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Repdigits as Product of Fibonacci and Pell numbers

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Abstract. In this paper, we find all repdigits which can be expressed as the product of a Fibonacci number and a Pell number. We use of a combined approach of lower bounds for linear forms in logarithms of algebraic numbers and a version of the Baker-Davenport reduction method to prove our main result.

1. Introduction

Diophantine equations involving recurrence sequences have been studied for a long time. One of the most interesting of these equations is the equations involving repdigits.

A repdigit (short for "repeated digit") T is a natural number composed of repeated instances of the same digit in its decimal expansion. That is, T is of the form

$$x \cdot \left(\frac{10^t - 1}{9}\right)$$

for some positive integers x, t with $t \geq 1$ and $1 \leq x \leq 9$.

Some of the most recent papers related to the repdigits with well known recurrence sequences are [3, 5, 6, 8]. In this note, we use Fibonacci and Pell sequences in our main result.

Binet's formula for Fibonacci numbers is

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

where $\varphi = \left(1 + \sqrt{5}\right)/2$ (the golden ratio) and $\psi = \left(1 - \sqrt{5}\right)/2$. From this formula, one can easiliy get

$$\varphi^{n-2} \leq F_n \leq \varphi^{n-1}. \tag{1}$$

Also, we can write

$$F_n = \frac{\varphi^n}{\sqrt{5}} + \theta \tag{2}$$

where $|\theta| \leq 1/\sqrt{5}$.

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Pell sequence, one of the most familiar binary recurrence sequence, is defined by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$. Some of the terms of the Pell sequence are given by $0, 1, 2, 5, 12, 29, 70, \ldots$. Its characteristic polynomial is of the form $x^2 - 2x - 1 = 0$ whose roots are $\alpha = 1 + \sqrt{2}$ (the silver ratio) and $\beta = 1 - \sqrt{2}$. Binet's formula enables us to rewrite the Pell sequence by using the roots α and β as

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}. \tag{3}$$

Also, it is known that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \tag{4}$$

and

$$P_n = \frac{\alpha^n}{2\sqrt{2}} + \lambda \tag{5}$$

where $|\lambda| \leq 1/\left(2\sqrt{2}\right)$.

In this study, our main result is the following:

Theorem 1.1. *The only positive integer triples (n, t, x) with $1 \leq x \leq 9$ satisfying the Diophantine equation*

$$F_n P_n = x \cdot \left(\frac{10^t - 1}{9}\right) \tag{6}$$

as follows:

$$(n, t, x) \in \{(1, 1, 1), (2, 1, 2)\}.$$

2. Preliminaries

Before proceeding with the proof of our main result, let us give some necessary information for proof. We give the definition of the logarithmic height of an algebraic number and its some properties.

Definition 2.1. *Let z be an algebraic number of degree d with minimal polynomial*

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \cdot \prod_{i=1}^{d} (x - z_i)$$

where a_i's are relatively prime integers with $a_0 > 0$ and z_i's are conjugates of z. Then

$$h(z) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|z_i|, 1\}\right)\right)$$

is called the logarithmic height of z. *The following proposition gives some properties of logarithmic height that can be found in [9].*

Proposition 2.2. *Let $z, z_1, z_2, \ldots, z_t$ be elements of an algebraic closure of $\mathbb{Q}$ and $m \in \mathbb{Z}$. Then*

1. $h(z_1 \cdots z_t) \leq \sum_{i=1}^{t} h(z_i)$
2. $h(z_1 + \cdots + z_t) \leq \log t + \sum_{i=1}^{t} h(z_i)$
3. $h(z^m) = |m| h(z)$.

We will use the following theorem (see [7] or Theorem 9.4 in [2]) and lemma (see [1] which is a variation of the result due to [4]) for proving our results.

Theorem 2.3. *Let $z_1, z_2, \ldots, z_s$ be nonzero elements of a real algebraic number field $\mathbb{F}$ of degree D, $b_1, b_2, \ldots, b_s$ rational integers. Set*

$$B := \max\{|b_1|, \ldots, |b_s|\}$$

and

$$\Lambda := z_1^{b_1} \ldots z_s^{b_s} - 1.$$

If Λ is nonzero, then

$$\log|\Lambda| > -3 \cdot 30^{s+4} \cdot (s+1)^{5.5} \cdot D^2 \cdot (1+\log D) \cdot (1+\log(sB)) \cdot A_1 \cdots A_s$$

where

$$A_i \geq \max\{D \cdot h(z_i), \left|\log z_i\right|, 0.16\}$$

for all $1 \leq i \leq s$. If $\mathbb{F} = \mathbb{R}$, then

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2 \cdot (1+\log D) \cdot (1+\log B) \cdot A_1 \cdots A_s.$$

Lemma 2.4. *Let A, B, μ be some real numbers with $A > 0$ and $B > 1$ and let γ be an irrational number and M be a positive integer. Take p/q as a convergent of the continued fraction of γ such that $q > 6M$. Set $\varepsilon := \left\|\mu q\right\| - M\left\|\gamma q\right\| > 0$ where $\|\cdot\|$ denotes the distance from the nearest integer. Then there is no solution to the inequality*

$$0 < \left|u\gamma - v + \mu\right| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \text{ and } w \geq \frac{\log \frac{Aq}{\varepsilon}}{\log B}.$$

3. The Proof of Theorem 1.1

Let us write Equations (2) and (5) in Equation (6). We get

$$\left(\frac{\varphi^n}{\sqrt{5}} + \theta\right)\left(\frac{\alpha^n}{2\sqrt{2}} + \lambda\right) = x \cdot \left(\frac{10^t - 1}{9}\right).$$

By using $|\theta| \leq 1/\sqrt{5}$ and $|\lambda| \leq 1/\left(2\sqrt{2}\right)$ we obtain

$$\left|\frac{(\varphi\alpha)^n}{\sqrt{5} \cdot 2\sqrt{2}} - \frac{x \cdot 10^t}{9}\right| < 0.8 \cdot \alpha^n.$$

To convert this inequality into form in Theorem 2.3, let us divide both sides by $(\varphi\alpha)^n / \left(\sqrt{5} \cdot 2\sqrt{2}\right)$. So, we have

$$\left|1 - 10^t \cdot (\varphi\alpha)^{-n} \cdot \left(\left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right)\right| < 5.06 \cdot \varphi^{-n}. \tag{7}$$

Set

$$\Gamma := 10^t \cdot (\varphi\alpha)^{-n} \cdot \left(\left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right) - 1.$$

We claim that $\Gamma \neq 0$. If $\Gamma = 0$, then one can easiliy see that $(\varphi)^{2n} \in \mathbb{Q}(\alpha)$. Since $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$ and φ is an quadratic algebraic number, the degree of $(\varphi)^{2n}$ is either 1 or 2. This means that $(\varphi)^{2n} \in \mathbb{Q}$ but from the Binomial theorem we know that $(\varphi)^{2n}$ is of the form $X_n + Y_n\sqrt{5}$ for some positive rational numbers X_n and Y_n which is a contradiction. Thus, we get $\Gamma \neq 0$.

Let us apply Theorem 2.3 to the inequality (7). Set

$$(z_1, z_2, z_3) = \left(10, \varphi\alpha, \left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right) \text{ and } (b_1, b_2, b_3) = (t, -n, 1).$$

Since $z_i \in \mathbb{Q}\left(\sqrt{2}, \sqrt{5}\right)$, we know that $D = 4$. So, we can take

$$\begin{aligned} 10 = A_1 &\geq 4 \cdot h(10) = 4 \cdot \log(10) \sim 9.21 \\ 3 = A_2 &\geq 4 \cdot h(\varphi\alpha) < 4 \cdot \log(2) \sim 2.77 \\ 25 = A_3 &\geq 4 \cdot h\left(x \cdot \sqrt{5} 2\sqrt{2}/9\right) < 24.96. \end{aligned}$$

Now, let us try to estimate the value of B. From the inequalities (1) and (4), we can write

$$\varphi^{n-1} \cdot \alpha^{n-1} \geq F_n P_n = x \cdot \left(10^{t-1} - 1\right)/9 > 10^{t-1}$$

and this inequality implies that

$$1.68t - 1 < n. \tag{8}$$

Since $t < 1.68t - 1$ for $t > 1$ we can write $t < n$ from the inequality (8). Thus, we take

$$B := n.$$

So, due to the Theorem 2.3 we have

$$|\Gamma| > \exp\left(-C \cdot (1 + \log n) \cdot 10 \cdot 3 \cdot 25\right)$$

where $C := 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 4^2 \cdot (1 + \log 4)$. From the inequality (7), we get

$$\frac{5.06}{\varphi^n} > \exp\left(-C \cdot (1 + \log n) \cdot 10 \cdot 3 \cdot 25\right).$$

Taking logarithm of both sides of the above inequality and considering $C < 5.5 \cdot 10^{12}$ and $1 + \log n < 2\log n$ for $n \geq 3$, we obtain

$$n < 7.1 \cdot 10^{17}. \tag{9}$$

By the inequality (8), we get

$$t < 4.3 \cdot 10^{17}. \tag{10}$$

Now, let us improve the bounds (9) and (10). Set

$$\Omega := t\log 10 - n\log(\varphi\alpha) + \log\left(\left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right).$$

So, we can rewrite the Inequality (7) as

$$\left|1 - e^{\Omega}\right| < \frac{5.06}{\varphi^n}.$$

If $\Omega > 0$, then

$$\Omega < e^{\Omega} - 1 < 5.06 \cdot \varphi^{-n}.$$

Otherwise, i.e., $\Omega < 0$, then

$$1 - e^{-|\Omega|} = \left|e^{\Omega} - 1\right| < 5.06 \cdot \varphi^{-n}.$$

Thus,

$$|\Omega| < e^{|\Omega|} - 1 < \varphi^{-n}/\left(1 - \varphi^{-n}\right) < \varphi^{-n+1}.$$

From this inequality, we get

$$|\Omega| < 5.06 \cdot \varphi^{-n+1}. \tag{11}$$

Now, without loss of the generality, suppose $\Omega > 0$ (operations for the case $\Omega < 0$ are similar). From the Inequality (11), we obtain

$$\begin{aligned} 0 &< t\log 10 - n\log(\varphi\alpha) + \log\left(\left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right) \\ &< 5.06 \cdot \varphi^{-(n-1)}. \end{aligned}$$

Dividing both sides of the above inequality by $\log(\varphi\alpha)$, we get

$$0 < t \cdot \frac{\log 10}{\log(\varphi\alpha)} - n + \frac{\log\left(\left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right)}{\log(\varphi\alpha)} < 3.72 \cdot \varphi^{-(n-1)}.$$

In here, $\gamma := \log 10/\log(\varphi\alpha)$ is an irrational number. Hence, we can apply the Lemma 2.4 to the above inequality with the parameters

$$\mu := \frac{\log\left(\left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right)}{\log(\varphi\alpha)}, \; A := 3.72, \; B := \varphi \text{ and } w := n-1.$$

We can choose $M := 4.3 \cdot 10^{17}$ from the bound (10). So, 41th convergence of γ is satisfies the condition $q > 6M$. From this convergent, we get the smallest ε as 0.00207249. Thus, we have

$$\frac{\log(3.72 \cdot 2714452526429576634/0.00207249)}{\log\varphi} \sim 103.775 \leq n-1$$

and so, we get $n < 104$. Considering this bound on n, we obtain $t < 63$ from the inequality (8). Hence, in Mathematica, for the values $1 \leq n \leq 103$ and $1 \leq t \leq 62$ we get the solutions of the equality (6) as follows:

$$(n, t, x) \in \{(1, 1, 1), (2, 1, 2)\}.$$

This completes the proof.

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