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Q-MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS RELATED WITH JANOWSKI FUNCTION

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Abstract. In the present paper, we introduce and explore certain new classes of meromorphic functions related to closed-to-convexity and q-calculus. Such results as coefficient estimates, grow the property and partial sums are derived. It is important to mention that our results are generalization of number of existing results in literature.

1. Introduction

Let $\sum_1$ denote the class of meromorphic functions of the form:

$$f(\omega) = \frac{1}{\omega} + \sum_{t=1}^{\infty} a_t \omega^t, \tag{1}$$

which are analytic in the punctured open unit disc $U^* = \{\omega : \omega \in \mathbb{C}$ and $0 < \{\omega\} < 1\} = U \backslash \{0\}$, where $U = U^* \cup \{0\}$.

In Geometric Function Theory, several subclasses of the meromorphic functions have already been examined and investigated through many perceptions, see([9,10, 12,18,21,22]). Ismail et al. [8] were the first to use the q-derivative operator Δ_q in order to study a certain q-analogue of the class T^* of starlike functions in U. Certain basic properties of the q-close-to-convex functions were studied by Raghavendar and Swaminathan [28], Aral et al. [2] successfully studied the applications of the

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q-calculus in operator theory. In fact, they found significant application of the q-calculus mainly in the Geometric Function Theory. Moreover, the generalized q-hypergeometric function was first introduced by Srivastava [26], see also([1, 3, 5, 6, 14, 16, 20]).

A function $f \in \sum_1$ is said to be meromorphic starlike of order α defined as:

$$f \in \sum^{MS}(\alpha) \Leftrightarrow \operatorname{Re}\left(\frac{\omega f'(\omega)}{f(\omega)}\right) < -\alpha \quad (0 \leq \alpha < 1;\ \omega \in U^*). \tag{2}$$

A related class of meromorphic convex function $\sum^{MC}(\alpha)$ is defined as:

$$f \in \sum^{MC}(\alpha) \Leftrightarrow \operatorname{Re}\left(1 + \frac{\omega f''(\omega)}{f'(\omega)}\right) < -\alpha \quad (\ \omega \in U^*). \tag{3}$$

By $\sum^{MK}(\alpha)$, we mean $f \in \sum_1$ and the class of all close-to-convex functions which satisfies the condition

$$\operatorname{Re}\left(\frac{\omega f'(\omega)}{g(\omega)}\right) < -\alpha, \quad \text{where } g \in \sum^{MS}(\alpha). \tag{4}$$

The study of operators plays main role in the theory of geometric functions. Many differential and integral operators can be written in terms of convolution of certain holomorphic functions.

For $g(\omega) = \frac{1}{\omega} + \sum\limits_{t=0}^{\infty} b_t \omega^t \in \sum_1$ and f given in (1). The Convolution (Hadamard product) is denoted by $f * g$ and defined as:

$$(f * g)(\omega) = \frac{1}{\omega} + \sum_{t=0}^{\infty} a_t b_t \omega^t = (g * f)(\omega). \tag{5}$$

A function h analytic in U and of the form

$$h(\omega) = 1 + \sum_{t=1}^{\infty} r_t \omega^t,$$

A given function Ψ with $\Psi(0) = 1$ is said to belong to the class $S^*[A, B]$ if and only if

$$\Psi(\omega) \prec \frac{1 + A\omega}{1 + B\omega} \qquad (-1 \leq B < A \leq 1).$$

This class was presented and studied by Janowski [11]. By taking $A = 1$ and $B = -1$, we obtain the class P of functions with a positive real part. It is important to mention that $\Psi(\omega) \in S^*[A, B]$ if and only if there exists $r \in P$ such that

$$\Psi(\omega) = \frac{(A+1) R(\omega) - (A-1)}{(B+1) R(\omega) - (B-1)} \qquad (-1 \leq B < A \leq 1).$$

Motivated by the works of Srivastava et al. see([7, 17, 19, 23, 25, 27])also see([4,13,15,24,29]). In this paper, we shall consider new subfamilies of q meromorphic close-to-convex functions with respect to Janowski functions.

Throughout in this paper, we assume

$$0 \leq \eta < 1, -1 \leq B < A \leq 1, 0 \leq q < 1, \ \omega \in U^{*}, \ f, g \in \sum_{1},$$

$$\begin{aligned} \Lambda(t, \eta, A, q) &= \left[|b_t| \left| (2[t]_q \eta + 2(1-\eta) + \eta(A+1)) - (A+1)(1-\eta) \right| \right], \\ \Lambda(t, B, q) &= [t]_q (2 + B + 1), \end{aligned}$$

and

$$\gamma(\eta, A, B, q) = |(B+1) + (A+1)\eta - (A+1)(1-\eta)q| + 2(1-\eta)(1-q),$$

unless otherwise mentioned.

Definition 1. *(see [9] and [10]) The q-derivative (q-difference) Δ_q of a function f is defined in a given subset of $\mathbb{C}$ by*

$$(\Delta_q f)(\omega) = \begin{cases} \frac{f(\omega) - f(q\omega)}{(1-q)\omega} & (\ \omega \neq 0), \\ f'(0) & (\ \omega = 0), \end{cases}$$

where $0 < q < 1$. This implies the following.

$$\lim_{q \to 1^-} (\Delta_q f)(\omega) = \lim_{q \to 1^-} \frac{f(\omega) - f(q\omega)}{(1-q)\omega} = f'(\omega),$$

provided that $f'(0)$ exists.

The function $\Delta_q f$ has Maclaurin's series representation

$$(\Delta_q f)(\omega) = \sum_{t=0}^{\infty} [t]_q a_t \omega^{t-1},$$

where $q \in (0, 1)$ and define the q-number $[\gamma]_q$ by

$$[\gamma]_q = \begin{cases} \frac{1-q^{\gamma}}{1-q} & (\ \gamma \in \mathbb{C}), \\ \sum_{k=0}^{t-1} q^{\gamma} = 1 + q^2 + q^3 + \ldots + q^{t-1} & (t \in \ \mathbb{N}). \end{cases}$$

For more details about q-derivatives, we refer the reader to (see [6]).

Definition 2. *For $f \in \sum_1$, let the q-derivative operator (q-difference operator) be defined by*

$$(\Delta_q f)(\omega) = \frac{f(q\omega) - f(\omega)}{(q-1)\omega} = -\frac{1}{q\omega^2} + \sum_{t=0}^{\infty} [t]_q a_t \omega^{t-1} \qquad (\omega \in U^{*}). \tag{6}$$

Similarly

$$(\Delta_q g)(\omega) = \frac{g(q\omega) - g(\omega)}{(q-1)\omega} = -\frac{1}{q\omega^2} + \sum_{t=0}^{\infty} [t]_q b_t \omega^{t-1} \qquad (\omega \in U^{*}). \tag{7}$$

Definition 3. *A function $f \in \sum_1$ is said to belong to the class $f \in T^*_{(q,\eta)}[A,B]$ if and only if*

$$\left|\frac{(B-1)\left(\frac{-\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}\right)-(A-1)}{(B+1)\left(\frac{-\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}\right)-(A+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.$$

Where $g \in \sum^{MS}(\alpha)$, It is easily observed that

$$\lim_{q\to 1^-} T^*_{(q,0)}[A,B] = S_q^{MK}[A,B],$$

secondly we have

$$\lim_{q\to 1^-} T^*_{(q,0)}[1,-1] = S_q^{MK},$$

where $S_q^{MK}[A,B]$ is the well-known function of meromorphic close-to-convex function.

2. Main Results

2.1. Coefficient estimates.

Theorem 1. *A function $f \in \sum_1$ of the form given by (1) is in the class $T^*_{(q,\eta)}[A,B]$ if it satisfies the following condition.*

$$\sum_{t=1}^{\infty}(\Lambda(t,B,q)\,|a_t|\,q+(t,\eta,A,q)\,|b_t|\,q)\leq\gamma(\eta,A,B,q), \tag{8}$$

where

$$\Lambda(t,B,q)=[t]_q(2+B+1), \tag{9}$$

$$\Lambda(t,\eta,A,q)=[|b_t|\,|(2[t]_q\eta+2(1-\eta)+\eta(A+1))-(A+1)(1-\eta)|] \tag{10}$$

and

$$\gamma(\eta,A,B,q)=|(B+1)+(A+1)\eta-(A+1)(1-\eta)q|+2(1-\eta)(1-q). \tag{11}$$

Proof. Assuming that (8) holds, it suffices to show that

$$\left|\frac{(B-1)\left(\frac{-\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}\right)-(A-1)}{(B+1)\left(\frac{-\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}\right)-(A+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.$$

Consider we have

$$\left|\frac{(B-1)\left(\frac{-\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}\right)-(A-1)}{(B+1)\left(\frac{-\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}\right)-(A+1)}-\frac{1}{1-q}\right|$$

which implies

$$=\left|\frac{-(B-1)\omega\Delta_q f(\omega)-(A-1)\left[(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)\right]}{-(B+1)\omega\Delta_q f(\omega)-(A+1)\left[(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)\right]}-1\right|+\frac{q}{1-q}$$

Thus

$$2\left|\frac{\omega\Delta_q f(\omega)+(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}{-(B+1)\omega\Delta_q f(\omega)-(A+1)\left[(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)\right]}\right|+\frac{q}{1-q}$$

Using (1), (6) and (7) in above equation.

$$\left|\frac{\begin{array}{c}2(1-\eta)(q-1)\\ +2\sum_{t=1}^{\infty}\left[[t]_q(a_t+\eta b_t)+(1-\eta)b_t\right]q\omega^{t+1}\end{array}}{\begin{array}{c}(B+1)+(A+1)\eta-(A+1)(1-\eta)q\\ -\sum_{t=1}^{\infty}\left[[t]_q((B+1)a_t+\eta(A+1)b_t)-(A+1)(1-\eta)b_t\right]q\omega^{t+1}\end{array}}\right|\leq 1,$$

we get

$$\sum_{t=1}^{\infty}[t]_q\,|a_t|\,(2+B+1)q+\sum_{t=1}^{\infty}\left[|b_t|\,|(2[t]_q\eta+2(1-\eta)+\eta(A+1))-(A+1)(1-\eta)|\right]q$$
$$\leq|(B+1)+(A+1)\eta-(A+1)(1-\eta)q|+2(1-\eta)(1-q). \tag{12}$$

The last expression become

$$\sum_{t=1}^{\infty}\Lambda(t,B,q)\,|a_t|\,q+\sum_{t=1}^{\infty}\Lambda(t,\eta,A,q)\,|b_t|\,q\ \leq\gamma(\eta,A,B,q).$$

This complete the proof of Theorem 2.1.

□

Corollary 1. *If a function $f\in\sum_1$ of the form given by (1) is in the class $T^*_{(q,\eta)}[A,B]$, then*

$$|a_t|\leq\frac{\gamma(\eta,A,B,q)}{\Lambda(t,B,q)}-\frac{\Lambda(t,\eta,A,q)}{\Lambda(t,B,q)}\,|b_t|\qquad(t\in N), \tag{13}$$

with equality for each t, we define the function of the form

$$f_t(\omega)=\frac{1}{\omega}+\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(t,B,q)}-\frac{\Lambda(t,\eta,A,q)}{\Lambda(t,B,q)}\,|b_t|\right)\omega^t,$$

where $\Lambda(t,B,q)$, $\Lambda(t,\eta,A,q)$ and $\gamma(\eta,A,B,q)$ are given by (9), (10) and (11) respectively.

2.2. **Distortion inequalities.**

Theorem 2. *If $f\in T^*_{(q,\eta)}[A,B]$, then*

$$\frac{1}{r}-\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(t,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)r\leq|f(\omega)|$$
$$\leq\frac{1}{r}+\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(t,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)r\ \ (|\omega|=r),$$

where equality holds for the function

$$f(\omega)=\frac{1}{\omega}+\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)}-\frac{\Lambda(1,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)\omega.$$

Proof. Let $f \in T^{*}_{(q,\eta)}[A,B]$. Then in view of Theorem (2.1), we have

$$\Lambda(1,B,q)\sum_{t=1}^{\infty}|a_t| \leq \sum_{t=1}^{\infty}\Lambda(t,B,q)\,|a_t| \quad \leq \gamma(\eta,A,B,q) - \sum_{t=1}^{\infty}\Lambda(1,\eta,A,q)\,|b_t|\,,$$

which yields

$$|f(\omega)| \leq \frac{1}{r} + \sum_{t=1}^{\infty}|a_t|\,r^t \leq \frac{1}{r} + r\sum_{t=1}^{\infty}|a_t| \leq \frac{1}{r} + \left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)} - \frac{\Lambda(1,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)r. \tag{14}$$

Similarly, we have

$$|f(\omega)| \geq \frac{1}{r} - \sum_{t=1}^{\infty}|a_t|\,r^t \geq \frac{1}{r} - r\sum_{t=1}^{\infty}|a_t| \geq \frac{1}{r} - \left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)} - \frac{\Lambda(1,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)r \tag{15}$$

which is required. □

Theorem 3. *If $f \in T^{*}_{(q,\eta)}[A,B]$, then*

$$\frac{1}{r^2} - 2\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)} - \frac{\Lambda(t,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)$$
$$\leq |f(\omega)| \leq \frac{1}{r^2} + 2\left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)} - \frac{\Lambda(t,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right) \quad (|\omega| = r)\,,$$

where equality holds for the function

$$f(\omega) = \frac{1}{\omega} + \left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)} - \frac{\Lambda(1,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right)\omega.$$

Proof. Let $f \in T^{*}_{(q,\eta)}[A,B]$. Then in view of theorem (2.1), we have

$$\Lambda(1,B,q)\sum_{t=1}^{\infty}|a_t| \leq \sum_{t=1}^{\infty}\Lambda(t,B,q)\,|a_t| \quad \leq \gamma(\eta,A,B,q) - \sum_{t=1}^{\infty}\Lambda(1,\eta,A,q)\,|b_t|\,.$$

Differentiate (14) and (15), we get

$$\left|f'(\omega)\right| \leq -\frac{1}{r^2} + \sum^{\infty} t\,|a_t|\,r^{t-1} \leq -\frac{1}{r^2} + \sum_{t=1}^{\infty}|a_t| \leq -\frac{1}{r^2} + \left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)} - \frac{\Lambda(1,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right). \tag{16}$$

Similarly, we have

$$\left|f'(\omega)\right| \geq -\frac{1}{r^2} - \sum^{\infty} t\,|a_t|\,r^{t-1} \geq -\frac{1}{r^2} - \sum_{t=1}^{\infty}|a_t| \geq -\frac{1}{r^2} - \left(\frac{\gamma(\eta,A,B,q)}{\Lambda(1,B,q)} - \frac{\Lambda(1,\eta,A,q)}{\Lambda(1,B,q)}\,|b_t|\right). \tag{17}$$

Comparing (16) and (17).

We have thus completed the proof of Theorem 2.4. □

2.3. **Partial sums.**
In this section, we examine the ratio of a function of the form (1) to its sequence of partial sums

$$f_t(\omega) = \frac{1}{\omega} + \sum_{t=1}^{k} a_t \omega^t,$$

when the coefficients of f are sufficiently small to satisfy condition (8). We will determine sharp lower bounds for

$$\mathrm{Re}\left(\frac{f(\omega)}{f_\nu(\omega)}\right), \quad \mathrm{Re}\left(\frac{f_\nu(\omega)}{f(\omega)}\right), \quad \mathrm{Re}\left(\frac{(\Delta_q f)(\omega)}{(\Delta_q f_\nu)(\omega)}\right) \quad \text{and} \quad \mathrm{Re}\left(\frac{(\Delta_q f_\nu)(\omega)}{(\Delta_q f)(\omega)}\right).$$

Theorem 4. *If f of the form (1) satisfies condition (8), then*

$$\mathrm{Re}\left(\frac{f(\omega)}{f_\nu(\omega)}\right) \geq 1 - \frac{1}{\kappa_{\nu+1}} \qquad (\omega \in U^*), \tag{18}$$

and

$$\mathrm{Re}\left(\frac{f_\nu(\omega)}{f(\omega)}\right) \geq \frac{\kappa_{\nu+1}}{1+\kappa_{\nu+1}} \qquad (\omega \in U^*), \tag{19}$$

where

$$\kappa_\nu = \frac{\gamma(\eta, A, B, q)}{\Lambda(t, B, q)} - \frac{\Lambda(t, \eta, A, q)}{\Lambda(t, B, q)} |b_t|. \tag{20}$$

Proof. In order to prove inequality (18), we set

$$\kappa_{\nu+1}\left[\frac{f(\omega)}{f_\nu(\omega)} - \left(1 - \frac{1}{\kappa_{\nu+1}}\right)\right] = \frac{1 + \sum_{t=1}^{\nu} a_t\omega^{t-1} + \kappa_{\nu+1}\sum_{t=\nu+1}^{\infty} a_t\omega^{t+1}}{1 + \sum_{t=1}^{\nu} a_t\omega^{t+1}}$$

$$= \frac{1 + h_1(\omega)}{1 + h_2(\omega)}.$$

Let

$$\frac{1 + h_1(\omega)}{1 + h_2(\omega)} = \frac{1 + g(\omega)}{1 - g(\omega)}.$$

Finally, to prove the inequality in (18), we get

$$\sum_{t=1}^{\nu} \left(1 - \kappa_{\nu+1}\right)|a_t| + \sum_{t=\nu+1}^{\infty} \left(\kappa_{\nu+1} - \kappa_t\right)|a_t| \geq 0.$$

The proof of inequality in (18) is now completed.
Similarly, we set

$$1 + \kappa_\nu\left[\frac{f_\nu(\omega)}{f(\omega)} - \left(\frac{\kappa_{\nu+1}}{1+\kappa_{\nu+1}}\right)\right] = \frac{1 + \sum_{t=1}^{\nu} a_t\omega^{t-1} - \kappa_{\nu+1}\sum_{t=\nu+1}^{\infty} a_t\omega^{t-1}}{1 + \sum_{t=1}^{\nu} a_t\omega^{t-1}}$$

$$= \frac{1+g(\omega)}{1-g(\omega)}.$$

We have completed the proof of (19), which complete the proof of Theorem 2.5. □

Theorem 5. *If f of the form (1) satisfies condition (8), then*

$$\mathrm{Re}\left(\frac{(\Delta_q f)(\omega)}{(\Delta_q f_\nu)(\omega)}\right) \geq 1 - \frac{[\nu+1]_q}{\kappa_{\nu+1}} \qquad (\omega \in U^*), \tag{21}$$

and

$$\mathrm{Re}\left(\frac{(\Delta_q f_\nu)(\omega)}{(\Delta_q f)(\omega)}\right) \geq \frac{\kappa_{\nu+1}}{\kappa_{\nu+1} + [\nu+1]_q} \qquad (\omega \in U^*), \tag{22}$$

where κ_ν is given by (20).

The proof of Theorem 2.6, is similar to that of Theorem 2.5.

2.4. **Radius of starlikeness.**

In the next theorem we find the radius of q-starlikeness for the class $T^*_{(q,\eta)}[A,B]$.

Theorem 6. *Let the function f given by (1) be in the class $T^*_{(q,\eta)}[A,B]$. Then f is meromorphic starlike of order α in $|\omega| \leq r$, where*

$$r = inf_{t\geq 1}\left[\frac{(1-\alpha)\Lambda(t,B,q)}{(n+2-\alpha)\left[\gamma(\eta,A,B,q) - \Lambda(t,\eta,A,q)\,|b_t|\right]}\right]^{\frac{1}{t+1}},$$

Proof. In order to prove above result, we must show that

$$\left|\frac{f'(\omega)}{f(\omega)} + 1\right| \leq 1-\alpha \quad (0 \leq \alpha < 1) \text{ and } |\omega| \leq r_1,$$

we have

$$\left|\frac{f'(\omega)}{f(\omega)} + 1\right| = \frac{\sum_{t=1}^{\infty}(t+1)a_t\omega^t}{\frac{1}{\omega} + \sum_{t=1}^{\infty} a_t\omega^t}.$$

$$\leq \frac{\sum_{t=1}^{\infty}(t+1)\,|a_t|\,|\omega|^{t+1}}{1 - \sum_{t=1}^{\infty}|a_t|\,|\omega|^{t+1}}. \tag{23}$$

Since the appropriate condition for a function f to be in the class $\sum^{MS}(\alpha)$ is given by

$$\sum_{t=1}^{\infty}(t+\alpha)\,|a_t| < 1-\alpha \quad (0 \leq \alpha < 1;\ \omega \in U^*). \tag{24}$$

Hence (23) holds true if

$$\sum_{t=1}^{\infty}(t+1)\,|a_t|\,|\omega|^{t+1} \leq (1-\alpha)\left(1 - \sum_{t=1}^{\infty}|a_t|\,|\omega|^{t+1}\right). \tag{25}$$

The inequality in (25) can be written as:

$$\sum_{t=1}^{\infty}\left(\frac{t+2-\alpha}{1-\alpha}\right)|a_t|\,|\omega|^{t+1}\leq 1. \tag{26}$$

With the aid of (8), inequality (26) is true if

$$\left(\frac{t+2-\alpha}{1-\alpha}\right)|\omega|^{t+1}\leq\frac{\Lambda(t,B,q)}{\gamma(\eta,A,B,q)-\Lambda(t,\eta,A,q)\,|b_t|}. \tag{27}$$

Solving (27) for $|\omega|$, we have

$$|\omega|=\left[\frac{(1-\alpha)\Lambda(t,B,q)}{(n+2-\alpha)\left[\gamma(\eta,A,B,q)-\Lambda(t,\eta,A,q)\,|b_t|\right]}\right]^{\frac{1}{t+1}}. \tag{28}$$

In view of (28) the proof of our theorem is now completed. □

Definition 4. *A function $f\in\sum_1$ is said to belong to the class $f\in T^*_{(q,\eta,1)}[A,B]$ if and only if*

$$\operatorname{Re}\left(\frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A+1)}\right)\geq 0.$$

We call $T^*_{(q,\eta,1)}[A,B]$ the class of q close-to-convex function of Type 1 related with the Janowski functions.

Definition 5. *A function $f\in\sum_1$ is said to belong to the class $f\in T^*_{(q,\eta,2)}[A,B]$ if and only if*

$$\left|\frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.$$

We call $T^*_{(q,\eta,2)}[A,B]$ the class of q close-to-convex function of Type 2 related with the Janowski functions.

Definition 6. *A function $f\in\sum_1$ is said to belong to the class $f\in T^*_{(q,\eta,3)}[A,B]$ if and only if*

$$\left|\frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A+1)}-1\right|<1.$$

We call $T^*_{(q,\eta,3)}[A,B]$ the class of q close-to-convex function of Type 3 related with the Janowski functions.

For Special Cases.

(1) For $\eta=0$ and $g(\omega)=f(\omega)$ then $T^*_{(q,0,)}[A,B]$, $T^*_{(q,0,1)}[A,B]$, $T^*_{(q,0,2)}[A,B]$ and $T^*_{(q,0,3)}[A,B]$ classes reduced to $S^*_q[A,B]$, $S^*_{(q,1)}[A,B]$, $S^*_{(q,2)}[A,B]$ and $S^*_{(q,3)}[A,B]$ studied by Srivastava et al [17, 27].

(2) For $\eta = 0$, $g(\omega) = f(\omega)$, $A = 1 - 2\alpha$ and $B = -1$ in $T^*_{(q,0,)}[A,B]$, $T^*_{(q,0,1)}[A,B]$, $T^*_{(q,0,2)}[A,B]$ and $T^*_{(q,0,3)}[A,B]$ we get the classes S^*_q, $S^*_{(q,1)}(\alpha)$, $S^*_{(q,2)}(\alpha)$ and $S^*_{(q,3)}(\alpha)$, which was introduced and studied by Wongsaijai and Sukantamala (see [30]).

2.5. **Main Results and Their Demonstration.**

We first derive the presence results for the succeeding generalized q-starlike functions:

$$T^*_{(q,\eta,1)}[A,B], \quad T^*_{(q,\eta,2)}[A,B] \quad and \quad T^*_{(q,\eta,3)}[A,B],$$

which are associated with the Janowski functions.

Theorem 7. *If* $-1 \leq B < A < 1$, *then*

$$T^*_{(q.\eta,3)}[A,B] \subset T^*_{(q.\eta,2)}[A,B] \subset T^*_{(q,\eta,1)}[A,B].$$

Proof. First of all, we suppose that $f \in T^*_{(q.\eta,3)}[A,B]$. Then, by Definition 2.10, we have

$$\left| \frac{(B-1)\frac{\omega \Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega \Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} - 1 \right| < 1,$$

so that

$$\left| \frac{(B-1)\frac{\omega \Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega D_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} - 1 \right| + \frac{q}{1-q} < 1 + \frac{q}{1-q}. \tag{29}$$

By using the triangle inequality and equation (29), we find that

$$\left| \frac{(B-1)\frac{\omega \Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega \Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}. \tag{30}$$

The last expression in (30) now implies that $f \in T^*_{(q.\eta,2)}[A,B]$, that is, that

$$T^*_{(q,3)}[A,B] \subset T^*_{(q,2)}[A,B].$$

Next, we let $f \in T^*_{(q,\eta,2)}[A,B]$, so that

$$f \in T^*_{(q,\eta,2)}[A,B] \Longleftrightarrow \left| \frac{(B-1)\frac{\omega \Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega \Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

As we know

$$\frac{1}{1-q} > \left| \frac{(B-1)\frac{\omega \Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega \Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} - \frac{1}{1-q} \right|$$

$$= \left| \frac{1}{1-q} - \frac{(B-1)\frac{\omega \Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega D_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} \right|,$$

we have

$$\operatorname{Re}\left(\frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}-(A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega D_q g(\omega)}-(A+1)}\right)>0 \qquad (\omega\in U^*). \tag{31}$$

This last equation now shows that $f\in T^*_{(q,\eta,1)}[A,B]$, that is, that

$$T^*_{(q.\eta,2)}[A,B]\subset T^*_{(q.\eta,1)}[A,B].$$

We have thus completed the proof of Theorem 2.11. □

Theorem 8. *Let $f\in\sum_1$, then $f\in T^*_{(q,\eta,2)}[A,B]$ if and only if*

$$\left|\frac{f(q\omega)}{(1-\eta)g(\omega)+\eta g(q\omega)}-\frac{\varkappa}{(B-1)q+B+3}\right|\leqq\frac{(A+1)(1-q)}{(B-1)q+B+3},$$

where

$$\varkappa=(A-1)q^2+(B-A+2)q+B+1.$$

Proof. Let

$$\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)}=\left(\frac{1}{1-q}\right)\left(1-\frac{f(q\omega)}{(1-\eta)g(\omega)+\eta g(q\omega)}\right).$$

Using Definition 2.9 of the class $T^*_{(q,\eta,2)}[A,B]$ associated with the Janowski functions.

$$\left|\frac{(B-1)\left(\frac{1}{1-q}\right)\left(1-\frac{f(q\omega)}{(1-\eta)g(\omega)+\eta g(q\omega)}\right)-(A-1)}{(B+1)\left(\frac{1}{1-q}\right)\left(1-\frac{f(q\omega)}{(1-\eta)g(\omega)+\eta g(q\omega)}\right)-(A+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.$$

We have thus completed the proof of Theorem 2.12. □

Corollary 2. *It is worth mentioning that the classes*

$$T^*_{(q,\eta,1)}[A,B],\quad T^*_{(q,\eta,2)}[A,B]\quad \textit{and}\quad T^*_{(q,\eta,3)}[A,B].$$

of the generalized q closed-to-convex functions of Type1, Type 2, and Type3, respectively, satisfy the following properties:

$$\cap_{q\in(0,1)}T^*_{(q,\eta,1)}[A,B]=\cap_{q\in(0,1)}T^*_{(q,\eta,2)}[A,B]=T^*[A,B].$$

and

$$\cap_{q\in(0,1)}T^*_{(q,\eta,1)}[A,B]=\cap_{q\in(0,1)}T^*_{(q,\eta,3)}[A,B]\subset T^*[A,B].$$

Let L be a subset of $\sum_1$ consisting of functions with a negative coefficient, that is,

$$f(\omega)=\frac{1}{\omega}-\sum_{t=1}^{\infty}|a_t|\,\omega^t \qquad (a_t\geq 0).$$

We also let

$$LT^*_{(q,\eta,t)}[A,B]=T^*_{(q,\eta,t)}[A,B]\cap L \quad (t=1,2,3).$$

Theorem 9. *For $-1 \leq B < A < 1$, then*

$$LT^{*}_{(q,\eta,1)}[A,B] = LT^{*}_{(q,\eta,2)}[A,B] = LT^{*}_{(q,\eta,3)}[A,B].$$

Proof. In view of Theorem 2.11, it is sufficient here to show that

$$LT^{*}_{(q,\eta,1)}[A,B] \subset LT^{*}_{(q.\eta,3)}[A,B].$$

Indeed, if we assume that , $f \in LT^{*}_{(q.\eta,1)}[A,B]$, then we have

$$\operatorname{Re}\left(\frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)}\right) \geq 0,$$

so that

$$\operatorname{Re}\left(\frac{(B-1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A-1)}{(B+1)\frac{\omega\Delta_q f(\omega)}{(1-\eta)g(\omega)+\eta\omega\Delta_q g(\omega)} - (A+1)} - 1\right) \geq -1.$$

After a simple calculation, we thus find that

$$2\left|\frac{-\omega\Delta_q f(\omega) + (1-\eta)g(\omega) + \eta\omega\Delta_q g(\omega)}{(B+1)\omega\Delta_q f(\omega) - (A+1)\left[(1-\eta)g(\omega) + \eta\omega\Delta_q g(\omega)\right]}\right| \geq -1$$

Using (1), (6) and (7) in above equation.

$$\left|\frac{\begin{array}{c} 2(2\eta-1) - 2(1-\eta)q + \\ 2\sum_{t=1}^{\infty}\left[[t]_q(a_t - \eta b_t) - (1-\eta)b_t\right]q\omega^{t+1} \end{array}}{\begin{array}{c} -(B+1) + (A+1)\eta - (A+1)(1-\eta)q \\ +\sum_{t=1}^{\infty}\left[[t]_q((B+1)a_t - \eta(A+1)b_t) - (A+1)(1-\eta)b_t\right]q\omega^{t+1} \end{array}}\right| < 1$$

This implies we get

$$\begin{aligned} &\sum_{t=1}^{\infty}|a_t|\,[t]_q(2-(B+1))q + \sum_{t=1}^{\infty}\left[\begin{array}{c} -2[t]_q\eta + (A+1)\eta \\ +(A+1)(1-\eta) - 2(1-\eta) \end{array}\right]|b_t|\,q \\ &\leq |(B+1) - (A+1)\eta - (A+1)(1-\eta)q| + 2(1-\eta)q - 2(2\eta-1), \end{aligned}$$

which satisfies $T^{*}_{(q,\eta,3)}[A,B]$. By Definition 2.10, the proof of Theorem 2.14 is completed. □

3. Conclusion

In our current investigation, we have presented and studied thoroughly some new subclasses of q meromorphic close-to-convex functions, which is connected with the Janowski functions. Then we discussed some interesting properties and characteristics of these new subclasses, including distortion theorem, radius problem and partial sum. Some special cases have been discussed as applications of our main results. The technique and ideas of this paper may stimulate further research in this dynamic field.

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