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Research Article

Stability and Period-Doubling Bifurcation in a Modified Commensal Symbiosis Model with Allee Effect

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Abstract

In this article, the qualitative behaviour of discrete-time commensal symbiosis model which is obtained by implementing the forward Euler's scheme is discussed in detail. Firstly, the local stability conditions of fixed points of the model are studied. It is proved that the considered model undergoes Period-Doubling bifurcation around coexistence fixed point with the help of bifurcation theory. In order to support the accuracy of obtained analytical finding, some parameter values have been determined and numerical simulations are carried out for these parameter values. Numerical simulations display new and rich nonlinear dynamical behaviours. More specifically, when the parameter δ is chosen as a bifurcation parameter, it is seen that the considered discrete-time commensal symbiosis model shows very rich nonlinear dynamical.

Keywords: Allee effect, stability, Period-Doubling bifurcation

Allee Etkisi Altında Modifiye Edilmiş Kesik Zamanlı Bir Ortak Yaşam Modelinin Kararlılığı ve Period-Doubling Çatallanması

Öz

Bu makalede, ileri Euler şeması uygulanarak elde edilen ayrık zamanlı ortak yaşam modelinin nitel davranışı ayrıntılı bir şekilde tartışılmıştır. İlk olarak bu modelin sabit noktalarının lokal kararlılık koşulları incelenmiştir. Çatallanma teorisi yardımı ile ele alınan modelin her iki türün bir arada olduğu sabit nokta civarında Period-Doubling çatallanmaya gittiği ispatlanmıştır. Elde edilen bu teorik sonuçların doğruluğunu desteklemek için bazı parametre değerleri belirlenmiş ve bu parametre değerleri için nümerik gösterimler verilmiştir. Nümerik simülasyonlar yeni ve zengin doğrusal olmayan dinamik davranışları göstermektedir. Özel olarak δ parametresi çatallanma parametresi olarak seçildiğinde ele alınan ayrık zamanlı ortak yaşam modelinin daha zengin doğrusal olmayan dinamik davranışlar sergilediği görülmektedir.

Anahtar Kelimeler: Allee Etkisi, kararlılık, Period-Doubling çatallanma

1. Introduction

Commensalism is a long-term biological symbiosis (interaction) in which members of one species gain benefits while those of the other species neither benefit nor harmed are (Wilson and Amin,1975). The commensal may obtain nutrients, shelter, support, or locomotion from the host species, which is substantially unaffected. The commensal relation is often between a larger host and a smaller commensal (Williams et al., 2003). An intraspecific commensal model was proposed by Sun and Wei (Sun and Wei, 2003). The authors studied the local stability of

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the fixed point for this model. Miao et al. (2015) analyzed the persistent property of the periodic Lotka-Volterra commensal symbiosis model including impulsive. During the last decades, there are many extensive articles on dynamic behaviours of the mutualism or commensalism model, for example, see (Sun and Wei, 2003; He and Chen, 2009; Miao et al., 2015) and the references therein. Xie et al. (2015) introduced a discrete commensal symbiosis model, and they studied the positive w-periodic solution of the considered model.

Differential and difference equations have been used to investigate a wide range of population models. The populations models governed by differential equations have been studied extensively by many researchers (Chen et al, 2009; Feng and Kang, 2015; Lin, 2018; Wu et al., 2018) and the reference therein. However, in recent years considerable number articles in literature discussed the dynamic of the discretization of the models governed by differential equations (Kartal, 2016; Kartal et al., 2016; Din, 2018). Because the discrete-time models present dynamics consistency according to continuous-time. In addition, the discrete-time population models are suitable for non-overlapping generations and are appropriate to describe the nonlinear dynamics and possibility their chaotic behaviour.

Bifurcation theory is research field that analyzes the change of dynamical models with respect to a control parameter. So, the behaviour the model according to a control parameter is observed. There are many studies on bifurcation analysis (Sen et al., 2012; Rana, 2015; Kangalgil and Kartal, 2018; Eskandri and Alidousti, 2020; Selvam et al., 2020).

The Allee effect, a reduction of the per capita growth rate of a population of biological species at densities smaller than a critical value, was first introduced by Allee in 1931 (Allee, 1931). Although there are many extensive research on Allee effect, many articles have not addressed the Allee effect with focus on bifurcation analysis of discrete-time model (Cheng and Cao, 2016; Kangalgil, 2020). Thus, in this work it is investigated that bifurcation and stability analysis of a modified a discrete-time model which developed from the inclusion of Allee effect in the second species.

In Wu et al. (2018), the authors have considered the following continuous-time model involving Allee effect in the second population:

$$\begin{cases} \dfrac{dx}{dt} = x\left(a_1 - b_1 x + \dfrac{c_1 y^p}{1+y^p}\right) \\ \dfrac{dy}{dt} = (a_2 - b_2 y)\dfrac{y}{m+y}, \end{cases} \tag{1}$$

where $a_i, b_i, i = 1,2$ p, m and c_1 are positive constants, $p \geq 1$. The term $f(y) = \frac{y}{m+y}$ is Allee effect. The authors investigated the dynamic behaviours of the model (1).

The aim of this article is to discuss the dynamics of the discrete-time model obtained by applying the forward Euler scheme to model (1)

$$x_{t+1} = x_t + \delta x_t[a_1 - b_1 x_t + \frac{c_1 {y_t}^p}{1 + {y_t}^p}]$$
$$y_{t+1} = y_t + \delta y_t[\frac{a_2 - b_2 y_t}{m + y_t}] \tag{2}$$

where $x(t)$ and $y(t)$ represent population densities of prey and predator at time t, respectively. a_1 and a_2 represent the instrinsic prey and predator growth rates, respectively. b_1 describes the competition among individuals of prey. b_2 has similar meaning to b_1. $\frac{a_2}{b_2}$ is the environment carrying capacity of the second species. Moreover, $\delta > 0$ is the step size. It is known that the Allee effect is a crucial phenomenon in the biological literature and has the following property (Zhou et al., 2005; Kangalgil, 2019):

1. $f'(y) = \frac{m}{(m+y)^2}$ for all $y\epsilon(0, \infty)$, it means that Allee effect decreases as density increases.
2. $lim_{y\to\infty}$f(y)=1, that is, the Allee effect vanishes at high densities.

This manuscript is organized as follows: in the next section, stability of the fixed points of the model will be introduced. Section 3 is Period-Doubling bifurcation analysis for the model (2). δ parameter is chosen as a bifurcation parameter. Moreover, direction of related to bifurcation is obtained by using normal form theory (Elaydi, 1996; Kuznetsov, 1998; Wiggins, 2003). At the end of each section, numerical simulations are presented to show the effectiveness of the acquired theoretical findings. Finally, in section 4, this paper is ended by a briefly discussion.

2. Material and Methods

In this section, it is investigated the stability conditions of the fixed points of the model (2). To determine the fixed points we have to solve the following nonlinear system:

$$\begin{cases} x^* + \delta x^* \left[a_1 - b_1 x^* + \dfrac{c_1 y^{*p}}{1 + y^{*p}}\right] = x^* \\ y^* + \delta y^* \left[\dfrac{a_2 - b_2 y^*}{m + y^*}\right] = y^* \end{cases} \tag{3}$$

From the definition of the fixed point, for all the parameters, the model (2) has four fixed points:

$$E_0 = (0,0), \qquad E_1 = \left(\frac{a_1}{b_1}, 0\right),$$

$$E_2 = \left(0, \frac{a_2}{b_2}\right), \quad E_3 = \left(x^*, \frac{a_2}{b_2}\right),$$

where

$$x^* = \frac{a_1 + a_1\left(\frac{a_2}{b_2}\right)^p + c_1\left(\frac{a_2}{b_2}\right)^p}{b_1(1+\left(\frac{a_2}{b_2}\right)^p)}.$$

On the other hand, the Jacobian matrix of the map (2) evaluated at any point (x,y) is given by:

$$J(x,y) = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

where

$$a_{11} = 1 + \delta(a_1 - 2b_1x) + \delta\frac{c_1y^p}{1+y^p},$$
$$a_{12} = \frac{pc_1y^{p-1}}{(1+y^p)^2}(1 - y^p),$$
$$a_{22} = 1 + \frac{\delta(a_2m - b_2y(2m+y)}{(m+y)^2}.$$

The characteristic polynomial is

$$F(\lambda) = \lambda^2 - tr\big(J(x,y)\big)\lambda + \det\big(J(x,y)\big). \tag{4}$$

Furthermore,

$$J(E_0) = \begin{pmatrix} 1 + \delta a_1 & 0 \\ 0 & 1 + \frac{\delta a_2}{m} \end{pmatrix},$$

$$J(E_1) = \begin{pmatrix} 1 - \delta a_1 & 0 \\ 0 & 1 + \frac{\delta a_2}{m} \end{pmatrix},$$

$$J(E_2) = \begin{pmatrix} 1 + \delta a_1 + \frac{\delta c_1 a_2^p}{a_2^p + b_2^p} & 0 \\ 0 & 1 - \frac{\delta a_2 b_2}{mb_2 + a_2} \end{pmatrix},$$

$$J(E_3) = \begin{pmatrix} 1 - \frac{2\delta}{\delta_2} & \Omega \\ 0 & 1 - \frac{2\delta}{\delta_1} \end{pmatrix}$$

are obtained. Where

$$\delta_1 = 2\left(\frac{m}{a_2} + \frac{1}{b_2}\right), \delta_2 = \frac{2\left(a_2^p + b_2^p\right)}{(a_1 + c_1)a_2^p + a_1 b_2^p},$$

$$\Omega = \frac{\delta c_1 p b_2 (\frac{a_2}{b_2})^p \left[a_1(1 + \left(\frac{a_2}{b_2}\right)^p) + c_1\left(\frac{a_2}{b_2}\right)^p\right]}{a_2 b_1 (1 + \left(\frac{a_2}{b_2}\right)^p)^3}.$$

Let’s give the following Definition 2.1 before analyzing stability.

Definition 2.1. (Khan, 2016; Kangalgil, 2019) A fixed point (x,y) is called

i) sink, if $|\lambda_1|<1$ and $|\lambda_2|<1$. A sink is always locally asymptotically stable,

ii) source, if $|\lambda_1|>1$ and $|\lambda_2|>1$. A source is locally unstable,

iii) saddle, if $|\lambda_1|<1$ and $|\lambda_2|>1$or $|\lambda_1|>1$and $|\lambda_2|<1$,

iv) non-hyperbolic, if either $|\lambda_1|=1$ or $|\lambda_2|=1$.

Using Jury’s criterion (Elaydi, 1996), it is obtained that the following stability conditions of the fixed points of the model (2).

Proposition 2.1. The trivial fixed point E_0 is an unstable.

Proof.

Because of the parameters δ, a_1, a_2, m are positive, $\lambda_1 = 1 + \delta a_1 > 1$ and $\lambda_2 = 1 + \frac{\delta a_2}{m}>1$ are valid. Hence, the trivial fixed point E_0 is unstable.

It is stated the following proposition about stability conditions of E_1 and E_2 without proof.

Proposition 2.2. For the predator free fixed point E_1, following topological classification holds:

(i) E_1 is not sink. Because one of the eigenvalues of associated with J(E_1) of the model (2) is bigger than zero for the all parameter.

(ii) E_1 is source, if $\delta > \frac{2}{a_1}$.

(iii) E_1 is saddle, if $0 < \delta < \frac{2}{a_1}$.

(iv) E_1 is non-hyperbolic, if $\delta = \frac{2}{a_1}$.

Proposition 2.3. For the prey free fixed point E_2 , following topological classification holds:

(i) E_2 is not sink. Because one of the eigenvalues of associated with $J(E_2)$ of the model (2) is bigger than zero for the all parameter.

(ii) E_2 is source, if $\delta > 2(\frac{m}{a_2} + \frac{1}{b_2})$.

(iii) E_2 is saddle, if $0 < \delta < 2(\frac{m}{a_2} + \frac{1}{b_2})$.

(iv) E_2 is non-hyperbolic, if $\delta = 2(\frac{m}{a_2} + \frac{1}{b_2})$.

On the other hand, the characteristic equation of the matrix $J(E_3)$ is

$$\lambda^2 - \left(2 - 2\delta\left(\frac{1}{\delta_1} + \frac{1}{\delta_2}\right)\right)\lambda + \left(1 - \frac{2\delta}{\delta_1}\right)\left(1 - \frac{2\delta}{\delta_2}\right) = 0 \tag{5}$$

The matrix $J(E_3)$ has two eigenvalues as

$$\lambda_1 = 1 - \frac{2\delta}{\delta_1} \text{ and } \lambda_2 = 1 - \frac{2\delta}{\delta_2}.$$

Using Definition 2.1, the local stability conditions of the model (2) can be presented as the following Proposition2.4:

Proposition 2.4. For the coexistence fixed point E_3 , following topological classification holds:

(i) E_3 is sink, if $0 < \delta < \min\{\delta_1, \delta_2\}$.

(ii) $E_3$is source, if $\delta > max\{\delta_1, \delta_2\}$.

(iii) E_3 is saddle, if $min\{\delta_1, \delta_2\} < \delta < max\{\delta_1, \delta_2\}$.

(iv) E_3 is non-hyperbolic, if $\delta = \frac{\delta_1}{2}$ or $\delta = \frac{\delta_2}{2}$.

Example 2.1. Consider the model (2) for the parameters values $a_1 = 1$, $a_2 = 2$,$b_1 = 1.2$,$b_2 = 1$, $c_1 = 1.4$, p=1, m=0.1, δ=0.05 and initial condition (x_0, y_0)=(1.3,2.2). The positive coexistence fixed point of the model (2) is obtained as $E_3 = (1.61111111, 2)$. So, Figure 1 support Propotion 2.4.

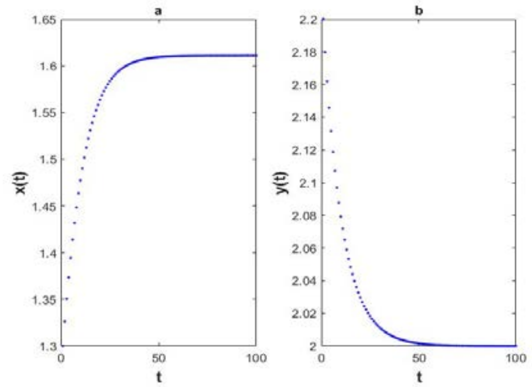

Figure 1. A stable coexistence fixed point for the model (2) for the parameters values $a_1 = 1, a_2 = 2$, $b_1 = 1.2$, $b_2 = 1$, $c_1 = 1.4$, p=1, m=0.1, δ=0.05 and initial condition (x_0, y_0)=(1.3,2.2).

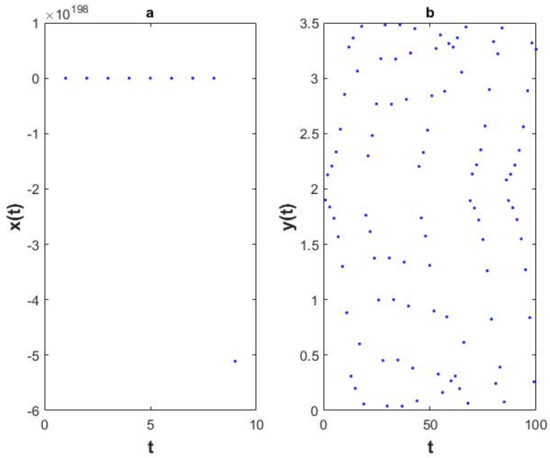

Figure 2. A unstable coexistence fixed point for the model (2) for the parameters values $a_1 = 1, a_2 = 2$, $b_1 = 1.2$, $b_2 = 1$, $c_1 = 1.4$, p=1, m=0.1, δ=2.39 and initial condition (x_0, y_0)=(1.6,1.9).

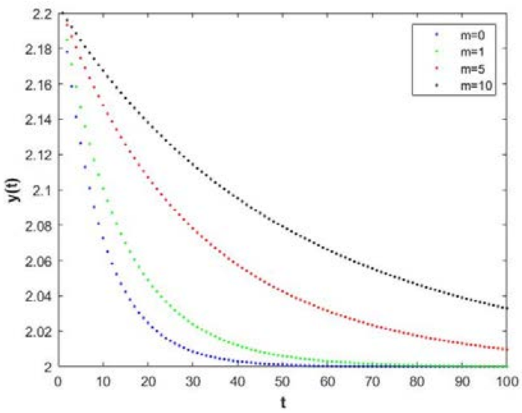

Figure 3. Numeric simulations of y(t) for the different values of m Allee constant and $a_1 = 1, a_2 = 2$, $b_1 = 1.2$, $b_2 = 1$, $c_1 = 1.4$, p=1, δ=0.05 and initial condition (x_0, y_0)=(1.3,2.2).

3. Results and Discussion

3.1. Period-Doubling Bifurcation

By using the bifurcation theory in (Elaydi, 1996; Kuznetsov, 1998; Wiggins, 2003), it will been investigated the conditions and direction of Period-Doubling bifurcation at the coexistence fixed point.

Theorem 3.1. (Kangalgil, 2019) Assume that $F(\lambda)$ is characteristic polynomial function of the model (2). For the model (2) one of the eigenvalues -1 and the other eigenvalues lie inside the unit circle if and only if

(i) $F(1) > 0$,
(ii) $F(-1) = 0$,
(iii) $1 + det\, J(E_3) > 0$,
(iv) $1 - \det \mathrm{J}(E_3) > 0$.

Lemma 3.1. (Eigenvalue Assignment)

Let $0 < \delta < \frac{\delta_1+\delta_2}{2}$. If $\delta_F = \delta_2$, then the eigenvalue assignment condition of Period-Doubling bifurcation in Theorem 3.1.

Proof.

Because of all the parameters are positive, it is satisfied $F(1) > 0$. On the other hand,

$$F(-1) = 4 - 4\delta\left(\frac{1}{\delta_1} + \frac{1}{\delta_2}\right) + \frac{4\delta^{\,2}}{\delta_1\delta_2} = 0 \tag{6}$$

which gives $\delta = \delta_1$ and $\delta = \delta_2$.

From condition (iii) of Theorem 3.1, it is obtained that the following inequality

$$\frac{4\delta^{\,2} - 2\delta(\delta_1+\delta_2) + 2\delta_1\delta_2}{\delta_1\delta_2} > 0 \tag{7}$$

The inequality (7) gives $2\delta^{\,2} + \delta_1\delta_2 > \delta(\delta_1+\delta_2)$.

From the condition (iv) of Theorem 1,

$$\frac{2\delta(\delta_1+\delta_2 - 2\delta)}{\delta_1\delta_2} > 0 \tag{8}$$

is obtained. The inequality (8) leads to $0 < \delta < \frac{\delta_1+\delta_2}{2}$.

This completes the proof.

Now, it is easy to check that the Jacobian matrix *J* has the eigenvalues $\lambda_1(\delta_2) = 1 - \frac{2\delta_2}{\delta_1}$, and $\lambda_2(\delta_2) = -1$.

So, it is showed that the correctness Lemma 2.

By the change of variables,

$$\begin{aligned} x &= x^* - X \\ y &= y^* - Y \end{aligned} \tag{9}$$

the coexistence fixed point E_3 converts the origin. Thus it can be computed the coefficients of the normal form. From (9), the model (2) can be rewritten as the following:

$$X_{t+1} = JX_t + \frac{1}{2}B(X_t, X_t) + \frac{1}{2}C(X_t, X_t, X_t) + O({X_t}^4)$$

where $J = J(\delta_F)$.

In addition, the multininear function *B* and *C* are defined by

$$B_i(x, y) = \sum\nolimits_{j,k=1}^{2} \frac{\partial^2 F_i(\varepsilon, 0)}{\partial\varepsilon_j \partial\varepsilon_k}\bigg|_{\varepsilon=0} x_j y_k, i = 1,2$$

and

$$C_i(x, y, z) = \sum\nolimits_{j,k,l=1}^{2} \frac{\partial^3 F_i(\varepsilon, 0)}{\partial\varepsilon_j \partial\varepsilon_k \partial\varepsilon_l}\bigg|_{\varepsilon=0} x_j y_k z_l,\ \ i = 1,2.$$

The value of *B* and *C* of the model (2) can be evaluated as

$$B_1(x, y) = -2\delta b_1^2 a_2^3 \frac{\left(b_2^4\left(1 + \left(\frac{a_2}{b_2}\right)^p\right)^5 + 5b_2^4\left(1 + \left(\frac{a_2}{b_2}\right)^p\right)^4 + a_2^4\right)}{\left(1 + \left(\frac{a_2}{b_2}\right)^p\right)^5 b_2^4 b_1 a_2^3} x_1 y_1$$

$$+\delta c_1 p a_2 b_2^6 (\frac{\left(b_1\left(1 + \left(\frac{a_2}{b_2}\right)^p\right)^3\left(1 + \left(\frac{a_2}{b_2}\right)^p + p\left(\frac{a_2}{b_2}\right)^p\left(\left(\frac{a_2}{b_2}\right)^{2p} - \left(\frac{a_2}{b_2}\right)^p - 1\right)\right)\right)}{\left(1 + \left(\frac{a_2}{b_2}\right)^p\right)^5 b_2^4 b_1 a_2^3}$$

$$+\frac{c_1(\left(1+\left(\frac{a_2}{b_2}\right)^p\right)^4-\left(\frac{a_2}{b_2}\right)^{2p}\left(2\left(\frac{a_2}{b_2}\right)^p-5\right)+p\left(\frac{a_2}{b_2}\right)^p\left(\left(\frac{a_2}{b_2}\right)^{2p}-1\right))}{\left(1+\left(\frac{a_2}{b_2}\right)^p\right)^5 b_2^4 b_1 a_2^3}$$

$$c(0)=\frac{1}{6}< p, C(q,q,q) > -\frac{1}{2}< p, B\big(q,(J-I)^{-1}B(q,q)\big) >. \tag{10}$$

$$C_1(x,y,z)=\frac{6\delta c_1 p b_2^3\left(\frac{a_2}{b_2}\right)^p}{b_1 a_2^3\left(1+\left(\frac{a_2}{b_2}\right)^p\right)^5}(c_1\left(\frac{a_2}{b_2}\right)^p(p^2\left(4\left(\frac{a_2}{b_2}\right)^p-1\right)+3p-2-4\left(\frac{a_2}{b_2}\right)^p-\left(\frac{a_2}{b_2}\right)^{2p}(p^2+5))$$

$$+a_1(-p\left(\frac{a_2}{b_2}\right)^{3p}(3+p)+3p\left(1+\left(\frac{a_2}{b_2}\right)^p\right)+$$
$$3p\left(\frac{a_2}{b_2}\right)^p\left(p-\left(\frac{a_2}{b_2}\right)^p\right)+\left(\frac{a_2}{b_2}\right)^{2p}(3p^2-6)-6\left(\frac{a_2}{b_2}\right)^p-2-p^2))x_2y_2z_2$$

$$+\frac{a_1\left(\left(1+\left(\frac{a_2}{b_2}\right)^p\right)^4-\left(\frac{a_2}{b_2}\right)^p\left(\left(\frac{a_2}{b_2}\right)^{2p}+3\left(\frac{a_2}{b_2}\right)^p+3\right)+p\left(\frac{a_2}{b_2}\right)^p\left(1+\left(\frac{a_2}{b_2}\right)^p\right)^2(\left(\frac{a_2}{b_2}\right)^p-1)\right)}{\left(1+\left(\frac{a_2}{b_2}\right)^p\right)^5 b_2^4 b_1 a_2^3})x_2y_2$$

$$B_2(x,y)=-\frac{2\delta b_2^3 m}{(mb_2+a_2)^3}x_2y_2$$

$$C_2(x,y,z)=\frac{6\delta b_2^4 m}{(mb_2+a_2)^3}x_2y_2z_2$$

and δ=δ_2.

It is well known that J(δ_2) has eigenvalue $\lambda_2(\delta_2)=-1$, and the corresponding eigenspace E^C is one dimensional and spanned by an eigenvector $q\epsilon R^2$ such that J(δ_2)q=-q. Assume that pϵR^2 be the adjoint eigenvector, that is $J^T(\delta_2)p$=-p. By a direct calculation it is obtained that

$$q\sim(1,0)^T,$$

$$p\sim(\frac{2(\delta_2-\delta_1)a_2b_1({a_2}^p+{b_2}^p)^3}{\delta_1\delta_2 p c_1 b_2((a_1+c_1{b_2}^p){a_2}^{2p}+a_1{a_2}^p{b_2}^{2p})},1)^T.$$

In order to p with respect to q, we can write

$$p=(1,\frac{\delta_1\delta_2 p c_1 b_2((a_1+c_1{b_2}^p){a_2}^{2p}+a_1{a_2}^p{b_2}^{2p})}{2(\delta_1-\delta_2)a_2b_1({a_2}^p+{b_2}^p)^3})^T$$

In order to present the direction of the Period-Doubling bifurcation, it is computed the critical normal form coefficient $c(0)$ by using the following equation:

So, it is given that the following theorem.

Theorem 3.2. Assume that E_3 is a positive coexistence fixed point of the model (2). Lemma 3.1 and $c(0) \neq 0$, then the model (2) undergoes Period-Doubling bifurcation at the coexistence fixed point E_3 when the parameter δ varies in a small neighborhood of δ_2. Furthermore, if $c(0) > 0$ (respectively, $c(0) < 0$) then the period-2 orbits that bifurcate from E_3 are stable (respectively unstable).

Now, we consider the following example in order to confirm the above theoretical analysis.

Example 3.1. For the parameters values $a_1 = 1,\ a_2 = 2\ , b_1 = 1.2\ , b_2 = 1,\ \ c_1 = 1.4$,*p*=1, *m*=0.1, δ= 1.02:0.001:1.07 and initial condition (x_0, y_0)=(1.3,2.2) the positive coexistence fixed point of the model (2) is obtained as $E_3 = (1.61111111, 2)$. In addition, the critical value of Period-Doubling bifurcation is obtained as

$$\delta_F = \delta_2 = 1.034482759.$$

The Jacobian matrix $J(\delta_2)$ of the model (2) is written by

$$J(\delta_2) = \begin{pmatrix} -1 & 0.2592592593 \\ 0 & 0.01477832476 \end{pmatrix}.$$

So, the characteristic polynomial of the model (2) at the coexistence fixed point $E_3 = (1.61111111, 2)$ is given;

$$F(\lambda) = \lambda^2 + 0.98522167 62\lambda - 0.01477832477.$$

Moreover, the eigenvalues of the model (2) are

$$\lambda_1 = -1 \text{ and } \lambda_2 = 0.01477832476 < 1.$$

In addition, the conditions of the Theorem 3.1 is verified. Also, the eigenvectors $q,\ p\epsilon R^2$ corresponding to

$\lambda_1 = -1$

$q \sim (1{,}0)^T$

and

$p \sim (3.914144966{,}1)^T$.

To reach the necessary normalization <*p*,*q*>=1, it can be obtained

$q = (1,0)^T$

p=$(1,0.2554836392)^T$.

To confirm the above theoretical analysis and show the complex dynamical behaviours the bifurcation diagram for the model (2) is given in the following Figure 4.

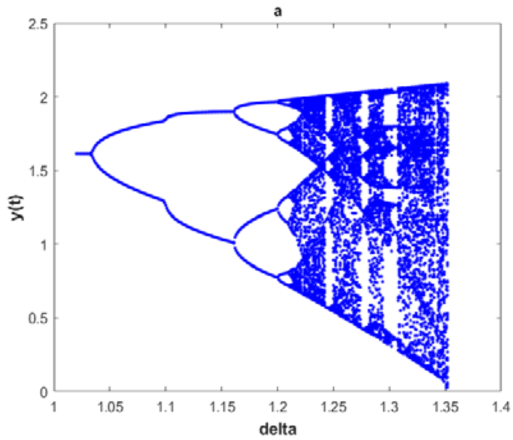

Figure 4. Bifurcation Diagram for the model (2).

4. Conclusion

In this article, the dynamic behaviour of the modified discrete-time model subject to Allee effect on second species. Firstly, it is proved that the considered model has the same fixed points as soon as the model (1). By using the method of linearization, the stability conditions of the fixed points of the model (2) is investigated. Although the considered model is similar to the model without Allee, the stability properties of the fixed points became complicated by adding the Allee effect into the model. For example, by using the Jacobian matrix the stability conditions can be analyzed for all the fixed points of the discrete-time model, howewer it can’t be analyzed for the fixed points E_0 and E_(1) of the continuous-time model. So, comparison with the continuous model in (Wu et al. 2018) says that the discrete-time model exhibits different and rich dynamical behaviours in the stability properties. According to Figure 1 and Figure 2 it can be observed that the small step size δ can stabilize the model (2), but large step may destabilize the model producing more complex dynamical behaviours. Moreover, by choosing the parameter δ is as a bifurcation parameter it is shown that the model (2) undergoes Period-Doubling bifurcation at the coexistence fixed point E_3 when δ >δ_F=1.034482759. And then, some numerical simulations are presented to verify obtained theoretical results via Matlab program. Because of consistency with the biological facts, the parameter values have been taken from literature (Kılıç, 2020).

It is shown in (Merdan and Duman, 2009) that the Allee effect reduced the densities of populations of both first and second species. Additionally, it is proved that Allee effect has no influence on the final density of the both species. (Wu et al., 2018) showed that the stronger the Allee effect, the considered model takes a longer time to reach its steady-state solution. (Lin, 2018) showed that the density of the first specie is the increasing function of the Allee effect.

Numerical simulations shows that there is no change on the final density of the prey population with increasing the Allee constant m, while the predator population is in the neighborhood of the coexistence fixed point. Such a case is expected for commensalism model.

In (Wu et al., 2018), the authors the dynamic behaviours of two species commensal symbiosis model is similar to the model without Allee Effect. On the other hand, they didn't investigate related to bifurcations of the model (2). So, the above results show that the model (2) the discrete version of the model (1) has far richer dynamics compared to the continuous model (1).

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

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