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Research Article

Some Bounds for the Weighted Energy

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Abstract

Energy of a graph is a concept defined in 1978 and originated from theoretical chemistry. Recently, the energy, Laplacian energy, signless Laplacian energy and normalized Laplacian energy of a graph have received much interest. In short, for an n-vertex connected unweighted graph G, the energy is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix.

For a simple connected matrix weighted graph G, the weighted energy is defined as the sum of the absolute values of the eigenvalues of its weighted adjacency matrix. In this paper, a brief overview for the notations and concepts of matrix weighted and number weighted graphs that will be used throughout this study is given. In the Main results section, the weighted energy of simple connected matrix weighted graphs are considered and some bounds for the weighted energy are found. Also, some results on number weighted and unweighted graphs are obtained by means of these bounds.

Keywords: Matrix weighted graph, number weighted graph, weighted energy, bound

Ağırlıklı Enerji İçin Bazı Sınırlar

Öz

Bir grafın enerjisi, 1978 yılında tanımlanmıştır ve teorik kimya kökenli bir kavramdır. Son zamanlarda, bir grafın enerjisi, Laplacian enerjisi, işaretsiz Laplacian enerjisi ve normalleştirilmiş Laplacian enerjisi çok ilgi görmektedir. Kısaca n-noktalı bağlantılı, ağırlıksız bir G grafı için enerji, komşuluk matrisinin özdeğerlerinin mutlak değerce toplamı olarak tanımlanır.

Basit, bağlantılı, matris ağırlıklı bir G grafı için ağırlıklı enerji, ağırlıklı komşuluk matrisinin özdeğerlerinin mutlak değerce toplamı olarak tanımlanır. Bu makalede, çalışma boyunca kullanılacak sayı ağırlıklı ve matris ağırlıklı grafların kavramları ve notasyonları için kısa bir özet verildi. "Temel sonuçlar" bölümünde basit, bağlantılı, matris ağırlıklı bir grafın ağırlıklı enerjisi dikkate alındı ve ağırlıklı enerji için bazı sınırlar bulundu. Ayrıca bu sınırlar yardımıyla sayı ağırlıklı ve ağırlıksız graflar üzerinde bazı sonuçlar elde edildi.

Anahtar Kelimeler: Matris ağırlıklı graf, sayı ağırlıklı graf, ağırlıklı enerji, sınır

Introduction

A matrix weighted graph is a graph in which each edge has been assigned a positive definite square matrix. Moreover, a number weighted graph is a graph in which each edge has been assigned a positive number.

Let $G=(V,E)$ be a simple connected matrix weighted graph on n vertices $(n\geq 2)$ and m edges. Let w_{ij} be the positive definite weight matrix of order t of the edge ij and assume that $w_{ij}=w_{ji}$. Let $w_i=\sum_{j:j\sim i} w_{ij}$, for all $i\in V$.

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The weight adjacency matrix $A(G)$ of G defined as $A(G)=(a_{ij})_{nt\times nt}$, where

$$a_{ij}=\begin{cases} w_{ij} & ; \quad \text{if } i \sim j \\ 0 & ; \quad \text{otherwise} . \end{cases}$$

The eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{nt}$ of $A(G)$ from the spectrum of G. The weighted energy of the graph G is defined as

$$E_W(G)=\sum_{i=1}^{nt}|\lambda_i| .$$

By assigning a positive-number instead of the matrix weight of each edge in a matrix weighted graph, number weighted graph is obtained. Also any unweighted graph can be thought as a number weighted graph in which we assign the weight of 1 to each edge.

In this paper, the weighted energy of simple connected matrix weighted graphs, where the edge weights are positive definite square matrices is considered. Besides some bounds for the weighted energy of the matrix weighted graphs are given.

Main results

In this section, some bounds for the weighted energy are found. Also, some results on number weighted and unweighted graphs are obtained by means of these bounds.

Now let us present the following lemma as the first preliminary result.

Lemma 1.

Let G be a simple connected matrix weighted graph. Then,

$$\sum_{i=1}^{nt}\lambda_i = 0$$

and

$$\sum_{i=1}^{nt}\lambda_i^2 = \sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^2\right).$$

Proof.

We clearly have

$$\sum_{i=1}^{nt}\lambda_i = tr[A(G)] = \sum_{i=1}^{n} tr(w_i) = 0 .$$

Moreover, for $i=1,2,...,n$ the $(i,i)-$th element of $A(G)$ is equal to $\sum_{j:j\sim i} w_{ij}^2$. Hence

$$\sum_{i=1}^{nt}\lambda_i^2 = tr\left[A(G)^2\right] = \sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^2\right).$$

This completes the proof.

Theorem 2.

Let G be a simple connected matrix weighted graph. Then,

$$\sqrt{\sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^2\right)} \leq E_W(G) \leq \sqrt{nt\sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^2\right)}.$$

Proof.

In the Cauchy-Schwartz inequality, if we choose $a_i = 1$ and $b_i = |\lambda_i|$, then we get

$$\left(\sum_{i=1}^{nt}|\lambda_i|\right)^2 \leq nt\sum_{i=1}^{nt}\lambda_i^2 .$$

From Lemma 1, we have

$$E_W(G)^2 = \left(\sum_{i=1}^{nt}|\lambda_i|\right)^2 \leq nt\sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^2\right),$$

i.e.,

$$E_W(G) \leq \sqrt{nt\sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^2\right)}. \qquad (1)$$

On the other hand, we can easily obtain the inequality

$$E_W(G)^2 = \left(\sum_{i=1}^{nt}|\lambda_i|\right)^2 \geq \sum_{i=1}^{nt}\lambda_i^2$$

$$= \sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^2\right),$$

i.e.,

$$\sqrt{\sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^2\right)} \leq E_W(G). \qquad (2)$$

From (1) and (2), we get

$$\sqrt{\sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^{2}\right)} \leq E_W(G) \leq \sqrt{nt\sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^{2}\right)}.$$

Hence the theorem is proved.

Corollary 3.

Let G be a simple connected number weighted graph, where each edge weight w_{ij} is a positive number. Then

$$\sqrt{\sum_{i=1}^{n}\sum_{j:j\sim i} w_{ij}^{2}} \leq E_W(G) \leq \sqrt{n\sum_{i=1}^{n}\sum_{j:j\sim i} w_{ij}^{2}}.$$

Proof.

For number weighted graph, where the edge weights w_{ij} are positive number, we have $tr\left(w_{ij}^{2}\right) = w_{ij}^{2}$, for all i, j. Using Theorem 2 we get the required result.

Corollary 4.

Let G be a simple connected unweighted graph. Then

$$\sqrt{2m} \leq E(G) \leq \sqrt{2mn}.$$

Proof.

For an unweighted graph, $w_{ij} = 1$ and $w_i = d_i$ for all i, j and $i \sim j$.

Using Corollary 3 we get the required result.

Theorem 5.

Let G be a simple connected matrix weighted graph. Then,

$$E_W(G) \leq \lambda_1 + \sqrt{(nt-1)\left(\sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^{2}\right) - \lambda_1^{2}\right)}.$$

Proof.

By the definition of weighted energy, combining Cauchy-Schwartz inequality and Lemma 1, we get

$$E_W(G) - \lambda_1 = \sum_{i=2}^{nt} |\lambda_i|,$$

$$\left(E_W(G) - \lambda_1\right)^2 = \left(\sum_{i=2}^{nt} |\lambda_i|\right)^2 \leq (nt-1)\sum_{i=2}^{nt} \lambda_i^{2}$$

$$= (nt-1)\left(\sum_{i=1}^{nt} \lambda_i^{2} - \lambda_1^{2}\right)$$

$$= (nt-1)\left(\sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^{2}\right) - \lambda_1^{2}\right).$$

Thus

$$E_W(G) \leq \lambda_1 + \sqrt{(nt-1)\left(\sum_{i=1}^{n}\sum_{j:j\sim i} tr\left(w_{ij}^{2}\right) - \lambda_1^{2}\right)}.$$

This completes the proof.

Corollary 6.

Let G be a simple connected number weighted graph where each edge weight w_{ij} is a positive number. Then

$$E_W(G) \leq \lambda_1 + \sqrt{(n-1)\left(\sum_{i=1}^{n}\sum_{j:j\sim i} w_{ij}^{2} - \lambda_1^{2}\right)}.$$

Proof.

For number weighted graph, where the edge weights w_{ij} are positive number, we have $tr\left(w_{ij}^{2}\right) = w_{ij}^{2}$, for all i, j. Using Theorem 5 we get the required result.

Corollary 7.

Let G be a simple connected unweighted graph. Then

$$E(G) \leq \lambda_1 + \sqrt{(n-1)\left(2m - \lambda_1^{2}\right)}.$$

Proof.

For an unweighted graph, $w_{ij} = 1$ and $w_i = d_i$ for all i, j and $i \sim j$.

Using Corollary 6 we get the required result.

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