1 - u^\dagger\| &\le (1-\alpha)\|u_0 - u^\dagger\| + \alpha\|T(u_0) - u^\dagger\| \\ &\le \|u_0 - u^\dagger\|.\end{aligned}$$

Consequently,

$$\|u_n - u^\dagger\| \le \|u_0 - u^\dagger\|, \quad \text{for all} \quad n \in \mathbb{N}\cup\{0\},$$

and the sequence (u_n) is bounded. Let $\lim\limits_{n\to\infty} \|u_n - u^\dagger\| = r \ge 0$. If $r = 0$, then there is nothing to prove. Now, we have

$$\begin{aligned}u_{n+1} - u^\dagger &= T_\alpha(u_n) - u^\dagger \\ &= (1-\alpha)u_n + \alpha T(u_n) - u^\dagger \\ &= (1-\alpha)(u_n - u^\dagger) + \alpha(T(u_n) - u^\dagger).\end{aligned}$$

From Proposition 1.8 it follows that $\|T(u_n) - u^\dagger\| \le \|u_n - u^\dagger\|$. Also $\lim\limits_{n\to\infty} \|u_n - u^\dagger\| = r = \lim\limits_{n\to\infty} \|u_{n+1} - u^\dagger\|$. Since T_α is asymptotically regular from Theorem 1.12, we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = \lim_{n\to\infty} \|T_\alpha^{n+1}(u_0) - T_\alpha^n(u_0)\| = 0. \tag{2}$$

Since X is reflexive and (u_n) is bounded, there exists a subsequence (u_{n_i}) of (u_n) converges weakly to $\tilde{u}$. From (2)

$$\lim_{i\to\infty} \|T_\alpha^{i+1}(u_0) - T_\alpha^i(u_0)\| = \lim_{i\to\infty} \|T_\alpha^i(u_0) - T_\alpha(T_\alpha^i(u_0))\| = 0,$$

and thus the demiclosedness of $I - T_\alpha$ we have $T_\alpha(\tilde{u}) = \tilde{u}$. By Lemma 1.10, $F(T_\alpha) = F(T)$ and since $F(T)$ is singleton, $\tilde{u} = u^\dagger$. This implies that every weakly convergent subsequence of (u_n) converges weakly to $\tilde{u}$. If (u_n) does not converge weakly to $\tilde{u}$ then there is a weak neighbourhood U of $\tilde{u}$ and a subsequence (u_{n_l}) of (u_n) with the property that $u_{n_l} \notin U$, $l = 1, 2, \ldots$. Again by reflexivity of X and boundedness of (u_{n_l}), there exists a subsequence of (u_{n_l}) converges weakly. By the same procedure, we can show that this subsequence must converge weakly to $\tilde{u}$. It follows that terms of the subsequence (u_{n_l}) must lie in U, a contradiction. Thus $(T_\alpha^n(u_0))$ converges weakly to $\tilde{u}$. This completes the proof. □

Theorem 2.3. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X which has the Opial property. Let $T : K \to K$ be a mapping satisfying condition (E) with $F(T) \neq \emptyset$. Then for each $u_0 \in K$, the sequence of iterates $(T_\alpha^n(u_0))$ converges weakly to a fixed point of T.*

Proof. We have shown in Theorem 2.2 that, the sequence (u_n) defined by $u_n = T_\alpha^n(u_0), n \in \mathbb{N} \cup \{0\}$ is bounded and $\lim_{n\to\infty} \|u_n - u^\dagger\|$ exists for each $u^\dagger \in F(T)$. Thus there exists a subsequence (u_{n_k}) of (u_n) such that $u_{n_k} \rightharpoonup \tilde{u} \in K$. By definition of mapping T_α, we have

$$\begin{aligned} \|u_{n_k} - T_\alpha(\tilde{u})\| &= \|u_{n_k} - (1-\alpha)\tilde{u} + \alpha T(\tilde{u})\| \\ &\leq (1-\alpha)\|u_{n_k} - \tilde{u}\| + \alpha\|u_{n_k} - T(\tilde{u})\|. \end{aligned}$$

Since T satisfies condition (E), we have

$$\|u_{n_k} - T_\alpha(\tilde{u})\| \leq (1-\alpha)\|u_{n_k} - \tilde{u}\| + \alpha\mu\|u_{n_k} - T(u_{n_k})\| + \alpha\|u_{n_k} - \tilde{u}\|.$$

Using the fact that $\|u_{n_k} - T_\alpha(u_{n_k})\| = \alpha\|u_{n_k} - T(u_{n_k})\|$, we get

$$\|u_{n_k} - T_\alpha(\tilde{u})\| \leq \|u_{n_k} - \tilde{u}\| + \mu\|u_{n_k} - T_\alpha(u_{n_k})\|.$$

From (2), we get

$$\liminf_{k\to\infty} \|u_{n_k} - T_\alpha(\tilde{u})\| \leq \liminf_{k\to\infty} \|u_{n_k} - \tilde{u}\|.$$

By the Opial property, it is evident that, $T_\alpha(\tilde{u}) = \tilde{u}$, and $\tilde{u} \in F(T_\alpha)$. Hence $\tilde{u} \in F(T)$. In order to show the weak convergence of sequence (u_n) to a point in $F(T)$, it is suffices to show that the set of all weak limits of (u_n) is singleton. Arguing by contradiction, let (u_{n_i}) and (u_{n_j}) be two subsequences of (u_n) such that $u_{n_i} \rightharpoonup \tilde{u}$ and $u_{n_j} \rightharpoonup \tilde{v}$, respectively with $\tilde{u} \neq \tilde{v}$. Since $\lim_{n\to\infty} \|u_n - u^\dagger\|$ exists for each $u^\dagger \in F(T)$, from the Opial property, we have

$$\begin{aligned} \lim_{n\to\infty} \|u_n - \tilde{u}\| &= \lim_{i\to\infty} \|u_{n_i} - \tilde{u}\| \\ &< \lim_{i\to\infty} \|u_{n_i} - \tilde{v}\| = \lim_{j\to\infty} \|u_{n_j} - \tilde{v}\| \\ &< \lim_{j\to\infty} \|u_{n_j} - \tilde{u}\| = \lim_{n\to\infty} \|u_n - \tilde{u}\|, \end{aligned}$$

a contradiction. This completes the proof. □

Theorem 2.4. *Let X be a Banach space and (u_n) be a sequence in X. Let $T : X \to X$ be a mapping satisfying condition (E) and v_n be the unique solution of the equation $u_n = w - T(w)$. If $\lim_{n\to\infty} \|u_n\| \to 0$ and sequence (v_n) converges to some $v \in X$ then v is a solution of the equation $w = T(w)$.*

Proof. By assumptions (v_n) is a sequence in X which is the unique solution to equation $u_n = w - T(w)$. Now, by the triangle inequality, we have

$$\begin{aligned} \|v - T(v)\| &\leq \|v - v_n\| + \|v_n - T(v_n)\| + \|T(v_n) - T(v)\| \\ &\leq \|v - v_n\| + \|v_n - T(v_n)\| + \|T(v_n) - v_n\| + \|v_n - T(v)\| \\ &\leq \|v - v_n\| + 2\|v_n - T(v_n)\| + \mu\|v_n - T(v_n)\| + \|v_n - v\| \\ &= 2\|v - v_n\| + (2+\mu)\|v_n - T(v_n)\|. \end{aligned}$$

Since $\lim_{n\to\infty} \|v_n - T(v_n)\| = 0$ and $\lim_{n\to\infty} v_n = v$, we get $\|v - T(v)\| = 0$. Therefore, v is a solution of the equation $w = T(w)$. □

Theorem 2.5. *Let K be a closed convex and bounded subset of a uniformly convex Banach space X and the mapping $T : K \to K$ be satisfying condition (E). If $u \in K$ and c is the asymptotic centre of $\{T^n(u) : n \in \mathbb{N} \cup \{0\}\}$, then $r(c) = \inf\{\|T^n(u) - c\| : n \in \mathbb{N} \cup \{0\}\} \leq \inf\{\|T^n(u) - u^\dagger\| : n \in \mathbb{N} \cup \{0\}\}$, where $u^\dagger \in F(T)$.*

Proof. Define $r_n(u^\dagger) = \|T^n(u) - u^\dagger\|$, $n \in \mathbb{N} \cup \{0\}$. Then

$$r_{n+1}(u^\dagger) = \|T^{n+1}(u) - u^\dagger\| = \|T(T^n(u)) - u^\dagger\| \leq \mu\|u^\dagger - T(u^\dagger)\| + \|T^n(u) - u^\dagger\|$$
$$= \|T^n(u) - u^\dagger\| = r_n(u^\dagger).$$

$$r_{n+1}(u^\dagger) \leq r_n(u^\dagger), \quad \text{for all} \quad u^\dagger \in F(T).$$

Thus $\{r_n(u^\dagger)\}$ is monotonically decreasing and convergent to $r(u^\dagger) = \inf\{\|T^n(u) - u^\dagger\|$. Since c is the asymptotic centre, it implies $r(c) \leq r(u)$ for all $u \in K$. Thus $r(c) = \inf\{\|T^n(u) - c\| : n \in \mathbb{N} \cup \{0\}\} \leq \inf\{\|T^n(u) - u^\dagger\| : n \in \mathbb{N} \cup \{0\}\}$ for each $u^\dagger \in F(T)$. □

Now, we present a strong convergence theorem for one parameter semigroup of mappings satisfying condition (E).

Theorem 2.6. *Let K be a convex and compact subset of a uniformly convex Banach space X. Let $\mathcal{S} = \{S(\zeta) : \zeta > 0\}$ be a semigroup of E_μ-nonexpansive mappings from K into itself with $F(\mathcal{S}) \neq \emptyset$. Suppose there exists a subclass $\{S(\zeta_n)\}$ of $\mathcal{S}$ such that the subclass $\{S(\zeta_n)\}$ satisfies NST***-condition with $\mathcal{S}$. For a given $u_0 \in K$, define a sequence (u_n) in K as follows:*

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n S(\zeta_n)(u_n),$$

for $n \in \mathbb{N} \cup \{0\}$ and $\alpha_n \in (0, 1)$. Then (u_n) converges strongly to a fixed point of one parameter semigroup of $E_\mu-$ nonexpansive mapping $\{S(\zeta) : \zeta \geq 0\}$.

Proof. Let $u^\dagger \in F(\mathcal{S})$. Then

$$\begin{aligned}
\|u_{n+1} - u^\dagger\| &= \|(1 - \alpha_n)u_n + \alpha_n S(\zeta_n)(u_n) - u^\dagger\| \\
&= \|(1 - \alpha_n)u_n + \alpha_n S(\zeta_n)(u_n) + \alpha_n u^\dagger - \alpha_n u^\dagger - u^\dagger\| \\
&\leq (1 - \alpha_n)\|u_n - u^\dagger\| + \alpha_n\|S(\zeta_n)(u_n) - u^\dagger\| \\
&\leq (1 - \alpha_n)\|u_n - u^\dagger\| + \alpha_n\|u_n - u^\dagger\| \\
&= \|u_n - u^\dagger\|.
\end{aligned}$$

So, $\lim_{n\to\infty} \|u_n - u^\dagger\|$ exists for all $u^\dagger \in F(\mathcal{S})$. Let $\lim_{n\to\infty} \|u_n - u^\dagger\| = d$ for some $d \geq 0$.

$$d = \lim_{n\to\infty} \|u_{n+1} - u^\dagger\| = \lim_{n\to\infty} \|(1 - \alpha_n)(u_n - u^\dagger) + \alpha_n(S(\zeta_n)(u_n) - u^\dagger)\|,$$

$$\lim_{n\to\infty} \|S(\zeta_n)(u_n) - u^\dagger\| \leq \|u_n - u^\dagger\| = d.$$

Using Lemma 1.2, we get

$$\lim_{n\to\infty} \|S(\zeta_n)(u_n) - u_n\| = 0.$$

Now, it follows that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = \lim_{n\to\infty} \alpha_n\|S(\zeta_n)(u_n) - u_n\| = 0.$$

Since $\{S(\zeta_n)\}$ satisfies the NST***-condition, so we get

$$\lim_{n\to\infty} \|u_n - S(\zeta)(u_n)\| = 0 \quad \text{for all} \quad \zeta > 0.$$

Since K is compact, $(u_{n_i}) \subset K$ be a subsequence of (u_n) converges strongly to $u^\dagger \in K$. Now for any $\zeta > 0$, we have

$$\lim_{i\to\infty} \|u_{n_i} - S(\zeta)(u^\dagger)\| \leq \lim_{i\to\infty} \mu\|u_{n_i} - S(\zeta)(u_{n_i})\| + \lim_{i\to\infty} \|u_{n_i} - u^\dagger\|$$
$$= 0 \quad \text{for all} \quad \zeta > 0.$$

Hence (u_{n_i}) converges to $S(\zeta)(u^\dagger)$. It implies that $S(\zeta)(u^\dagger) = u^\dagger$. Therefore $u^\dagger$ is a fixed point of $\{S(\zeta) : \zeta > 0\}$. It completes the proof.

□

3. Examples

In this section, we present a couple of nontrivial examples for mappings which are not nonexpansive but does satisfy condition (E). Further, we illustrate our results by showing convergence behaviour for different choices of initial guesses and coefficients.

Example 3.1. *Let $T : [0,1] \to [0,1]$ be a mapping defined as*

$$T(p) = \begin{cases} \frac{p}{3}, \ p \in \left[0, \frac{2}{3}\right), \\ \frac{p}{6} + \frac{1}{5}, \ p \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

For this we consider following cases:

Case(a): $p, q \in \left[0, \frac{2}{3}\right)$. *Then* $|T(p) - T(q)| = \frac{1}{3}|p - q| < |p - q|$ *and we have,*

$$|p - T(q)| \leq |p - T(p)| + |p - q|.$$

Case(b): $p, q \in \left[\frac{2}{3}, 1\right]$. *Then* $|T(p) - T(q)| = \frac{1}{6}|p - q| < |p - q|$ *and we have,*

$$|p - T(q)| \leq |p - T(p)| + |p - q|.$$

Case(c): $p \in \left[0, \frac{2}{3}\right), q \in \left[\frac{2}{3}, 1\right]$. *Then we have,*

$$\begin{aligned} |q - T(p)| = \left|q - \frac{p}{3}\right| &\leq \left|5q - \frac{6}{5}\right| + |p - q| \\ &= 6\left|q - \frac{q}{6} - \frac{1}{5}\right| + |p - q| = 6|q - T(q)| + |p - q|. \end{aligned}$$

$$\begin{aligned} |p - T(q)| = \left|p - \frac{q}{6} - \frac{1}{5}\right| &\leq |p| + \left|\frac{q}{6} + \frac{1}{5}\right| \leq |p| + \left|\frac{1}{6} + \frac{1}{5}\right| \\ &\leq |p| + \frac{11}{30}. \end{aligned}$$

Case(i): *If* $p \in \left[0, \frac{9}{30}\right)$,

$$|p - T(q)| \leq |p| + \frac{11}{30} \leq 4|p| + \frac{11}{30} \leq 6|p - T(p)| + |p - q|.$$

Case(ii): *If* $p \in \left(\frac{9}{30}, \frac{2}{3}\right)$,

$$6|p - T(p)| = 4|p| \geq \frac{36}{30},$$

$$|p - T(q)| \leq \frac{2}{3} + \frac{11}{30} = \frac{31}{30},$$

$$|p - T(q)| \leq \frac{31}{30} \leq \frac{36}{30} \leq 6|p - T(p)| + |p - q|.$$

Therefore in all the cases T satisfies condition (E).

On the other hand, for $p = \frac{3}{5}, q = \frac{2}{3}$, *we have* $T(p) = \frac{1}{5}, T(q) = \frac{31}{100}$, *and*

$$|T(p) - T(q)| = 0.11 > 0.06 = |p - q|.$$

Hence T is not a nonexpansive mapping.

Example 3.2. *Let $X = \mathbb{R}^2$ and $K = \{p = (p_1, p_2) \in [0,1] \times [0,1]\}$ be a subset of X with norm $\|p\| = \|(p_1,p_2)\| = (|p_1|^2 + |p_2|^2)^{1/2}$. The mapping $T : K \to K$ is defined by*

$$T(p_1,p_2) = \begin{cases} (1-p_1, 1-p_2), & (p_1,p_2) \in \left[0, \frac{1}{2}\right] \times [0,1], \\ \frac{1}{3}(1+p_1, 1+p_2), & (p_1,p_2) \in \left(\frac{1}{2}, 1\right] \times [0,1]. \end{cases}$$

For this we consider following cases:

Case(a): $p, q \in \left[0, \frac{1}{2}\right] \times [0,1]$. *Then*

$$\begin{aligned} \|p - T(q)\| &\le \|p - T(p)\| + \|T(p) - T(q)\| \\ &= \|p - T(p)\| + \left(|p_1 - q_1|^2 + |p_2 - q_2|^2\right)^{1/2} \\ &= \|p - T(p)\| + \|p - q\|. \end{aligned}$$

Case(b): $p, q \in \left(\frac{1}{2}, 1\right] \times [0,1]$. *Then*

$$\begin{aligned} \|p - T(q)\| &\le \|p - T(p)\| + \|T(p) - T(q)\| \\ &= \|p - T(p)\| + \frac{1}{3}\left(|p_1 - q_1|^2 + |p_2 - q_2|^2\right)^{1/2} \\ &\le \|p - T(p)\| + \left(|p_1 - q_1|^2 + |p_2 - q_2|^2\right)^{1/2} \\ &= \|p - T(p)\| + \|p - q\|. \end{aligned}$$

Case(c): $p \in \left[0, \frac{1}{2}\right] \times [0,1],\ q \in \left(\frac{1}{2}, 1\right] \times [0,1]$. *Then*

$$\begin{aligned} \|p - T(q)\| &= \left\|(p_1,p_2) - \left(\frac{1+q_1}{3}, \frac{1+q_2}{3}\right)\right\| \\ &= \left\|\left(p_1 - \left(\frac{1+q_1}{3}\right)\right), \left(p_2 - \left(\frac{1+q_2}{3}\right)\right)\right\| \\ &= \left\|\left(\frac{3p_1 - q_1 - 1}{3}, \frac{3p_2 - q_2 - 1}{3}\right)\right\| \\ &= \left(\left|\frac{3p_1 - q_1 - 1}{3}\right|^2 + \left|\frac{3p_2 - q_2 - 1}{3}\right|^2\right)^{1/2}, \end{aligned}$$

and

$$\|p - T(p)\| = \left(|2p_1 - 1|^2 + |2p_2 - 1|^2\right)^{1/2}.$$

Since

$$\begin{aligned} \left|\frac{3p_1 - q_1 - 1}{3}\right| &\le |2p_1 - 1 + p_1 - q_1| \le |2p_1 - 1| + |p_1 - q_1|, \\ \left|\frac{3p_2 - q_2 - 1}{3}\right| &< |2p_2 - 1 + p_2 - q_2| \le |2p_2 - 1| + |p_2 - q_2|. \end{aligned}$$

we have,

$$\|p - T(q)\| \le \|p - T(p)\| + \|p - q\|.$$

Therefore in all the cases T satisfies condition (E).

On the other hand, for $p = (0,0)$ and $q = \left(\frac{51}{100}, \frac{25}{100}\right)$, we have

$$\|T(p) - T(q)\| = 0.766 > 0.567 = \|p - q\|.$$

Therefore T is not a nonexpansive mapping.

Now, we present the convergence behaviour for sequence of iterates of the mapping T_α considered in Example (3.2). The convergence behaviours are presented in Table 1, 2 and figures 1, 2 below. In Fig. 1 the convergence behaviour is illustrated for different choices of initial guesses and in Fig. 2 for different choices of coefficients. The stopping criteria is $\|u_n - u^\dagger\| < 10^{-8}$, (where $u^\dagger \in F(T)$).

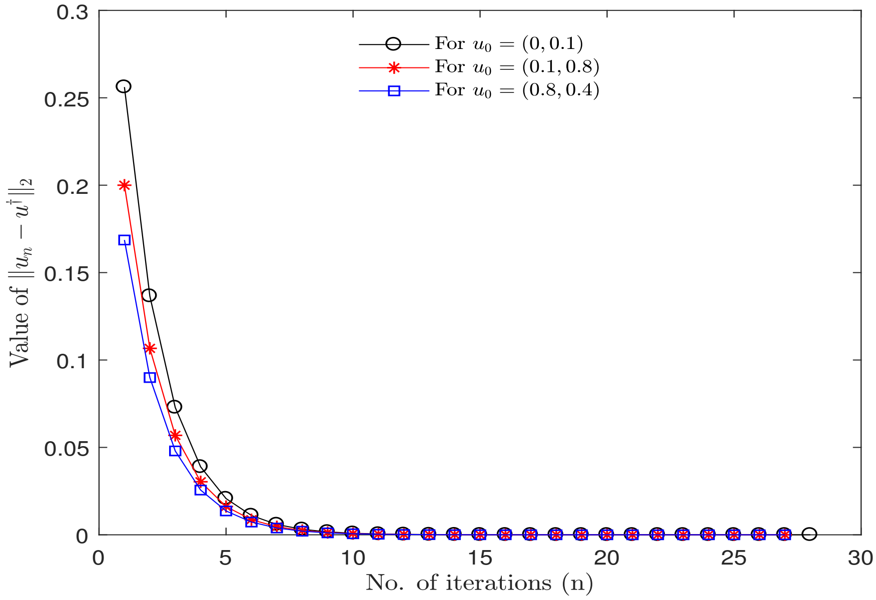

Figure 1: Convergence behaviour for different choices of initial guesses.

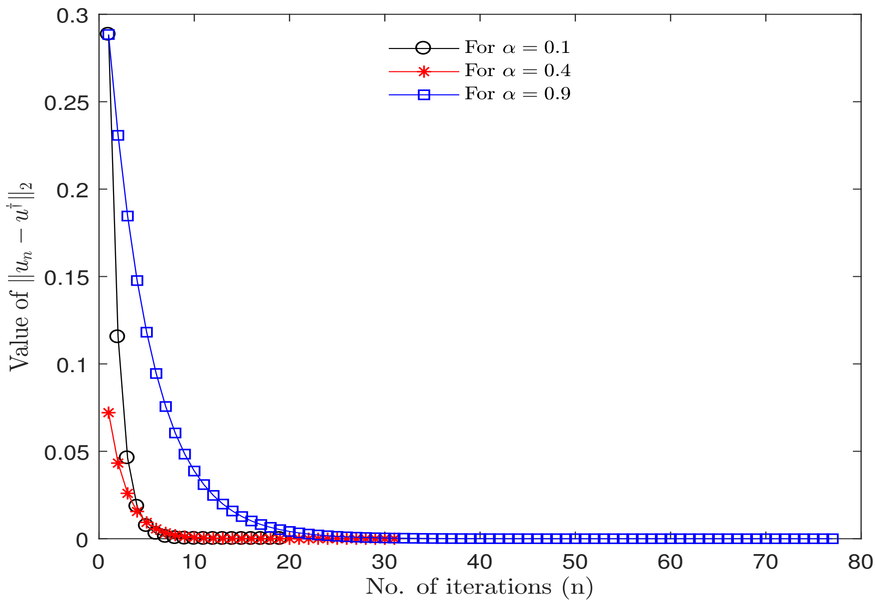

Figure 2: Convergence behaviour for different choices of coefficients.

Table 1: Influence of initial guess with coefficient $\alpha = 0.3$.

Initial guesses	**Number of iterations**
$u_0 = (0, 0.1)$	30
$u_0 = (0.1, 0.8)$	29
$u_0 = (0.8, 0.4)$	29

Table 2: Influence of coefficients $\alpha = 0.1, 0.4, 0.9$ for fixed initial guess.

Coefficients	**Number of iterations**
$\alpha = 0.1$	21
$\alpha = 0.4$	33
$\alpha = 0.9$	79

Now we compare the α-Krasnosel'skiĭ iteration procedure with Picard iteration procedure for the mapping considered in Example (3.2).

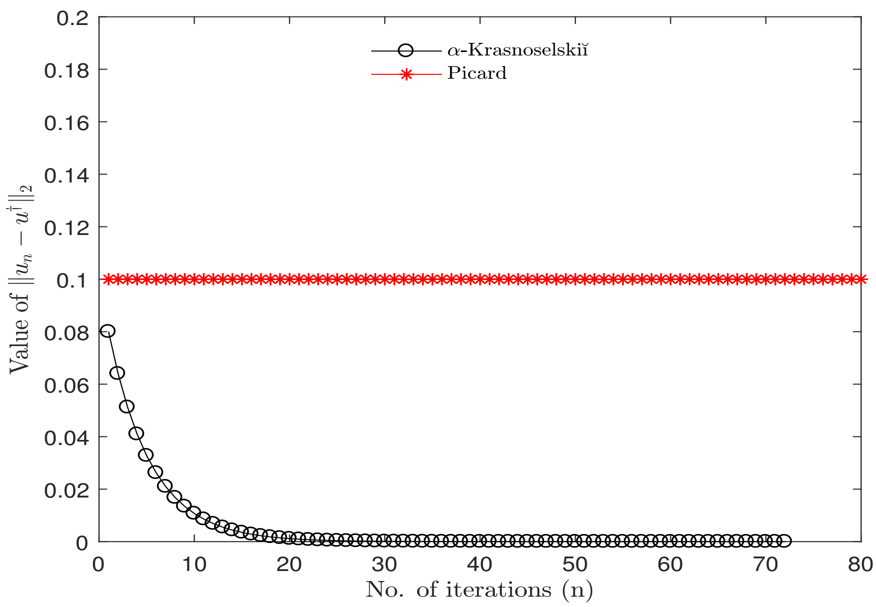

Figure 3: Comparison between α-Krasnosel'skiĭ iteration procedure and Picard iteration procedure for $u_0 = (0.5, 0.6)$.

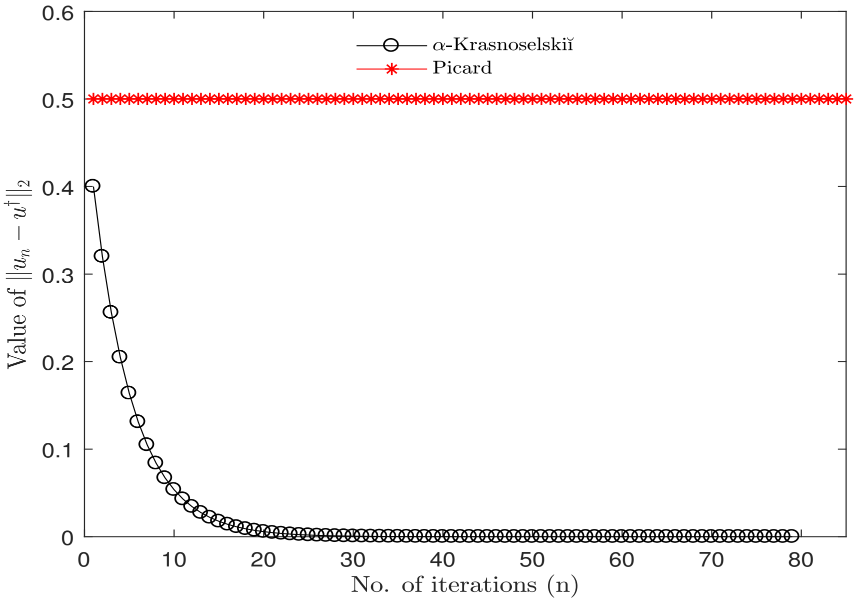

Figure 4: Comparison between α-Krasnosel'skiĭ iteration procedure and Picard iteration procedure for $u_0 = (0.5, 1)$.

Table 3: The value of $\|u_n - u^\dagger\|_2$ for $u_0 = (0.5, 0.6)$.

No. of Iterations	**α-Krasnosel'skiĭ iteration**	**Picard iteration**
1	0.0800000000000000	0.100000000000000
2	0.0640000000000001	0.100000000000000
3	0.0512000000000000	0.100000000000000
4	0.0409600000000000	0.100000000000000
5	0.0327680000000000	0.100000000000000
......	......	......
68	2.57110086554491e-08	0.100000000000000
69	2.05688068799503e-08	0.100000000000000
70	1.64550455483692e-08	0.100000000000000
71	1.31640364164909e-08	0.100000000000000
72	1.05312291109883e-08	0.100000000000000

Table 4: The value of $\|u_n - u^\dagger\|_2$ for $u_0 = (0.5, 1)$.

No. of Iterations	**α-Krasnosel'skiĭ iteration**	**Picard iteration**
1	0.400000000000000	0.500000000000000
2	0.320000000000000	0.500000000000000
3	0.256000000000000	0.500000000000000
4	0.204800000000000	0.500000000000000
5	0.163840000000000	0.500000000000000
......	......	......
75	2.69599466085069e-08	0.500000000000000
76	2.15679573090100e-08	0.500000000000000
77	1.72543658250035e-08	0.500000000000000
78	1.38034926822073e-08	0.500000000000000
79	1.10427941235614e-08	0.500000000000000

4. Conclusions

(i) From Table 1, 2 and Fig. 1, 2 we conclude that the convergence behaviour of sequence of iterates of α-Krasnosel'skiĭ mapping depends more on the coefficients and less on initial guesses. We can see that while changing the initial guesses there is slightly difference in number of iterations but when we are changing the coefficient α then there is a huge difference in number of iterations.

(ii) In Fig. 3, 4 we presented the comparison between Picard iteration procedure and α-Krasnosel'skiĭ iteration procedure and from Table 3, 4 we can conclude that for initial guess $u_0 = (0.5, 0.6)$ and $u_0 = (0.5, 1)$ the Picard iteration procedure does not converge but α-Krasnosel'skiĭ iteration procedure is converging.

Acknowledgement

The authors are thankful to the reviewers and the editor for their constructive comments and suggestions that have been useful for the improvement of this paper. The second and third authors acknowledge the support from the GES 4.0 fellowship, University of Johannesburg, South Africa.

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