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Research Article

Regular Γ-irresolvable spaces

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Abstract

In this paper by using regular open sets and Γ-local functions, we introduce and investigate the notions of $\mathcal{I}_R$-dense sets, $\mathcal{I}_R$-hyperconnectedness, $\mathcal{I}^*_R$-hyperconnectedness, Γ-resolbablity and regular Γ-irresobability in ideal topological spaces.

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1. Introduction

Throughout the present paper, (X, τ) or X will denote a topological space and $Cl(A)$ (resp. $Int(A)$) will denote the closure (resp. interior) of a subset A of X. A subset A of X is said to be regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). We denote by $Ro(X)$ (resp. $Rc(X)$) the family of all regular open (resp. regular closed) sets in a space (X, τ). It is well known that $Ro(X)$ is a base for a topology τ_S on X which is coarser than τ. The space (X, τ_S) is called the semi-regularization [12] of (X, τ) and (X, τ) is said to be semi-regular if $\tau = \tau_S$. The closure (resp. interior) of a subset A of (X, τ_s) will be denoted by $Cl_{\tau_s}(A)$ (resp. $Int_{\tau_s}(A)$).

A nonempty collection $\mathcal{I}$ of subsets on a topological space (X, τ) is called a topological ideal on (X, τ) [11] if it satisfies the following two conditions:

(1) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$ (heredity).
(2) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$ (finite additivity).

Let $(X, \tau, \mathcal{I})$ be an ideal topological space, that is, a topological space (X, τ) with an ideal $\mathcal{I}$ on X. For a subset A of X, the local function $A^*(\mathcal{I}, \tau)$ [9] and the $\Gamma^*(A)(\mathcal{I}, \tau)$ [3] are defined as follows:

$A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin I \text{ for every } U \in \tau(x)\}$ and
$\Gamma^*(A)(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin I \text{ for every } U \in Ro(x)\}$, respectively,
where $\tau(x) = \{U \in \tau : x \in U\}$ and $Ro(x) = \{U \in Ro(X) : x \in U\}$.
Hereafter, $A^*(\mathcal{I}, \tau)$ and $\Gamma^*(A)(\mathcal{I}, \tau)$ are simply denoted by A^* and $\Gamma(A)$, respectively. Hatir et al. [7] defined the δ-local function $A^{\delta *}(\mathcal{I}, \tau)$ of a subset A of X. However, it is shown in [3] that $A^{\delta *}(\mathcal{I}, \tau) = \Gamma^*(A)(\mathcal{I}, \tau)$ for every subset A of X. A subset A of an ideal topological

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space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_R$-dense (resp. $\mathcal{I}$-dense) if every point of X is in $\Gamma(A)$ (resp. A^*), i.e., if $\Gamma(A) = X$ (resp. $A^* = X$). If $Ro(X) \cap \mathcal{I} = \{\emptyset\}$ (resp. $\tau \cap \mathcal{I} = \{\emptyset\}$), then $\mathcal{I}$ is said to be regular [5] (resp. codense). Several characterizations of ideal topological spaces were provided in [1, 2, 4].

Remark 1.1. For an ideal space $(X, \tau, \mathcal{I})$, if $D \subseteq X$ is $\mathcal{I}_R$-dense, then X is also $\mathcal{I}_R$-dense, i.e., $\Gamma(X) = X$.

Theorem 1.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space, then the following properties are equivalent:*

(1) *$\mathcal{I}$ is regular;*
(2) *If $I \in \mathcal{I}$, then $\delta Int(I) = \phi$;*
(3) *For every $G \in Ro(X)$, $G \subseteq \Gamma(G)$;*
(4) *$X = \Gamma(X)$.*

In this paper by using regular open sets and Γ-local functions, we introduce and investigate the notions of $\mathcal{I}_R$-dense sets, $\mathcal{I}_R$-hyperconnectedness, $\mathcal{I}^*_R$-hyperconnectedness, Γ-resolbablity and regular Γ-irresobability in ideal topological spaces. Hereafter, an ideal topological space $(X, \tau, \mathcal{I})$ will be simply called an ideal space.

2. $\mathcal{I}_R$-hyperconnected spaces

Definition 2.1. An ideal space $(X, \tau, \mathcal{I})$ is said to be

(1) R-hyperconnected if every pair of nonempty regular open sets of X has nonempty intersection.
(2) $\mathcal{I}_R$-hyperconnected if every nonempty regular open set is $\mathcal{I}_R$-dense in X.
(3) R-hyperconnected modulo $\mathcal{I}$ if the intersection of every two nonempty regular open sets is not in $\mathcal{I}$.

Lemma 2.2. *For an ideal space $(X, \tau, \mathcal{I})$, the following statements are equivalent:*

(1) *$(X, \tau, \mathcal{I})$ is R-hyperconnected modulo $\mathcal{I}$;*
(2) *There are no proper regular closed sets G and H such that $X - (G \cup H) \in \mathcal{I}$.*

Proof. Suppose that there are proper regular closed G and H such that $X - (G \cup H) \in \mathcal{I}$. If H is empty, then $X - G \in \mathcal{I}$. Since $X - G$ and X are nonempty regular open sets with $X \cap (X - G) = (X - G) \in \mathcal{I}$. This is a contradiction. Hence, G and H both are nonempty proper regular closed sets. Then $X - G$ and $X - H$ are nonempty regular open sets. However, $(X - G) \cap (X - H) = X - (G \cup H) \in \mathcal{I}$ which contradicts to (1).
Conversely, let A and B be any nonempty regular open sets in X. Then $X - A$ and $X - B$ are proper regular closed sets in X and $X - [(X - A) \cup (X - B)] \notin \mathcal{I}$. This implies that $X - [X - (A \cap B)] \notin \mathcal{I}$. Thus, $(A \cap B) \notin \mathcal{I}$. □

Theorem 2.3. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $\mathcal{I}$ be a regular ideal in X. Then X is R-hyperconnected modulo $\mathcal{I}$ if and only if X is R-hyperconnected.*

Proof. Let X be R-hyperconnected modulo $\mathcal{I}$. Then, since $\emptyset \in \mathcal{I}$, X is R-hyperconnected. Conversely, let X be R-hyperconnected and $\emptyset \neq A, B \in Ro(X)$. Then $\emptyset \neq A \cap B \in Ro(X)$. Since $\mathcal{I}$ is regular, $A \cap B \notin \mathcal{I}$. Thus, X is R-hyperconnected modulo $\mathcal{I}$. □

Theorem 2.4. *An ideal space $(X, \tau, \mathcal{I})$ is R-hyperconnected if and only if the union of two not τ_s-dense sets is a not τ_s-dense set.*

Proof. Let $(X, \tau, \mathcal{I})$ be R-hyperconnected and E, F be two not τ_s-dense sets in $(X, \tau, \mathcal{I})$. Then there exist two nonempty regular open sets U and V such that $U \cap E = \emptyset$ and $V \cap F = \emptyset$. Since $(X, \tau, \mathcal{I})$ is R-hyperconnected, $U \cap V \neq \emptyset$. But $(U \cap V) \cap (E \cup F) = \emptyset$ and hence $E \cup F$ is not τ_s-dense in $(X, \tau, \mathcal{I})$.
Conversely, let the condition hold in $(X, \tau, \mathcal{I})$ but $(X, \tau, \mathcal{I})$ is not R-hyperconnected. Then there exist $\emptyset \neq U, V \in Ro(X)$ such that $U \cap V = \emptyset$. Hence $U \subseteq X - V$ and $V \subseteq X - U$.

Then $X-U$ and $X-V$ are not τ_s-dense in $(X,\tau,\mathcal{I})$. But $(X-U)\cup(X-V)=X$ which contradicts the fact that the union of two not τ_s-dense sets is a not τ_s-dense set. Hence the theorem is now proved. $\square$

Lemma 2.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then X is $\mathcal{I}_R$-hyperconnected if and only if X is R-hyperconnected and $\mathcal{I}$ is regular.*

Proof. Clearly every $\mathcal{I}_R$-hyperconnected space is R-hyperconnected. Let U be regular open, nonempty and a member of the ideal. Then $\Gamma(U)=X$. On the other hand, since $U\in\mathcal{I}$, $\Gamma(U)=\emptyset$. Hence $X=\emptyset$. This is a contradiction. Therefore, $\mathcal{I}$ is regular.

Conversely, let $\emptyset\neq U\in Ro(X)$. Let $x\in X$. Due to the R-hyperconnectedness of X, every regular open set V containing x meets U. Moreover, $U\cap V$ is regular open and $U\cap V\notin\mathcal{I}$ because $\mathcal{I}$ is regular. Thus $x\in\Gamma(U)$. This shows that U is $\mathcal{I}_R$-dense. $\square$

Theorem 2.6. *Let $(X,\tau,\mathcal{I})$ be an ideal space, where $\mathcal{I}$ is regular. Then a set D is $\mathcal{I}_R$-dense if and only if $(U-A)\cap D\neq\emptyset$ whenever $\emptyset\neq U\in Ro(X)$ and $A\in\mathcal{I}$.*

Proof. Let D be $\mathcal{I}_R$-dense. Then $U\cap D\notin\mathcal{I}$ for all nonempty regular open set U. Hence for all $A\in\mathcal{I}$, $(U-A)\cap D\neq\emptyset$, for otherwise $(U-A)\cap D=\emptyset$ and hence $\emptyset=U\cap(X-A)\cap D=(U\cap D)\cap(X-A)$. Therefore, $U\cap D\subseteq A$. Since $A\in\mathcal{I}$, $U\cap D\in\mathcal{I}$ which is contrary to $U\cap D\notin\mathcal{I}$. Therefore, $(U-A)\cap D\neq\emptyset$.

Conversely let $(U-A)\cap D\neq\emptyset$ whenever $\emptyset\neq U\in Ro(X)$ and $A\in\mathcal{I}$. Then we claim that D is $\mathcal{I}_R$-dense. Let D be not $\mathcal{I}_R$-dense. Then there exists some nonempty regular open set U such that $U\cap D\in\mathcal{I}$. Let $U\cap D=A$. Then, since $\mathcal{I}$ is regular, $U-A$ is nonempty but $(U-A)\cap D=\emptyset$. This is contrary to our assumption. $\square$

Theorem 2.7. *Let $(X,\tau,\mathcal{I})$ be an ideal space, where $\mathcal{I}$ is regular. Then X is R-hyperconnected modulo $\mathcal{I}$ if and only if $(U-A)\cap D\neq\emptyset$ whenever $\emptyset\neq U,D\in Ro(X)$ and $A\in\mathcal{I}$.*

Proof. The proof follows from Lemma 2.5 and Theorem 2.6. $\square$

3. Γ-resolvable spaces

A topological space (X,τ) is said to resolvable [8] if X is the union of two disjoint dense subsets. An ideal space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}$-resolvable [6] if X has two disjoint $\mathcal{I}$-dense subsets. If $\mathcal{I}$ and $\mathcal{J}$ are ideals with $\mathcal{I}\subseteq\mathcal{J}$ and X is $\mathcal{J}$-resolvable, then X is $\mathcal{I}$-resolvable. An ideal space $(X,\tau,\mathcal{I})$ is said to be Γ-resolvable if it has two disjoint $\mathcal{I}_R$-dense sets; otherwise it is said to be Γ-irresolvable.

Example 3.1. Let $(\mathbb{R},\tau)$ be the real numbers with the left-ray topology, i.e. $\tau=\{(-\infty,a):a\in\mathbb{R}\}\cup\{\mathbb{R},\emptyset\}$. Let $\mathcal{I}_f$ be the ideal of all finite subsets of $\mathbb{R}$. Let $A=[0,1]$ and $B=[2,3]$. We have the following

(1) $\Gamma(A)=\{x\in\mathbb{R}:A\cap U=A\notin\mathcal{I}_f$ for every $U\in Ro(x)\}=\mathbb{R}$.
(2) $\Gamma(B)=\{x\in\mathbb{R}:B\cap U=B\notin\mathcal{I}_f$ for every $U\in Ro(x)\}=\mathbb{R}$.
(3) Then A and B are disjoint $\mathcal{I}_R$-dense. Hence $(\mathbb{R},\tau,\mathcal{I})$ is Γ-resolvable.
(4) Also $\mathcal{I}\cap Ro(\mathbb{R})=\{\emptyset\}$, then $\mathcal{I}$ is regular.

Example 3.2.
Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset,X,\{a,c\},\{d\},\{a,c,d\}\}$, $\mathcal{I}=\{\emptyset,\{c\}\}$ and $Ro(X)=\{\emptyset,X,\{a,c\},\{d\}\}$. We have the following

(1) $\mathcal{I}\cap Ro(X)=\{\emptyset\}$, then $\mathcal{I}$ is regular.
(2) $\Gamma(\{a,d\})=\Gamma(\{a,b,d\})=\Gamma(\{a,c,d\})=\Gamma(X)=X$, then the collection of all $\mathcal{I}_R$-dense are only $\{a,d\},\{a,b,d\},\{a,c,d\},X$ and it is clear that are not disjoint.
(3) Hence $(X,\tau,\mathcal{I})$ is Γ-irresolvable.

Lemma 3.3. *Let $(X,\tau,\mathcal{I})$ be an ideal space.*
(1) $(X,\tau,\mathcal{I})$ is Γ-resolvable if and only if X is the union of two disjoint $\mathcal{I}_R$-dense sets.
(2) If $(X,\tau,\mathcal{I})$ is Γ-resolvable, then $\mathcal{I}$ is regular.

Proof. (1) Let A and B be disjoint $\mathcal{I}_R$-dense sets. Then $\Gamma(A) = X$ and $X = \Gamma(B) \subseteq \Gamma(X - A)$ and hence $X = \Gamma(X - A)$. Therefore, X is the union of $\mathcal{I}_R$-dense sets A and $X - A$. The converse is obvious.

(2) Let A and B be disjoint $\mathcal{I}_R$-dense sets. Then, by Lemma 2.3(1) of [3], we have $X = \Gamma(A) \subseteq \Gamma(X)$. Therefore, X is $\mathcal{I}_R$-dense. Thus, by Theorem 1.2, $\mathcal{I}$ is regular. □

The converse of Lemma 3.3 (2) is not true as shown by Example 3.2.

Theorem 3.4. *An ideal space $(X, \tau, \mathcal{I})$ is Γ-resolvable if and only if (X, τ_Γ) is resolvable and $\mathcal{I}$ is regular.*

Proof. Let $(X, \tau, \mathcal{I})$ be Γ-resolvable. Then, by Lemma 3.3(1), $X = A \cup B$, where A and B are disjoint $\mathcal{I}_R$-dense sets of X. Note that $Cl_\Gamma(A) = A \cup \Gamma(A) = A \cup X = X$. Hence A and B are τ_Γ-dense. Thus, (X, τ_Γ) is resolvable. By Lemma 3.3(2), $\mathcal{I}$ is regular. Conversely, Let (X, τ_Γ) be resolvable and $\mathcal{I}$ be regular. Suppose that $X = A \cup B$, $A \cap B = \emptyset$ and both A and B are τ_Γ-dense. Let $x \in X$ and $x \notin \Gamma(A)$, then there exists a regular open set U containing x such that $V = U \cap A \in \mathcal{I}$. Since B is τ_Γ-dense and $\mathcal{I}$ is regular, V is nonempty and moreover $U \nsubseteq A$. It follows from [3, Theorem 2.6] that $\emptyset \neq W = U - V \in \tau_\Gamma$ and $W \cap A = \emptyset$. This contradicts that A is τ_Γ-dense. Thus $x \in \Gamma(A)$ and hence A is $\mathcal{I}_R$-dense. A similar argument shows that B is $\mathcal{I}_R$-dense. Thus $(X, \tau, \mathcal{I})$ is Γ-resolvable. □

Theorem 3.5. *A space $(X, \tau, \mathcal{I})$ is Γ-resolvable if and only if there exists an $\mathcal{I}_R$-dense set D such that for all nonempty $U \in Ro(X)$ and all $A \in \mathcal{I}$, $U - A \neq \emptyset$ implies $(U - A) \nsubseteq D$.*

Proof. Let $(X, \tau, \mathcal{I})$ be Γ-resolvable. Then, by Remark 1.1 and Theorem 1.2, $\mathcal{I}$ is regular. Now there exist two disjoint $\mathcal{I}_R$-dense sets, say D_1 and D_2. We show that $(U - A) \nsubseteq D_1$ whenever $U - A \neq \emptyset$ for all $\emptyset \neq U \in Ro(X)$ and $A \in \mathcal{I}$. If possible let $(U - A) \subseteq D_1$ for some $\emptyset \neq U \in Ro(X)$ and $A \in \mathcal{I}$. Then $(U - A) \cap D_2 = \emptyset$. Now since $\mathcal{I}$ is regular, by Theorem 2.6 D_2 is not $\mathcal{I}_R$-dense. This is contrary that D_2 is $\mathcal{I}_R$-dense. Hence $(U - A) \nsubseteq D_1$ whenever $U - A \neq \emptyset$ for all $\emptyset \neq U \in Ro(X)$ and $A \in \mathcal{I}$.

Conversely let the condition hold in $(X, \tau, \mathcal{I})$. Then there exists an $\mathcal{I}_R$-dense set D such that $(U - A) \nsubseteq D$ if $U - A \neq \emptyset$ for all $\emptyset \neq U \in Ro(X)$ and $A \in \mathcal{I}$. We show that $X - D$ is $\mathcal{I}_R$-dense. Let $X - D$ be not $\mathcal{I}_R$-dense. Then there exists $\emptyset \neq V \in Ro(X)$ such that $V \cap (X - D) \in \mathcal{I}$. Clearly $V \cap (X - D) \neq \emptyset$, for otherwise $V \subseteq D$, which is contrary to our assumption. Let $A = V \cap (X - D)$. Then $V - A \neq \emptyset$. For if $V - A = \emptyset$ then $V \subseteq A$ and hence $V \in \mathcal{I}$ which implies $V \cap D \in \mathcal{I}$. This is contrary that D is $\mathcal{I}_R$-dense. Therefore, $V - A \subseteq D$, which is again contrary to our assumption. Thus $X - D$ is $\mathcal{I}_R$-dense and hence $(X, \tau, \mathcal{I})$ is Γ-resolvable. □

Corollary 3.6. *An ideal space $(X, \tau, \mathcal{I})$ is Γ-irresolvable if and only if for each $\mathcal{I}_R$-dense set D, there exist $U \in Ro(X)$ and $A \in \mathcal{I}$ such that $\emptyset \neq (U - A) \subseteq D$.*

Lemma 3.7. *Let Y be a subspace of a topological space X and $\mathcal{I}$ be an ideal in X. Then $\mathcal{I}_Y = \{I \in \mathcal{I} : I \subseteq Y\} = \{I \cap Y : I \in \mathcal{I}\}$ is an ideal in Y.*

Theorem 3.8. *Let $(X, \tau, \mathcal{I})$ be an ideal space such that $\mathcal{I}$ is regular. If D is $\mathcal{I}_R$-dense in $(X, \tau, \mathcal{I})$, then for all $Y = U - A$, where $\emptyset \neq U \in Ro(X)$ and $A \in \mathcal{I}$, $Y \cap D$ is $\mathcal{I}_R$-dense in $(Y, \tau_Y, \mathcal{I}_Y)$.*

Proof. Clearly we suppose $\mathcal{I}$ is regular. A regular open set in Y is of the form $Y \cap O = (U - A) \cap O = (U \cap O) - A$, where O is regular open in (X, τ). Let $\emptyset \neq U \cap O - A$. Consider $\emptyset \neq ((U \cap O) - A) - B$, $B \in \mathcal{I}_Y$. Then since D is $\mathcal{I}_R$-dense and $U \cap O$ is regular open in (X, τ), by Theorem 2.6 $(U \cap O - (A \cup B)) \cap D \neq \emptyset$. Hence $(((U \cap O) - A) - B) \cap D \neq \emptyset$. Therefore again by Theorem 2.6 $Y \cap D$ is $\mathcal{I}_R$-dense in $(Y, \tau_Y, \mathcal{I}_Y)$. □

Lemma 3.9 ([10]). *Let (X, τ) be a topological space and Y be open or dense in X. Then*

(1) $Ro(Y) = \{A \cap Y : A \in Ro(X)\}$.

(2) $Rc(Y) = \{A \cap Y : A \in Rc(X)\}$.
(3) $(\tau_Y)_S = (\tau_S)_Y$.

Theorem 3.10. *Let $(X, \tau, \mathcal{I})$ be an ideal space such that $\mathcal{I}$ is regular and $P \subseteq Y = U - A$, where $\emptyset \neq U \in Ro(X)$, $A \in \mathcal{I}$, and Y is open or dense in X. Then P is $\mathcal{I}_R$-dense in $(Y, \tau_Y, \mathcal{I}_Y)$ if and only if $P = Y \cap D$, where D is $\mathcal{I}_R$-dense in $(X, \tau, \mathcal{I})$.*

Proof. Let P be $\mathcal{I}_R$-dense in $(Y, \tau_Y, \mathcal{I}_Y)$. Consider the set $P \cup (X - Y)$. Then $(P \cup (X - Y)) \cap O = (P \cap O) \cup ((X - Y) \cap O)$, where $\emptyset \neq O \in Ro(X)$. Now if $O \subseteq X - Y$, then $P \subseteq Y$ and $P \cap O = \emptyset$ and we have $(P \cup (X - Y)) \cap O = O$ which is not in $\mathcal{I}$ because $\mathcal{I}$ is regular. Finally if $O \cap Y \neq \emptyset$, then since P is $\mathcal{I}_R$-dense in $(Y, \tau_Y, \mathcal{I}_Y)$, $P \cap (O \cap Y) \notin \mathcal{I}_Y$ and hence $P \cap O \notin \mathcal{I}$. Therefore $(P \cup (X - Y)) \cap O \notin \mathcal{I}$. Thus $(P \cup (X - Y)) = D$ say is $\mathcal{I}_R$-dense in $(X, \tau, \mathcal{I})$ and hence $P = Y \cap D$. Conversely, let $P = Y \cap D$, where D is $\mathcal{I}_R$-dense in $(X, \tau, \mathcal{I})$. Then, by Theorem 3.8, P is $\mathcal{I}_R$-dense in $(Y, \tau_Y, \mathcal{I}_Y)$. This completes the proof of the theorem. □

4. Regular Γ-irresolvable spaces

We shall now define and discuss properties of a regular Γ-irresolvable space.

Definition 4.1. An ideal space $(X, \tau, \mathcal{I})$ is said to be regular Γ-irresolvable if for each $\mathcal{I}_R$-dense set D and each $\emptyset \neq U \in Ro(X)$ and $A \in \mathcal{I}$ such that $\emptyset \neq U - A$, there exist $\emptyset \neq V \in Ro(X)$ and $B \in \mathcal{I}$ such that $\emptyset \neq (V - B) \subseteq (U - A) \cap D$.

Theorem 4.2. *An ideal space $(X, \tau, \mathcal{I})$ is regular Γ-irresolvable, where $\mathcal{I}$ is regular, if and only if the intersection of any two $\mathcal{I}_R(X)$-dense sets is an $\mathcal{I}_R(X)$-dense set.*

Proof. Let $(X, \tau, \mathcal{I})$ be regular Γ-irresolvable and $\mathcal{I}$ be regular. Let D_1 and D_2 be two $\mathcal{I}_R(X)$-dense sets in $(X, \tau, \mathcal{I})$. We show that $D_1 \cap D_2$ is $\mathcal{I}_R(X)$-dense. Consider $U - A$, where $\emptyset \neq U \in Ro(X)$ and $A \in \mathcal{I}$. We show that $(U - A) \cap D_1 \cap D_2 \neq \emptyset$. Since D_1 is $\mathcal{I}_R(X)$-dense, by Theorem 2.6 $(U - A) \cap D_1 \neq \emptyset$. Since $(X, \tau, \mathcal{I})$ is regular Γ-irresolvable, there exist $\emptyset \neq V_1 \in Ro(X)$ and $B_1 \in \mathcal{I}$ such that $\emptyset \neq (V_1 - B_1) \subseteq (U - A) \cap D_1$. Again since D_2 is $\mathcal{I}_R(X)$-dense, there exist $\emptyset \neq V_2 \in Ro(X)$ and $B_2 \in \mathcal{I}$ such that $\emptyset \neq (V_2 - B_2) \subseteq (V_1 - B_1) \cap D_2$. Hence $\emptyset \neq V_2 - B_2 \subseteq (U - A) \cap D_1 \cap D_2$. Therefore, $(U - A) \cap (D_1 \cap D_2) \neq \emptyset$ and by Theorem 2.6 $D_1 \cap D_2$ is $\mathcal{I}_R(X)$-dense.

Conversely let the intersection of any two $\mathcal{I}_R(X)$-dense sets is $\mathcal{I}_R(X)$-dense. Suppose that $(X, \tau, \mathcal{I})$ is not regular Γ-irresolvable. Then there exist an $\mathcal{I}_R(X)$-dense set D_1, $\emptyset \neq U \in Ro(X)$ and $A \in \mathcal{I}$, where $\emptyset \neq U - A$, such that $(U - A) \cap D_1$ does not contain $V - B$, for any $\emptyset \neq V \in Ro(X)$ and $B \in \mathcal{I}$. Consider the set $D_2 = (X - (U - A)) \cup ((U - A) - (U - A) \cap D_1)$. By Theorem 2.6, D_2 is $\mathcal{I}_R(X)$-dense since $(V - B) \cap D_2 \neq \emptyset$. But $(U - A) \cap D_1 \cap D_2 = \emptyset$. This is contrary to the fact that the intersection of two $\mathcal{I}_R(X)$-dense sets is an $\mathcal{I}_R(X)$-dense set. Hence $(X, \tau, \mathcal{I})$ must be regular Γ-irresolvable. This completes the proof of the theorem. □

Theorem 4.3. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $\mathcal{I}$ be regular. If $(X, \tau, \mathcal{I})$ is regular Γ-irresolvable, then $(Y, \tau_Y, \mathcal{I}_Y)$ is regular Γ-irresolvable whenever $Y = U - A$ is open or dense in X, for every $\emptyset \neq U \in Ro(X)$ and $A \in \mathcal{I}$.*

Proof. Let D and G be $\mathcal{I}_R(Y)$-dense sets in $(Y, \tau_Y, \mathcal{I}_Y)$. Then, by Theorem 3.10, $D = (U - A) \cap D_1$ and $G = (U - A) \cap D_2$, where D_1 and D_2 are $\mathcal{I}_R(X)$-dense sets in $(X, \tau, \mathcal{I})$. Hence $D \cap G = (U - A) \cap D_1 \cap D_2$ and, since $D_1 \cap D_2$ is an $\mathcal{I}_R(X)$-dense set in $(X, \tau, \mathcal{I})$, again by Theorem 3.10, $D \cap G$ is $\mathcal{I}_R(Y)$-dense in $(Y, \tau_Y, \mathcal{I}_Y)$. Hence by Theorem 4.2 $(Y, \tau_Y, \mathcal{I}_Y)$ is regular Γ-irresolvable. □

Definition 4.4. An ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_R^*$-hyperconnected if each $\emptyset \neq U - A$, where $U \in Ro(X)$ and $A \in \mathcal{I}$, is $\mathcal{I}_R(X)$-dense.

Theorem 4.5. *An ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}_R^*$-hyperconnected if and only if $(X, \tau, \mathcal{I})$ is $\mathcal{I}_R$-hyperconnected and $\mathcal{I}$ is regular.*

Proof. Let $(X, \tau, \mathcal{I})$ be $\mathcal{I}_R^*$-hyperconnected. Clearly $(X, \tau, \mathcal{I})$ is $\mathcal{I}_R$-hyperconnected. Let $\emptyset \neq U$ be regular open and a member of the ideal. Then $\Gamma(U) = X$ since $(X, \tau, \mathcal{I})$ is $\mathcal{I}_R$-hyperconnected. On the other hand, since $U \in \mathcal{I}$, $\Gamma(U) = \emptyset$, which is a contradiction. Hence $\mathcal{I}$ is regular.

Conversely let $(X, \tau, \mathcal{I})$ be $\mathcal{I}_R$-hyperconnected and $\mathcal{I}$ be regular. Consider $U - A$, where $\emptyset \neq U \in Ro(X)$ and $A \in \mathcal{I}$. Then $\emptyset \neq U - A$ because $\mathcal{I}$ is regular. We show that $U - A$ is $\mathcal{I}_R(X)$-dense. Let $x \in X$ and V be a regular open set containing x. By Lemma 2.5, (X, τ) is R-hyperconnected and $V \cap (U - A) \neq \emptyset$ because $V \cap (U - A) = V \cap U - A \neq \emptyset$ and $\mathcal{I}$ is regular. Thus $(X, \tau, \mathcal{I})$ is $\mathcal{I}_R^*$-hyperconnected. □

Theorem 4.6. *If an ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}_R^*$-hyperconnected and Γ-irresolvable, then it is regular Γ-irresolvable.*

Proof. By Theorem 4.5, $\mathcal{I}$ is regular. Let D_1 and D_2 be two $\mathcal{I}_R(X)$-dense sets in $(X, \tau, \mathcal{I})$. We show that $D_1 \cap D_2$ is $\mathcal{I}_R(X)$-dense. By Theorem 2.6 it is sufficient to prove that $(D_1 \cap D_2) \cap (U - A) \neq \emptyset$ for all $\emptyset \neq U \in Ro(X)$ and $A \in \mathcal{I}$. Since $(X, \tau, \mathcal{I})$ is Γ-irresolvable, by Corollary 3.6 there exist $\emptyset \neq V \in Ro(X)$ and $B \in \mathcal{I}$ such that $\emptyset \neq V - B \subseteq D_1$. Similarly there exist $\emptyset \neq W \in Ro(X)$ and $C \in \mathcal{I}$ such that $\emptyset \neq W - C \subseteq D_2$. Now (X, τ) is R-hyperconnected by Theorem 4.5 and $V \cap W \neq \emptyset$. Since $\mathcal{I}$ is regular, $(V - B) \cap (W - C) = (V \cap W) - (B \cup C) \neq \emptyset$ and hence $(V \cap W) - (B \cup C) \subseteq D_1 \cap D_2$. Therefore, by $\mathcal{I}_R^*$-hyperconnectedness of $(X, \tau, \mathcal{I})$, $(V \cap W) - (B \cup C)$ is $\mathcal{I}_R(X)$-dense and, by Theorem 2.6, we have $\emptyset \neq (U - A) \cap [(V \cap W) - (B \cup C)]$ and hence $(U - A) \cap (D_1 \cap D_2) \neq \emptyset$. Therefore, $D_1 \cap D_2$ is $\mathcal{I}_R(X)$-dense. Thus by Theorem 4.2, $(X, \tau, \mathcal{I})$ is regular Γ-irresolvable. □

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