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# On the Essential Element Graph of a Lattice

Alper Ülker

*Department of Mathematics and Computer Science, Istanbul Kültür University, 34158, Bakirköy, Istanbul, Turkey.*

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Abstract. Let $\mathcal{L}$ be a bounded lattice. The essential element graph of $\mathcal{L}$ is a simple undirected graph $\varepsilon_{\mathcal{L}}$ such that the elements $x, y$ of $\mathcal{L}$ form an edge in $\varepsilon_{\mathcal{L}}$, whenever $x \vee y$ is an essential element of $\mathcal{L}$. In this paper, we study properties of the essential elements of lattices and essential element graphs. We study the lattices whose zero-divisor graphs and incomparability graphs are isomorphic to its essential element graphs. Moreover, the line essential element graphs are investigated.

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**Keywords:** Lattice, zero-divisor graph, essential element, annihilator.

## 1. Introduction

The study of graphs associated to the algebraic structures plays significant roles in both algebra and combinatorics. Beck [7] introduced the coloring of graphs in commutative rings as follows: Let $G$ be a simple undirected graph which has its vertex set as elements of the ring $R$ and elements $x, y$ of $R$ form an edge in $G$ whenever $xy = 0$. Later, Anderson and Livingston introduced the zero-divisor graph of a commutative ring on non-zero elements of a commutative ring with identity [6]. Recently, there have been many research done related to graphs of algebraic structures [4, 11, 19].

The notion of zero-divisor graph of a partially ordered set was introduced by Liu and Xue in [21]. Later, in [1], the authors studied planarity of zero-divisor graphs of partially ordered sets. Lu and Wu in [15] gave an application to semigroups. In [13], the authors introduced the reduced zero-divisor graphs of posets.

Simple and undirected graphs associated to lattices such as zero-divisor graphs, incomparability graphs and comaximal graphs of lattices are studied by many authors [2, 3, 10, 17–20]. Essential elements of a lattice has been studied in [5]. Inspired by the notion of an essential element graph introduced in [16], we study the properties of these elements and combinatorial properties of its associated graphs.

The necessary background on lattices and the graph theory is given in Section 2. In Section 3, we study properties of essential elements and its graphs. In Section 4, we study the lattices whose essential element graphs are isomorphic to its zero-divisor graphs and incomparability graphs. In Section 5, we investigated the lattices whose essential element graphs are not line graphs of some graphs.

## 2. Preliminaries

In this section, we give basic terminology, definitions and notations on lattices and graph theory. For undefined terms and notations see [12] on lattice-theoretic concepts and see [9] on the graph theory.

Email address: a.ulker@iku.edu.tr (A. Ülker)

2.1. **Lattice Theory Foundations.** A lattice $\mathcal{L}$ is an algebra $(\mathcal{L}, \vee, \wedge)$ satisfying the following conditions; for all $u, v, w \in \mathcal{L}$,

(1) $u \vee u = u, u \wedge u = u$.
(2) $u \vee v = v \vee u, u \wedge v = v \wedge u$.
(3) $(u \vee v) \vee w = u \vee (v \vee w), (u \vee v \wedge w = u \wedge (v \wedge w)$.
(4) $u \vee (u \wedge v) = u \wedge (u \vee v) = u$.

$(\mathcal{L}, \leq)$ is an ordered set and for every $u, v \in \mathcal{L}$ the least upper bound which is called join $u \vee v$ and greatest lower bound which is called meet $u \wedge v$ exist. If there are elements 0 and 1 in $\mathcal{L}$ such that $u \vee 1 = 1$ and $u \wedge 0 = 0$ for all $u \in \mathcal{L}$ then $\mathcal{L}$ is a bounded lattice.

For any two elements $u$ and $v$, if $u \leq v$ or $v \leq u$ then elements $u$ and $v$ are said to be comparable. Otherwise they are incomparable elements.

**Definition 2.1.** Let $\mathcal{L}$ be a lattice. If there is no element $a$ in $\mathcal{L}$ such that $u \leq a \leq v$, then we say that $v$ covers $u$ and this is denoted by $u \leq v$.

**Definition 2.2.** Let $\mathcal{L}$ be a bounded lattice with bottom element 0 and top element 1. If $a \in \mathcal{L}$ covers 0, then $a$ is called *atom*. If $b \in \mathcal{L}$ covered by 0, then $b$ is called *coatom*.

**Definition 2.3.** An element in a lattice $\mathcal{L}$ is said to be join-irreducible if it covers exactly one element, and meet-irreducible if it is covered by exactly one element. An element that is both join- and meet-irreducible is said to be doubly irreducible.

**Definition 2.4.** Let $\mathcal{L}$ is a bounded lattice with 0 and 1. Then, any element in $\mathcal{L} \backslash \{0, 1\}$ is called *proper element* of $\mathcal{L}$.

**Definition 2.5.** If $L_i$ are bounded lattices, then the *horizontal sum* of lattices $L_i$ are obtained by identifying top and bottom elements.

2.2. **Graph Theory Foundations.** Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. If $x$ and $y$ are distinct vertices in $G$, the length of the shortest path between $x$ and $y$ is denoted by $\mathrm{d}(x, y)$. The diameter of $G$ is defined by $\mathrm{diam}(G) := sup\{\mathrm{d}(x, y) : x, y \in G\}$. If a subgraph is obtained by only vertex deletion, then this subgraph is called induced subgraph. A graph is said to be complete, if for all $x, y \in G$, $xy \in E(G)$. An induced complete subgraph of $G$ with $k$ vertices is called *k-clique* of $G$. A vertex $x$ of $G$ is called isolated if and only if $xy \notin E(G)$ for any $y \in G$.

The following definition is from [14], the authors characterized the $k$-chordal graphs which have no induced cycles of length greater than $k$.

**Definition 2.6.** [14] A graph is *k-chordal* if and only if it has no induced cycle of length greater than $k$ for some $k \geq 3$. A chordless path $v_1 - v_2 - ... - v_i$ is called a simplicial path if it does not extend to any chordless path $v_0 - v_1 - v_2 - ... - v_i - v_{i+1}$. A graph is *k-chordal* if and only if each of its non-empty induced subgraphs contains a simplicial path with at most $k - 2$ vertices.

**Definition 2.7.** A graph $G = (X, Y; E)$ is bipartite if its vertex set $V$ can be partitioned into two subsets X and Y so that every edge has one end in $X$ and one end in $Y$. If $V$ can be partitioned into $n$ subsets then $G = (X_1, ..., X_n; E)$ is said to be an *n-partite* graph.

It is well-known that by D. König, a graph is bipartite if and only if it contains no cycle of odd length.

## 3. Properties of Essential Element Graphs

In this section, we prove certain properties of the essential elements of a lattice and combinatorial properties of essential element graph of a lattice.

**Definition 3.1.** Let $\mathcal{L}$ be a lattice. The annihilator of an element $x$ of $\mathcal{L}$ is the set $\mathrm{Ann}(x) := \{y \in \mathcal{L} : x \wedge y = 0\}$.

**Definition 3.2.** An element $a$ of a lattice $\mathcal{L}$ is said to be essential, if there is no non-zero $x \in \mathcal{L}$ such that $a \wedge x = 0$. In other words, an element $a$ of a lattice $\mathcal{L}$ is an essential element if and only if there exists no non-zero $x$ such that $x \in \mathrm{Ann}(a)$.

The following definition of essential element graph of a lattice is due to Nimbhorkar and Deshmukh [16].

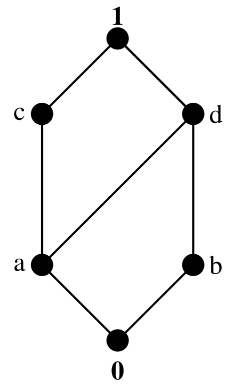


FIGURE 1. A lattice $\mathcal{L}$

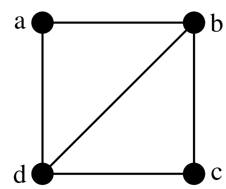


FIGURE 2. Essential element graph $\varepsilon_{\mathcal{L}}$

**Definition 3.3.** Let $\mathcal{L}$ be a lattice. The essential element graph of $\mathcal{L}$ is a simple undirected graph with vertex set $\mathcal{L}\backslash\{0, 1\}$ and any vertices $x$ and $y$ are adjacent if and only if $x \vee y$ is an essential element.

**Definition 3.4.** Let $\mathcal{L}$ be a lattice with least element 0. The zero-divisor graph $\Gamma(\mathcal{L})$ of $\mathcal{L}$ is a simple undirected graph with vertex set $\mathcal{L}\backslash\{0\}$ and any vertices $x$ and $y$ are adjacent if and only if $x \wedge y = 0$.

Two elements $x$ and $y$ of a lattice are said to be comparable whenever $x \leq y$ or $y \leq x$, otherwise they are incomparable. Next we give the definition of a incomparability graph of a lattice.

**Definition 3.5.** Let $\mathcal{L}$ be a lattice. The incomparability graph of $\mathcal{L}$ is a simple undirected graph with vertex set $\mathcal{L}$ and any two elements $x$ and $y$ are adjacent if and only if they are incomparable.

**Example 3.6.** The lattice shown in Figure 1 has an essential element $d$. Its essential element graph is Figure 2. Since $a \vee c = c$ and $c$ is non-essential, then $a$ and $c$ are not adjacent.

**Lemma 3.7.** *Let $\mathcal{L}$ be a bounded lattice with a unique atom. Then every element of $\mathcal{L}$ is an essential element.*

*Proof.* Let $a$ be a unique atom of $\mathcal{L}$. Then, for any $x \in \mathcal{L}$, we have either $x \wedge y = a$ or $x \wedge y = b$ for $a \leq b$ for every $y \in \mathcal{L}$, since $a \leq u$ for every element $u$ of $\mathcal{L}$ by definition. Thus any element of $\mathcal{L}$ is an essential element of $\mathcal{L}$.

□

Now, we can give the following corollary.

**Corollary 3.8.** *If a lattice $\mathcal{L}$ has a unique atom, then $\varepsilon_{\mathcal{L}}$ is a complete graph.*

*Proof.* Since every element of $\mathcal{L}$ is essential by Lemma 3.7, one can conclude that, for any $x, y \in \mathcal{L}$, $x \vee y$ is an essential element. Thus $xy \in E(\varepsilon_{\mathcal{L}})$.

□

**Proposition 3.9.** *Let $\mathcal{L}$ be a bounded lattice. Then, every subset of coatoms of $\mathcal{L}$ forms a clique.*

*Proof.* Assume that $\{x_1, ..., x_k\}$ is a set of coatoms of $\mathcal{L}$. So $x_i \vee x_j = 1$ for any $i, j \in \{1, \ldots, k\}$. Thus, $x_i x_j \in E(\varepsilon_{\mathcal{L}})$. Hence $\{x_1, ..., x_k\}$ is a $k$-clique in $\varepsilon_{\mathcal{L}}$.

□

The following theorem is about induced cycles that any $\varepsilon_{\mathcal{L}}$ can contain. We study this in the view of Definition 2.6.

**Theorem 3.10.** *If $\mathcal{L}$ is a bounded lattice, then $\varepsilon_{\mathcal{L}}$ is 4-chordal graph.*

*Proof.* Let $x_1 - x_2$ be a chordless path with two vertices. If we extend this chordless path to the path $x_0 - x_1 - x_2 - x_3$ with four vertices, then we have to show that this path has a chord. We claim that not all $x_0, x_1, x_2, x_3$ are comparable. Assume the contrary that they are all comparable. Without loss of generality, let $x_0 \leq x_1 \leq x_2 \leq x_3$. Since $x_2 \vee x_3 = x_3$, then $x_3$ is essential. The fact that $x_1 \vee x_3 = x_3$, implies that $x_1$ is adjacent to $x_3$, a contradiction. We claim that not all $x_0, x_1, x_2, x_3$ are incomparable. Assume that they are all incomparable. Let $x_1 \vee x_2 = p$ and $x_2 \vee x_3 = q$. Then $p$ and $q$ are distinct essential elements. It follows that $p \vee q$ is an essential element. Thus $x_1$ is adjacent to $x_3$, a contradiction. Now suppose that, $x_0 \leq x_2$ and $x_1 \leq x_3$. Note that $x_0 \vee x_1$ and $x_2 \vee x_3$ are essential, since they are adjacent. Then $x_1 \vee x_2$ is essential. Therefore $x_0 \vee x_3$ is essential and the path $x_0 - x_1 - x_2 - x_3$ has a chord. This argument implies that $\varepsilon_{\mathcal{L}}$ is 4-chordal.

□

**Lemma 3.11.** *If $\mathcal{L}$ is a lattice with a unique coatom $x$, then $x$ is an essential element.*

*Proof.* Let $x$ be a unique coatom, then for any element $y$ in $\mathcal{L}$, we have that $y \leq x$. Thus, clearly $\mathrm{Ann}(x) = 0$.

□

**Lemma 3.12.** *Let $x$ be a non-essential element of a lattice $\mathcal{L}$. If $y \leq x$, then $y$ is non-essential.*

*Proof.* Since $x$ is non-essential, there exist some $u$ in $\mathcal{L}$ such that $x \wedge u = 0$. We claim that $y \wedge u = 0$, otherwise there is $0 \neq v = y \wedge u$ which also satisfies $u \wedge x = v$, a contradiction.

□

An isolated vertex of graph is a vertex which has no edge between any other vertex of that graph. The following proposition shows elements that forms isolated vertices.

**Proposition 3.13.** *Let $\mathcal{L}$ be a bounded lattice and $\varepsilon_{\mathcal{L}}$ be its essential element graph. Then, $x$ is an isolated vertex of* $\varepsilon_{\mathcal{L}}$ if and only if $x = a \wedge b$ such that $a$ and $b$ are non-essential coatoms and $\mathrm{Ann} \cap \mathrm{Ann}(b) = 0$.

*Proof.* Assume that $x$ is an isolated vertex of $\varepsilon_{\mathcal{L}}$. It follows that $x \vee u$ is non-essential for every $u$. If $x$ is a coatom, then $xy \in E(\varepsilon_{\mathcal{L}})$ for some coatoms $y$ of $\mathcal{L}$, a contradiction. If $x$ is a unique coatom, then $x$ is an essential element and $xy \in E(\varepsilon_{\mathcal{L}})$ for any $x \leq y$, a contradiction. Hence $x$ is not a coatom of $\mathcal{L}$. Thus, there exist some elements $a$ and $b$ in $\mathcal{L}$ such that $x = a \wedge b$. If $a = b$ (which means that only one element covers $x$, say $a$), then any element $u$ which satisfies $u \wedge a = 0$ also satisfies $u \wedge x = 0$. We claim that $x \vee u = 1$, otherwise $x \vee u = b \geq a$. If $b$ is essential, then $x$ is adjacent to $b$, since $x \vee b = b$. This is a contradiction. Thus, there exists such an element $u$ in $\mathcal{L}$ such that $x \vee u = 1$. This implies that $x$ is adjacent to $u$, a contradiction. Hence we get that $a \neq b$. Now assume that there exists a non-zero element $u \in \mathrm{Ann}(a) \cap \mathrm{Ann}(b)$, then $a \wedge u = 0$ and $b \wedge u = 0$. It follows that $(a \wedge b) \wedge u = 0$. Hence we get $u \in Ann(x)$, since $x = a \wedge b$. The fact that $u \in \mathrm{Ann}(x)$, $u \in \mathrm{Ann}(a)$ and $u \in \mathrm{Ann}(b)$ implies that $x \vee u = v$ for some $v \geq a$ or $v \geq b$ or both. If $v$ is essential, then $v$ is adjacent to $x$, since $x \vee v = v$. This contradicts to our assumption. If $v$ is non-essential, then there exists a non-zero $w$ such that $v \wedge w = 0$. Again a contradiction by induction. Therefore, $u = 0$ and $a$ and $b$ are coatoms. Assume that $x = a \wedge b$ such that $a$ and $b$ are non-essential coatoms and $\mathrm{Ann} \cap \mathrm{Ann}(b) = 0$ . Then, $a$, $b$ and $x$ are non-essential elements. Thus, clearly $x$ is an isolated vertex of $\varepsilon_{\mathcal{L}}$. □

The next theorem is about the diameter of $\varepsilon_{\mathcal{L}}$ which is an important combinatorial property for a graph.

**Theorem 3.14.** *Let $\mathcal{L}$ be a lattice and $\varepsilon_{\mathcal{L}}$ be its essential element graph. If $\varepsilon_{\mathcal{L}}$ is a connected graph, then* $\mathrm{diam}(\varepsilon_L) \leq 2$.

*Proof.* If $x$ and $y$ are adjacent then $\mathrm{d}(x, y) = 1$. Assume that $x$ and $y$ are elements of $\mathcal{L}$ such that $xy \notin E(\varepsilon_{\mathcal{L}})$. Since $\varepsilon_{\mathcal{L}}$ is connected, there exists elements $a$ and $b$ such that $x \vee a$ and $y \vee b$ are both essential. Let $x \vee a = p$ and $y \vee b = q$. Since $p$ and $q$ are both essential, then $p \vee q = u$ is essential. Thus, $x \vee u = u$ and $y \vee u = u$ and $x - u - y$ is a path in $\varepsilon_{\mathcal{L}}$ which connects $x$ and $y$. Therefore, $\mathrm{diam}(\varepsilon_L) \leq 2$.

□

**Theorem 3.15.** *Let $\mathcal{L}$ be a lattice. Then the followings hold:*

(1) $\text{girth}(\varepsilon_{\mathcal{L}}) \in \{3, 4, \infty\}$.
(2) $\text{girth}(\varepsilon_{\mathcal{L}}) = 4$ *if and only if* $\varepsilon_{\mathcal{L}}$ *is bipartite but not a star graph.*
(3) $\text{girth}(\varepsilon_{\mathcal{L}}) = 3$ *if and only if* $\varepsilon_{\mathcal{L}}$ *contains an odd cycle.*

*Proof.* (1) Suppose that $\text{girth}(\varepsilon_{\mathcal{L}}) \neq \infty$. Then, there exist a cycle of minimal length $n$ in $\varepsilon_{\mathcal{L}}$, that is, $x_1 - x_2 - x_3 - \cdots - x_n - x_1$. Let $n \geq 5$. The minimality of $n$ implies that $x_1$ is not adjacent to $x_5$. So $x_1 \vee x_5 = u$ for a non-essential $u \in \mathcal{L}$. Since $x_1 \vee u = u$ and $x_5 \vee u = u$, we have $x_1 - u - x_5 - \cdots - x_n - x_1$ is a cycle of length $n-2$ in $\varepsilon_{\mathcal{L}}$. This contradicts the minimality of $n$. Thus, $n = 3$ or $n = 4$. Therefore, $\text{girth}(\varepsilon_{\mathcal{L}}) \in \{3, 4\}$.

(2) If we assume $\text{girth}(\varepsilon_{\mathcal{L}}) = 4$, then clearly $\varepsilon_{\mathcal{L}}$ is not a star graph, since $\varepsilon_{\mathcal{L}}$ is not a tree. We shot that $\varepsilon_{\mathcal{L}}$ has no cycle of odd length. Then by the result of König, $\varepsilon_{\mathcal{L}}$ is bipartite graph. On the contrary, we assume that $x_1 - x_2 - \cdots - x_n - x_1$ is an odd cycle of minimum length in $\varepsilon_{\mathcal{L}}$. Since $\text{girth}(\varepsilon_{\mathcal{L}}) \neq 3$, it is clear that $n \geq 5$. The minimality of $n$ implies that $x_1$ is not adjacent to $x_5$. So $x_1 \vee x_5 = u$ for a non-essential $u \in \mathcal{L}$. Since $x_1 \vee u = u$ and $x_5 \vee u = u$, we have $x_1 - u - x_5 - \cdots - x_n - x_1$ is a cycle of length $n-2$ in $\varepsilon_{\mathcal{L}}$. This contradicts the minimality of $n$. Thus $\varepsilon_{\mathcal{L}}$ has no odd cycle. Hence $\varepsilon_{\mathcal{L}}$ is bipartite graph.

Conversely, let $\varepsilon_{\mathcal{L}}$ be a bipartite but not a star graph. Then by König Theorem $\text{girth}(\varepsilon_{\mathcal{L}}) \neq 3$. Since $\varepsilon_{\mathcal{L}}$ is not a star graph, $\text{girth}(\varepsilon_{\mathcal{L}}) \neq \infty$. Therefore $\text{girth}(\varepsilon_{\mathcal{L}}) \neq 4$.

(3) Result is obvious by 1-2. □

## 4. Zero-divisor and Incomparability Graphs

In this section, we study the properties of lattices for which its incomparability graphs and zero-divisor graphs are isomorphic to the their essential element graphs.

**Lemma 4.1.** *Let $\mathcal{L}$ be the horizontal sum of bounded lattices $L_i$ for $i \geq 2$. Then, every proper element of $\mathcal{L}$ is a non-essential element.*

*Proof.* Let $\mathcal{L}$ be the horizontal sum of lattices $L_i$ and $x$ is an element of $\mathcal{L}$. If $x \in L_i$, then $x \wedge y = 0$ for some $y \in L_j$ with $j \neq i$, since $L_i \cap L_j = \{0, 1\}$. Otherwise, there exists $z \neq 1$ in $\mathcal{L}$ such that $x \vee y = z$ which contradicts to definition of the horizontal sum. □

**Remark 4.2.** If $\mathcal{L}$ is the horizontal sum of lattices $L_1, \ldots, L_n$. Then, for any two elements $x \in L_i$ and $y \in L_j$ with $i \neq j$, one can deduce that $x \vee y = 1$.

**Theorem 4.3.** *Let $\mathcal{L}$ be a bounded lattice. Then, $\mathcal{L}$ is the horizontal sum of the lattices $L_1, ..., L_n$ with each of the lattices $L_i$ has a unique atom and a unique coatom if only if its zero-divisor graph $\Gamma(\mathcal{L})$ is isomorphic to its essential element graph $\varepsilon_{\mathcal{L}}$.*

*Proof.* Assume that each $L_i$ has only one proper element which is also an atom and a coatom. Then, $x_i \vee x_j = 1$ and $x_i \wedge x_j = 0$ for $x_i \in \mathcal{L}_i$ and $x_j \in \mathcal{L}_j$. Hence $x_i x_j \in E(\varepsilon_{\mathcal{L}})$ and $x_i x_j \in E(\Gamma(\mathcal{L}))$. Now assume that each $L_i$ has more than one proper element and $a_i$ is an atom, $b_i$ is a coatom of $\mathcal{L}_i$. If $x$ and $y$ of $\mathcal{L}$ belongs to same $L_i$, then $x \vee y = u$ for $u \leq b_i$ and $x \wedge y = v$ for $a_i \leq v$. Thus, $xy \notin E(\varepsilon_{\mathcal{L}})$ and $xy \notin E(\Gamma_{\mathcal{L}})$. If $x$ and $y$ of $\mathcal{L}$ belong to $L_i$ and $L_j$ respectively, then $x \vee y = 1$ and $x \wedge y = 0$. Thus $xy \in E(\varepsilon_{\mathcal{L}})$ and $xy \in E(\Gamma_{\mathcal{L}})$. Therefore, we can conclude that $\Gamma(\mathcal{L})$ is isomorphic to $\varepsilon_{\mathcal{L}}$.

Conversely, suppose that $\Gamma(\mathcal{L})$ is isomorphic to $\varepsilon_{\mathcal{L}}$. Then $xy \in E(\Gamma_{\mathcal{L}})$ if and only if $xy \in E(\varepsilon_{\mathcal{L}})$. Thus, $x \vee y$ essential element and $x \wedge y = 0$ in $\mathcal{L}$. Now assume that $x \wedge y = 0$, then we have $x \vee y \neq x$ or $x \vee y \neq y$. Let $z \neq 1$ be an element of $\mathcal{L}$ such that $x \vee y = z$. Since $z$ is essential, there is no non-zero element in $\mathcal{L}$ which is a zero-divisor of $z$. So $xz$ and $yz$ are edges in $\varepsilon_{\mathcal{L}}$, but $xz$ and $yz$ are not edges in $\Gamma_{\mathcal{L}}$, since we have $x \vee z = z$ and $y \vee z = z$, also we have $x \wedge z = x$ and $y \wedge z = y$. This is a contradiction, thus $z = 1$. Hence $\bigvee_{i=1}^{n} L_i = 1$ and $\bigwedge_{i=1}^{n} L_i = 0$. Therefore, $\mathcal{L}$ is the horizontal sum of lattices. □

**Theorem 4.4.** *Let $G = (X, Y; E)$ be a connected bipartite graph. Then, $G$ is both zero-divisor and essential element graph of a lattice $\mathcal{L}$ if and only if $\mathcal{L}$ has exactly two atoms $a_1$ and $a_2$ and exactly two coatoms $b_1$ and $b_2$ such that $b_1 \wedge b_2 = 0$.*

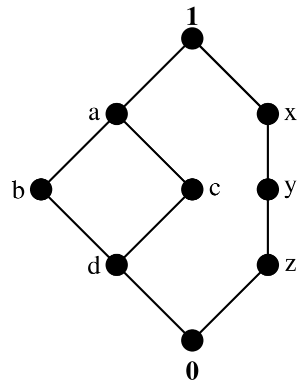


Figure 3. A lattice $\mathcal{L}$ with $\varepsilon_{\mathcal{L}}$ is bipartite

*Proof.* Assume that $G = X \cup Y$ is a bipartite graph such that $G = \varepsilon_{\mathcal{L}}$ and $G = \Gamma_{\mathcal{L}}$. First we show that $\mathcal{L}$ has two atoms. If $\mathcal{L}$ has one atom then $\varepsilon_{\mathcal{L}}$ is a complete graph by Corrollary 3.8, and $\Gamma_{\mathcal{L}}$ is a discrete graph since $x \wedge y \neq 0$ for every $x, y \in \mathcal{L}$ a contradiction. If we assume that $\mathcal{L}$ has atom other than $a_1$ and $a_2$, then this atom makes a $C_3$ with $a_1$ and $a_2$ in $\Gamma_{\mathcal{L}}$. This is a contradiction because $G$ is bipartite. Hence $\mathcal{L}$ has two atoms. Now we show that $\mathcal{L}$ has two coatoms. If $\mathcal{L}$ has one coatom, then this vertex is adjacent to all other vertices in $\varepsilon_{\mathcal{L}}$ and isolated vertex in $\Gamma_{\mathcal{L}}$, a contradiction. If there exists a coatom other than $b_1$ and $b_2$, then by Proposition 3.9 they form a $C_3$ in $\varepsilon_{\mathcal{L}}$, a contradiction. Hence $\mathcal{L}$ has two coatoms. If $b_1 \wedge b_2 \neq 0$, then at least one of the $b_1$ and $b_2$ is non-essential, say $b_1$. So some elements less than $b_1$ are adjacent and this is a contradiction since partition sets must be independent.

Now assume that $\mathcal{L}$ is a bounded lattice with two atoms $a_1$ and $a_2$ and two coatoms $b_1$ and $b_2$ such that $b_1 \wedge b_2 = 0$. Let $a_1 \leq b_1$ and $a_2 \leq b_2$. Then, the sets $X = \{x \in \mathcal{L} : a_1 \leq x \leq b_1\}$ and $Y = \{y \in \mathcal{L} : a_2 \leq y \leq b_2\}$ form a bipartite graph $\varepsilon_{\mathcal{L}}$ with bipartition $G = X \cup Y$, because every $x_i \vee x_j$ and $y_i \vee y_j$ is non-essential and $x_i \vee y_j = 1$. □

Incomparability graphs of lattices and partially ordered sets are well-studied graphs [18, 20]. We include the following important theorem:

**Theorem 4.5.** *Let $\varepsilon_{\mathcal{L}}$ be the essential element graph and* IC($\mathcal{L}$) *be the incomparability graph of a bounded lattice $\mathcal{L}$. If $\varepsilon_{\mathcal{L}} \cong$* IC($\mathcal{L}$)*, then* 1 *is the only join-reducible essential element in $\mathcal{L}$.*

*Proof.* Let $\varepsilon_{\mathcal{L}} \cong \mathrm{IC}(\mathcal{L})$. If we assume that there exists an element $u \neq 1$ in $\mathcal{L}$ which is join-reducible, then $u = a \vee b$ for some $a, b \in \mathcal{L}$. This implies that $ua$ and $ub$ are edges in $\varepsilon_{\mathcal{L}}$, since $u$ is essential. However, $u$ is not adjacent to $a$ and $b$ in IC($\mathcal{L}$), since they are comparable. This is a contradiction. Therefore, there is no essential element other than 1 in $\mathcal{L}$ which is join-reducible.

□

## 5. Line Essential Element Graphs of Lattices

In this section, we study the essential element graphs arising from bounded lattices which are, in fact, line graphs of some simple graphs. Next, we give fundamental forbidden induced subgraph characterization of line graphs.

**Theorem 5.1.** *[8] Let G be a graph. If G is a line graph of some simple graph if and only if none of the nine graphs in Figure 5 is an induced subgraph of G.*

In the following we give a condition on a lattice whose essential element graph is not a line graph of any graph.

**Theorem 5.2.** *Let $\mathcal{L}$ be a bounded lattice. If $\mathcal{L}$ has a chain of non-essential elements of length $\geq 2$, then $\varepsilon_{\mathcal{L}}$ is not a line graph.*

*Proof.* Assume that $a_1 \leq ... \leq a_k$ for $k \geq 3$ is a chain of length at least 2. Note that there exist an $x$ in $\mathcal{L}$ such that $x \wedge a_i = 0$ for $i \in \{1, .., k\}$, since $a_1, ..., a_k$ are non-essential. Then, it is obvious that the set $\{a_1, ..., a_k\}$ forms an independent set in $\varepsilon_{\mathcal{L}}$. Notice that $a_i \vee x = 1$ for all $i \in \{1, .., k\}$. It follows that $\{a_1, .., a_k, x\}$ has an induced $K_{1,3}$ in $\varepsilon_{\mathcal{L}}$ which implies that $\varepsilon_{\mathcal{L}}$ is not a line graph of any graph. □

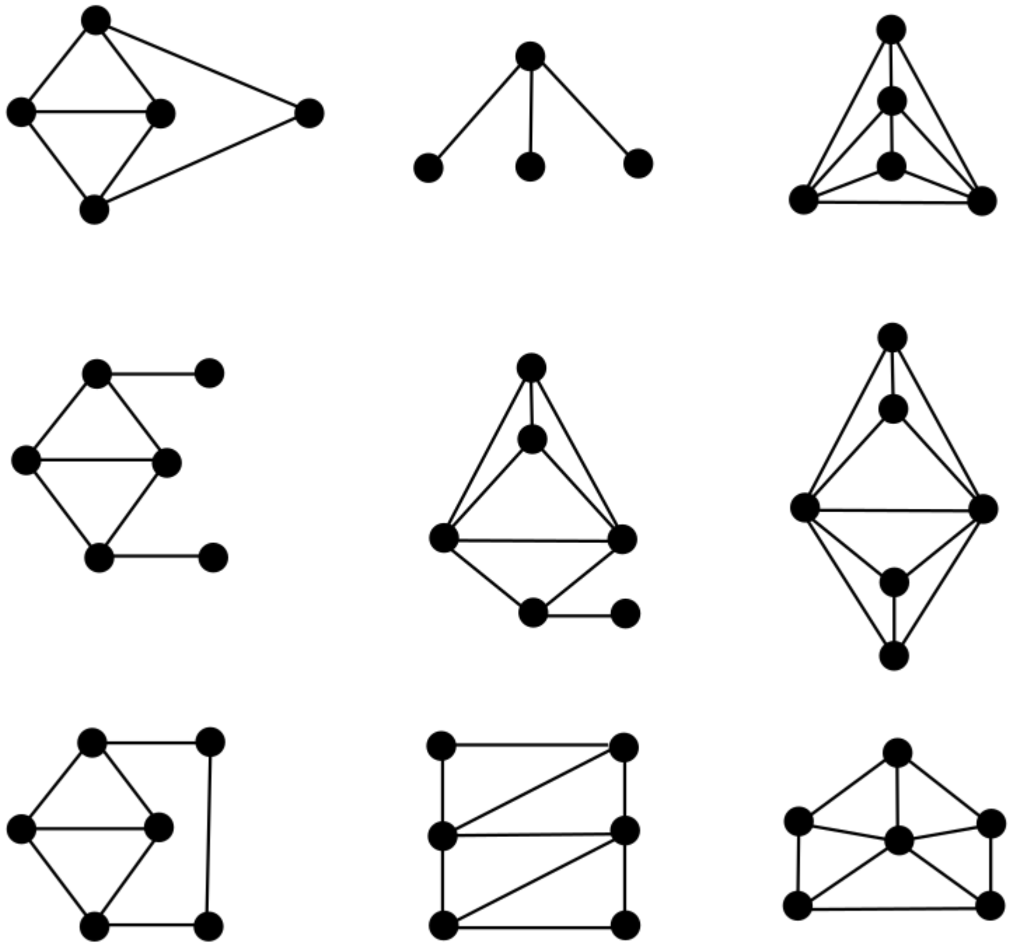

FIGURE 4. Forbidden induced subgraphs of line graphs

**Theorem 5.3.** *Let $\mathcal{L}$ be a bounded lattice with coatom c. Let the set $\{x_1, ..., x_k\}$ consists of proper elements of $\mathcal{L}$ with each $x_i \leq c$ for $i \in \{1, ..., k\}$, where $k \geq 3$. If there exists an element u which is both atom and coatom. Then, $\varepsilon_{\mathcal{L}}$ is not a line graph.*

*Proof.* Since $u$ is both atom and coatom, it follows that $c \wedge u = 0$ which implies that $c$ is a non-essential element. So either $x_i \vee u = c$ or $x_i \vee u = b$ for some $b \leq c$. Combining the $c \wedge u = 0$ and $b \leq c$ implies that $b$ is non-essential as well, since $b \wedge u = 0$. Hence $K_{1,k}$ is an induced subgraph of $\varepsilon_{\mathcal{L}}$. Therefore, by Theorem 5.1, $\varepsilon_{\mathcal{L}}$ is not a line graph. □

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## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

## Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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